NOTATION

Let:

N be the total number of distinct stocks (companies) in which we can invest

K be the desired number of distinct stocks in the TP

X_i be the number of units of stock i in the current TP

V_{it} be the value of one unit of stock i at time t

T be such that we have observed historical values for stocks and the index over the time period 0,1,2,...,T. The time T represents a decision point, a time at which we **may** switch from our current TP [X_i] to a new TP.

I_t be the value of the index at time t

C be the total value of the current TP [X_i] at time T plus any **cash change** in the portfolio (either new cash available for investment or cash being withdrawn at time T), i.e.

$$C = \sum_{i=1}^{N} V_{iT}X_i + \text{cash change}$$

Then our decision variables are:

x_i the number of units of stock i that we choose to hold in the new TP

z_i = 1 if any of stock i is held in the new TP
= 0 otherwise

Without significant loss of generality we allow [x_i] to take fractional values.

CONSTRAINTS

The constraints associated with the index tracking problem are:

$$\sum_{i=1}^{N} z_i = K$$

$$V_{iT}x_i/C \leq z_i \quad i=1,...,N$$

$$\sum_{i=1}^{N} V_{iT}x_i = C$$

$$x_i \geq 0 \quad i=1,...,N$$

$$z_i \in [0,1] \quad i=1,...,N$$

OBJECTIVES - TRACKING

In time period t we get a return associated with the index, $R_t = \log_e(I_t/I_{t-1})$, where we define return using continuous time.

If, in each and every time period, the return associated with the TP:

$$r_t = \log_e[[\sum_{i=1}^{N} V_{it}x_i]/[\sum_{i=1}^{N} V_{it-1}x_i]]$$

was EXACTLY equal to R_t then this might seem ideal.

A possible objective in terms of index tracking is therefore:

$$\text{minimise} \sum_{t=1}^{T} (r_t - R_t)^2/T$$

i.e. minimise average squared error

EXCESS RETURN

Suppose we are interested in excess return, return over and above the return on the index. Here we seek to out-perform the index (*enhanced indexation*).

How then can we construct TPs that "both track the index and exceed it"? One way is:

- take the return R_t given by the current index
- create an **artificial (enhanced return) index** whose return is $A_t = R_t+R^*$ where R^* is the desired excess return per time period
- track this enhanced return index

The advantage of this approach is:

- the constraints we have given for the index tracking problem automatically apply to the enhanced indexation problem
- any algorithm developed to find TPs for the index tracking problem can “automatically” be applied to the problem of deciding a portfolio of stocks (a TP) so as to outperform the index.

OBJECTIVES - ENHANCED INDEXATION

To define different objectives for enhanced indexation we consider three cases:

Specified out-performance:

- we require $r_t = A_t \; \forall t$
- minimise $\sum_{t=1}^{T} (r_t - A_t)^2/T$

Semi-specified out-performance:

- we require $r_t \geq A_t \; \forall t$
- minimise $\sum_{t=1}^{T} (\min[0, r_t - A_t])^2/T$

Unspecified out-performance:

- we simply wish to out-perform the index
- minimise $\lambda\sqrt{[\sum_{t=1}^{T} (r_t - R_t)^2]}/T$

$$- (1 - \lambda) \sum_{t=1}^{T} (r_t - R_t)/T$$

As an illustration we also consider here objectives related to the Sharpe and Sortino ratios:

Maximise the modified Sharpe ratio:

$$\left(\sum_{t=1}^{T} r_t/T - R^{mean}\right)/\sqrt{\text{Variance}\{r_t\}}$$

where

$$R^{mean} = \left(\sum_{t=1}^{T} R_t/T\right) + R^{*} = \left(\sum_{t=1}^{T} A_t/T\right)$$

Here R^{mean}, a constant equal to the average return on the artificial (enhanced return) index, replaces the risk-free rate in the original Sharpe ratio.

Maximise the modified Sortino ratio:

