

Cardinality constrained portfolio optimisation

We consider what happens when we explicitly constrain the total number of assets that can be present in a portfolio, where that portfolio is derived based upon standard Markowitz mean-variance criteria.

Recall that the (unconstrained) Markowitz portfolio optimisation problem is:

$$\text{minimise} \quad \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij} \quad (1)$$

subject to

$$\sum_{i=1}^N w_i \mu_i = R \quad (2)$$

$$\sum_{i=1}^N w_i = 1 \quad (3)$$

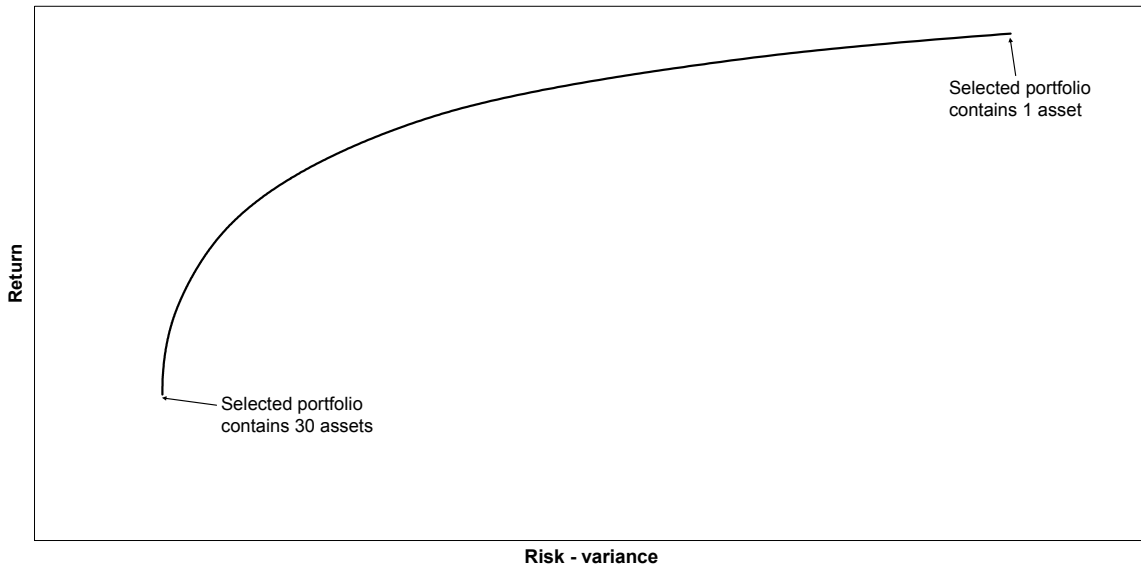
$$0 \leq w_i \leq 1 \quad i=1, \dots, N \quad (4)$$

Equation (1) minimises the total variance (risk) associated with the portfolio (where σ_{ij} is the covariance between the returns associated with assets i and j) whilst equation (2) ensures that the portfolio has an expected return of R . Equation (3) ensures that the proportions add to one.

The point of the above optimisation problem is to construct an *efficient frontier*, (unconstrained efficient frontier, UEF) a smooth non-decreasing curve that gives the best possible tradeoff of **risk** against **return**, i.e. the curve represents the set of **Pareto-optimal (non-dominated)** portfolios.

We have already seen some UEF's previously (for example as below).

Efficient frontier for the FTSE 100



As the above illustrates with respect to the two end points the number of assets in a portfolio on the UEF changes as we move along the frontier.

If we wish to control the number of assets in any portfolio we choose to invest in then we need to add constraints to the problem, the precise nature of which will become apparent below.

Be clear here – restricting the number of assets can only (for a given level of return) increase the risk. This is because the above is the unconstrained efficient frontier, so for a given level of return the risk is at a minimum (as the mathematics for the Markowitz model explicitly requires). The practical reason why we might allow an increase in risk is that we may find it more convenient/less costly in terms of transaction cost to have a portfolio with just a few assets. For example if a portfolio on the UEF contains 25 assets, but there is a portfolio with only a small increase in risk (same return) with 10 assets which do you think you might be tempted to invest in?

CARDINALITY CONSTRAINED PROBLEM

In order to extend our formulation to the cardinality constrained case let:

K be the desired number of assets in the portfolio

Introducing zero-one decision variables:

$z_i = 1$ if any of asset i is held
 $= 0$ otherwise

the cardinality constrained portfolio optimisation problem is:

$$\text{minimise} \quad \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij}$$

subject to

$$\sum_{i=1}^N w_i \mu_i = R$$

$$\sum_{i=1}^N w_i = 1$$

$$\sum_{i=1}^N z_i = K$$

$$0 \leq w_i \leq z_i \quad i=1, \dots, N$$

$$z_i \in [0, 1] \quad i=1, \dots, N$$

The changes from the Markowitz model we have seen before are:

- $\sum_{i=1}^N z_i = K$, meaning we must invest in precisely K assets
- $0 \leq w_i \leq z_i$ meaning that if we do not invest in asset i (so $z_i = 0$) then w_i will also be zero. If $z_i = 1$ then this constraint merely says $0 \leq w_i \leq 1$

Here we have (for simplicity) not introduced constraints on the proportion of the portfolio associated with (invested in) any particular asset. This can be easily done if we so wish.

You should note here that the material presented here relating to cardinality constrained portfolio optimisation is relatively up to date (such material only being reported in the academic literature in 2000). By contrast the (unconstrained) Markowitz model dates from the 1950's. In terms of practical application the main changes since the 1950's for Markowitz are:

- better (cheaper and faster) computers
- better software for solving the quadratic program involved in the Markowitz model
- improved data availability and better quality data

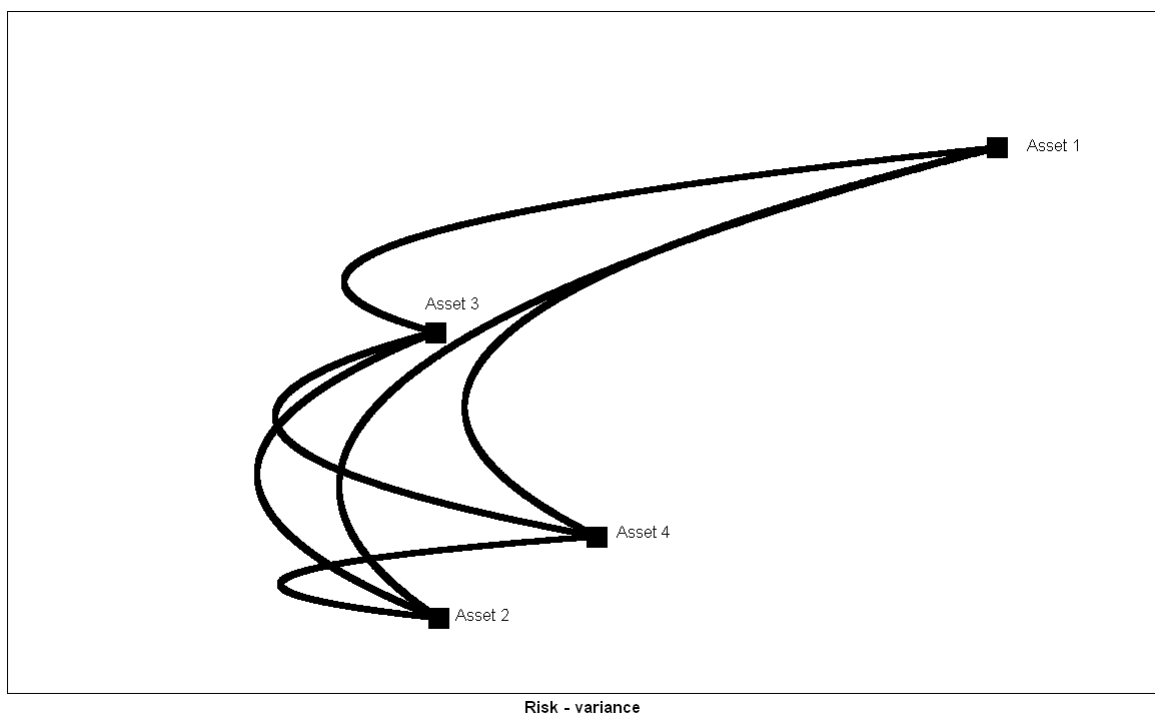
FRONTIER SHAPE

For the unconstrained portfolio optimisation problem the frontier was a nice smooth continuous curve. In particular note that for every possible return (between the minimum return achievable and the maximum return achievable) we had a portfolio on the efficient frontier with an associated risk

In the presence of cardinality constraints the efficient frontier may become **discontinuous**, where the discontinuities imply that there are certain returns which no rational investor would consider (since there exist portfolios with less risk and greater return). This means the efficient frontier in the cardinality constrained case is distinctly different from that in the unconstrained case.