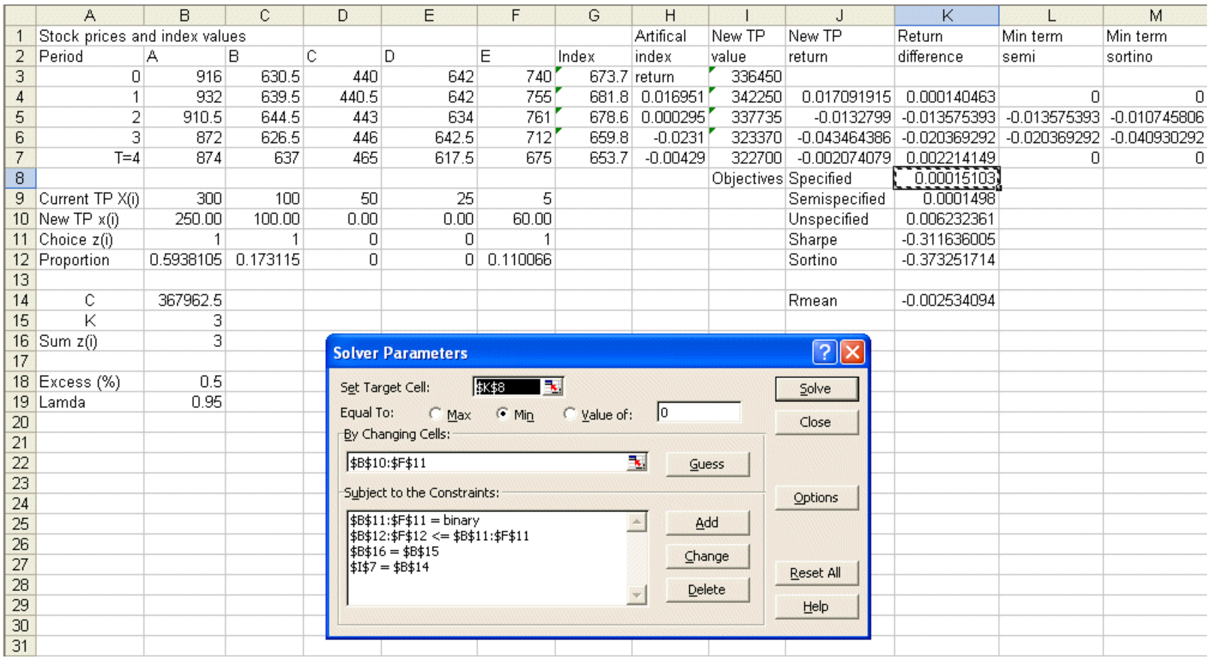
$$\left(\sum_{t=1}^{T} r_t/T - R^{mean}\right)/\sqrt{\left[\sum_{t=1}^{T} [\min(0, r_t - R^{mean})]^2/T\right]}$$

Here R^{mean} replaces the minimum acceptable return in the original Sortino ratio.

The example below is the one we will deal with in class

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Stock prices and index values							Artifical	New TP	New TP	Return	Min term	Min term
2	Period	A	B	C	D	E	Index	index	value	return	difference	semi	sortino
3	0	916	630.5	440	642	740	673.7	return	336450				
4	1	932	639.5	440.5	642	755	681.8	0.016951	342250	0.017091915	0.000140463	0	0
5	2	910.5	644.5	443	634	761	678.6	0.000295	337735	-0.0132799	-0.013575393	-0.013575393	-0.010745806
6	3	872	626.5	446	642.5	712	659.8	-0.0231	323370	-0.043464386	-0.020369292	-0.020369292	-0.040930292
7	T=4	874	637	465	617.5	675	653.7	-0.00429	322700	-0.002074079	0.002214149	0	0
8									Objectives	Specified	0.00015103		
9	Current TP X(i)	300	100	50	25	5				Semispecified	0.0001498		
10	New TP x(i)	250.00	100.00	0.00	0.00	60.00				Unspecified	0.006232361		
11	Choice z(i)	1	1	0	0	1				Sharpe	-0.311636005		
12	Proportion	0.5938105	0.173115	0	0	0.110066				Sortino	-0.373251714		
13													
14	C	367962.5								Rmean	-0.002534094		
15	K	3											
16	Sum z(i)	3											
17													
18	Excess (%)	0.5											
19	Lamda	0.95											