To illustrate this point the figure below shows four assets (stocks) drawn from the FTSE 100. In that figure all possible portfolios involving **exactly** two assets ($K=2$) are shown.

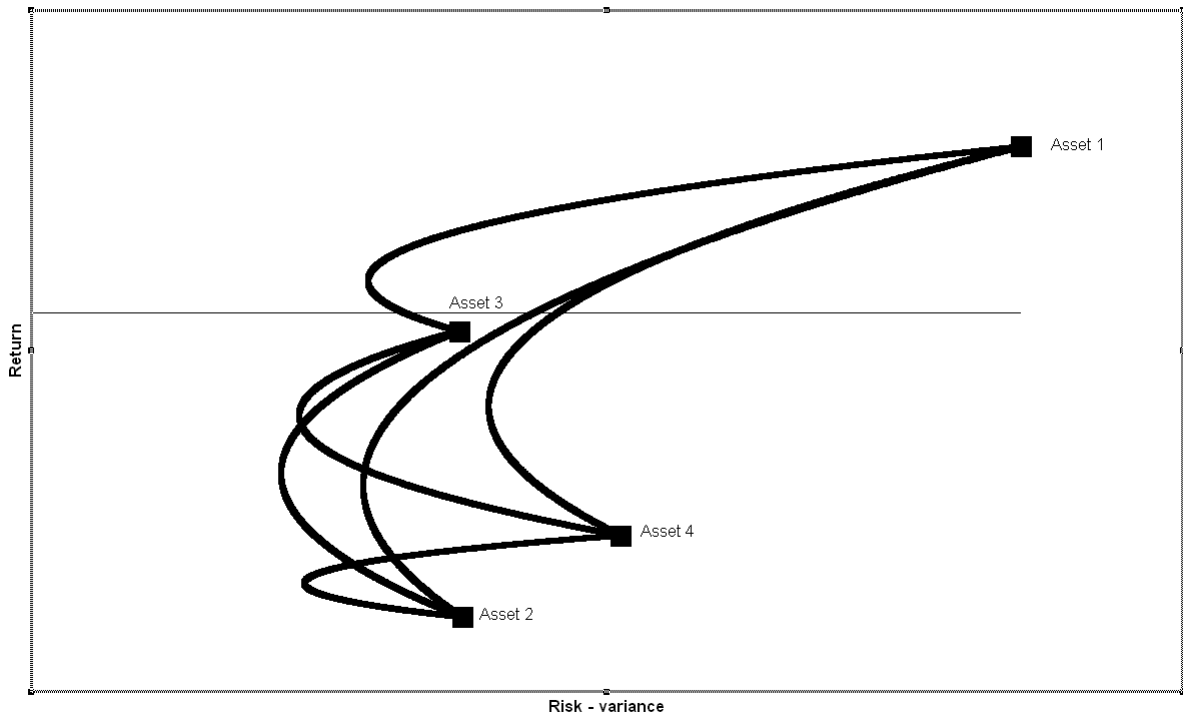
For four assets there are $4 \times 3 / 2 = 6$ possible choices of pairs of assets, each of which leads to a different curve in the figure below.



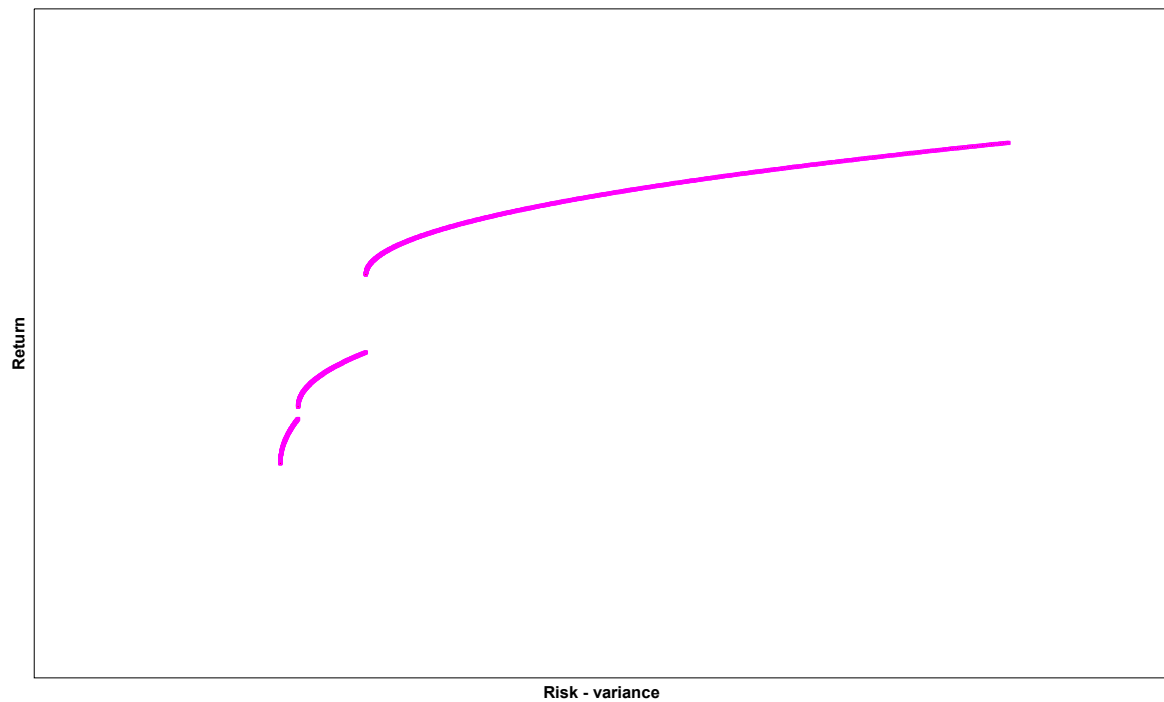
Suppose we are interested in the return shown on the horizontal line in the diagram below. There are only three possible portfolios that achieve this return:

- one where the horizontal line intersects the curve between assets 3 and 1 (so a portfolio consisting of those two assets in some proportion)
- one where the horizontal line intersects the curve between assets 2 and 1 (so a portfolio consisting of those two assets in some proportion)
- one where the horizontal line intersects the curve between assets 4 and 1 (so a portfolio consisting of those two assets in some proportion)

Are these three portfolios dominated or not ?



The figure below shows the efficient frontier as derived from the figure above. Note the discontinuities – where there are return values for which there is no portfolio having that return which is not dominated. We refer to the cardinality constrained efficient frontier as the CCEF.



The example below is the one we will deal with in class

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Stock prices							Returns					
2	A	B	C	D	E	Period		A	B	C	D	E	Period
3	916	630.5	440	642	740	0		1.73	1.42	0.11	0.00	2.01	0
4	932	639.5	440.5	642	755	1		-2.33	0.78	0.57	-1.25	0.79	2
5	910.5	644.5	443	634	761	2		-4.32	-2.83	0.67	1.33	-6.66	3
6	872	626.5	446	642.5	712	3		0.23	1.66	4.17	-3.97	-5.34	T=4
7	874	637	465	617.5	675	T=4		-1.17	0.26	1.38	-0.97	-2.30	
8						Average							
9													
10	Weights w()							Covariance COVAR (2dp)					
11	A	B	C	D	E			A	B	C	D	E	
12	0.00	0.00	0.00	0.00	1.00			5.41	3.61	0.85	-2.08	4.59	A
13	z()							3.61	3.28	1.05	-2.59	3.95	B
14	0	0	0	1	1			0.85	1.05	2.64	-2.75	-3.34	C
15								-2.08	-2.59	-2.75	3.83	0.60	D
16	Working							4.59	3.95	-3.34	0.60	14.07	E
17	4.59	3.95	-3.34	0.60	14.07								
18								CCEF					
19	sum z()	2						14.07	-2.30	-2.30	1.38	14.07	Risk
20	sum w()	1						5.18	-1.69	-1.07		4.98	Return
21	risk	14.07						1.26	-1.07			1.24	-1.07
22	return	actual	desired	K				0.61	-0.46			0.34	-0.46
23		-2.30	-2.30	2				0.27	0.15			0.02	0.15
24								0.64	0.77			0.64	0.77
25	For an array formula:							2.64	1.38			2.64	1.38
26	highlight cells where the results will go												
27	type formula												
28	do ctrl-shift-enter												
29													
30													
31													
32													
33													
34													
35													
36													
37													
38													
39													
40													
41													
42													
43													