with the Solver model for this example being shown below



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RESULTS – EXAMPLE

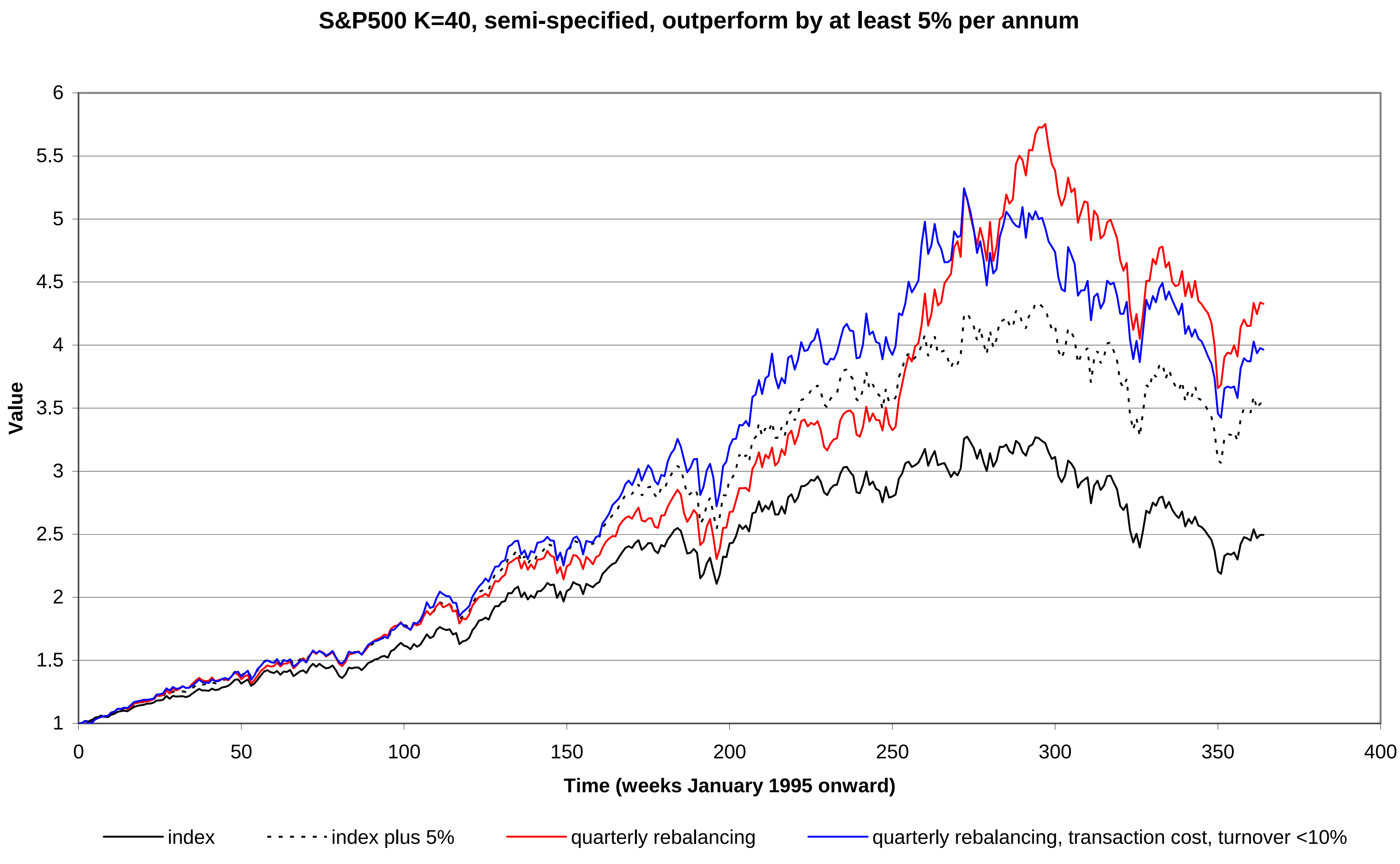
These results below are for a different heuristic algorithm.

Weekly price/index data for the S&P500 (500 stocks) from 1994 to 2002.

The approach adopted was:
repeat
- Decide a TP using our heuristic looking one year into the past
- Hold that TP unchanged into the future for a fixed time period and then rebalance

Results based on:
- a value of K=40
- rebalancing every quarter
- either zero transaction cost or a transaction cost of 15 basis points (0.15%)

Computation time is of the order of 10 minutes per rebalance and the results below show the accumulated value over time of our TPs as compared to the index.


S&P500 K=40, semi-specified, outperform by at least 5% per annum
Value
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0
50
100
150
200
250
300
350
400
Time (weeks January 1995 onward)
index
index plus 5%
quarterly rebalancing
quarterly rebalancing, transaction cost, turnover <10%

RESULTS - GENERAL

Utilising historical back-testing for a number of indices (DAX, Hang Seng, FTSE 100, Nikkei 225, S&P500, Russell 2000 and Russell 3000) over an eight year time period we found:

- The (modified) Sharpe and Sortino objectives hold the greatest promise, offering significant returns for the Hang Seng, FTSE 100 and S&P500.

- The specified and semi-specified out-performance objectives were in general not successful. Out-of-sample, the target out-performance levels were not achieved.

- For the unspecified objective higher risks and returns were associated with lower values of λ (greater emphasis on out-performance). For these cases, significant returns were achieved for most indices.