Solver Parameters

Set Target Cell: To: Max Min Value of:

By Changing Variable Cells:

Subject to the Constraints:

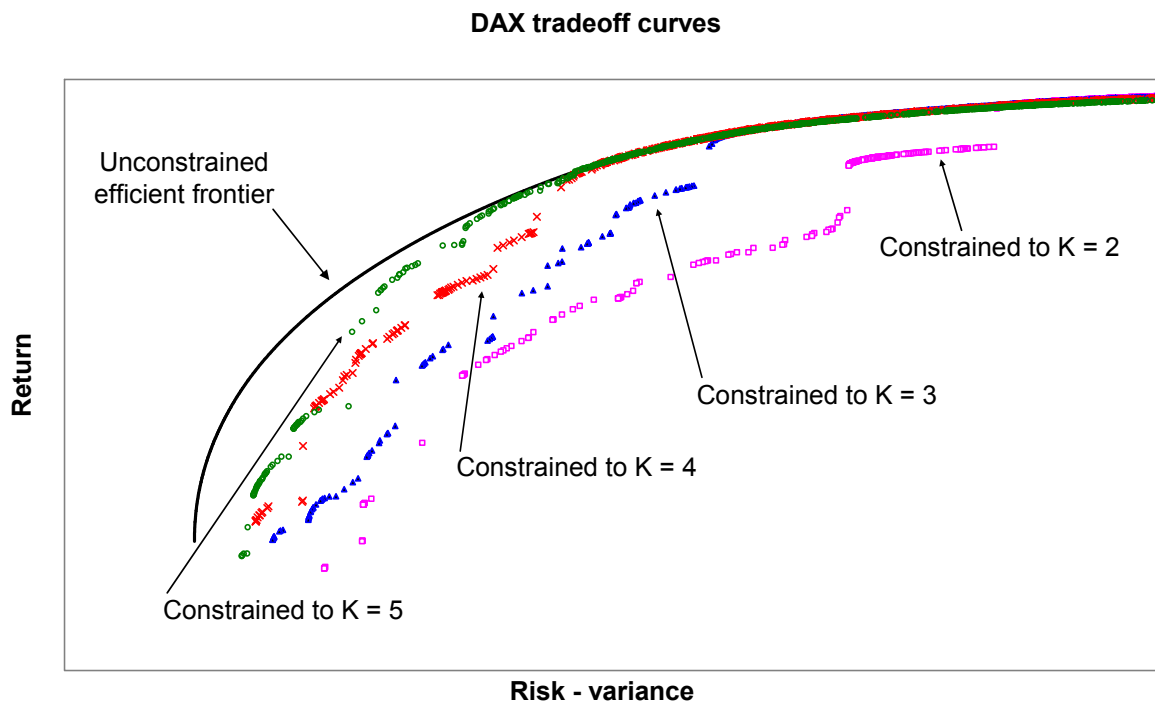
- Add
- Change
- Delete
-
-
-

do 5 intermediate points
return step 0.61

UEF Risk

Solve Close Options Reset All Help

As an illustration of what can be achieved with cardinality constrained portfolio optimisation we show below some tradeoff curves for assets chosen from the DAX index.



On a practical and computational note you should be aware that computing the CCEF is a much harder task than computing the UEF. Technically this is because the mathematical model for cardinality constrained portfolio optimisation is a zero-one (binary) quadratic programming problem.

In particular note that in regard to our Solver model above Solver is using a **heuristic** solution technique (unlike linear/integer programming dealt with previously where we had an optimal solution technique). As such Solver cannot guarantee to find the optimal (best possible) solution.