An Asymptotic Model For The Dynamic Motion In An Incompressible Plate

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1. Introduction

Modern material technology now enables materials to be manufactured which are able to withstand large deformation and support high loads prior to failure. In this paper we seek to further elucidate the dynamic characteristics of elastic structures subject to large homogeneous primary deformations. For an excellent account of the appropriate underlying theory the reader is referred to [1]. The specific problem to be considered concerns long wave high frequency motions of a pre-stressed incompressible elastic plate. As such this generalises an earlier study for which a plane strain assumption was made, see [2], and complements another previous investigation concerning short wave approximations, see [3]. The theory has applications to various problems involving radiation and scattering and seeks to derive an asymptotically consistent two-dimensional model.

We shall consider a body B composed of an incompressible isotropic elastic material. It is subjected to a purely homogeneous primary static deformation, which transforms the solid from its initial unstressed state B_0 to a finetely stressed equilibrium configuration B_e, with resulting configuration termed the current and denoted by B_t. The position vectors of a representative particle in B_0, B_e and B_t are denoted by X_A, $x_i(X_A)$ and $\tilde{x}_i(X_A) = x_i(X_A) + u_i x_i(X_A), t)$ respectively, where $u_i(x_i(X_A), t)$ are the components of the incremental displacement associated with the deformation $B_e \to B_t$. The deformation gradients $\overline{F}$ and F, arising from the deformations and $\mathrm{B_e} \to \mathrm{B_t}$ and defined through the component relations $\overline{F}_{iA} = \dfrac{\partial x_i}{\partial X_A}$, $F_{iA} = \dfrac{\partial \tilde{x}_i}{\partial X_A}$, furthermore the definition of $\tilde{x}(X_A)$ allows us to conclude that $F_{iA} = (\delta_{ij} + u_{ij})\overline{F}_{jA}$, where the comma denotes differentiation with respect to the implied spatial coordinate component in B_e and δ_{ij} is the Kronecker delta. Every possible material deformation is isotropic, from which it may be deduced that $J - 1 = 0$, $J = \det \mathbf{F}$ [4]. For problems involving internal constraints it is usual to introduce the pseudo strain energy function and in the case of an incompressible material it takes the from $W(\mathbf{F}, p) = W_0(\mathbf{F}) - p(J - 1)$. The first term on the right hand side generates the constitutive part of the stress whilst the latter term generates a workless reaction stress to support the constraint. This extra term is constrained to be zero throughout all possible material deformations, thus the scalar p acts as a Lagrange multiplier. It is interpreted as a pressure and must ultimately be chosen so as to satisfy the equations of motion and the boundary conditions. The pressure may be decomposed into a static pressure $\overline{p}$ in B_e and a small increment $p_t = p_t(t)$, so $p = \overline{p} + p_t$. In the absence of body forces the equations of motion are given by $(\pi_{iA}\overline{F}_{mA})_{,m} = \rho \ddot{u}_i$, where π_{iA} is the first Piola-Kirchhoff stress tensor, ρ is the material density, a superimposed dot denotes differentiation with respect to the time and summation over the repeated suffices is assumed. In view of the fact that the time-dependent part of the deformation is assumed to be sufficiently small and $\pi_{iA} = \partial W_0 / \partial F_{iA}$, see Spencer [4], we may linearise the equations of motion about the static state B_e, this yields

$$B_{milk} u_{k,lm} - p_{t,i} = \rho \ddot{u}_i, \tag{1}$$

within which B_{milk} is the fourth order elasticity tensor, defined by $B_{milk}\overline{F}_{lB}\overline{F}_{mA} \left.\dfrac{\partial^2 W_0}{\partial F_{aB}\partial F_{iA}}\right|_{B_e}$. An analogous approach may be used to obtain an appropriate measure of incremental surface traction

$$\tau_i = B_{milk} u_{k,l} n_m + \overline{p} u_{m,i} n_m - p_t n_i, \tag{2}$$

where $\mathbf{n}$ is the outward unit normal to a material surface in B_e.

Consider now the case in which the aforementioned solids forms an infinit elastic layer of half-thickness h. An appropriate Cartesian Coordinate System $Ox_1x_2x_3$ is chosen with the origin O in the mid-plane, the x_2 axis being orthogonal to the plane of the layer and the axes x_1, x_3 assumed coincident with the principal axes of the in plane pre-stress. All non-zero components of the elasticity tensor $\boldsymbol{B}$ for an incompressible isotropic material have one of three general forms B_{iijj}, B_{ijij} and B_{ijji}, (i, j ∈ {1, 2, 3}), [1]. Thus the equation of motion (1) may now be expressed in the explicit form

$$B_{1111}u_{1,11} + (B_{1122} + B_{2112})u_{2,12} + (B_{1133} + B_{3113})u_{3,13} + B_{2121}u_{1,22} + B_{3131}u_{1,33} - p_{t,1} = \rho\ddot{u}_1,$$
$$B_{2222}u_{2,22} + (B_{2211} + B_{1221})u_{1,12} + (B_{2233} + B_{3223})u_{3,23} + B_{1212}u_{2,11} + B_{3232}u_{2,33} - p_{t,2} = \rho\ddot{u}_2, \quad (3)$$
$$B_{3333}u_{3,33} + (B_{3311} + B_{1331})u_{1,13} + (B_{3322} + B_{2332})u_{2,23} + B_{1313}u_{3,11} + B_{2323}u_{3,22} - p_{t,3} = \rho\ddot{u}_3,$$

and must be solved in conjuction with the linearlised incompressibility constraints

$$u_{1,1} + u_{2,2} + u_{3,3} = 0. \quad (4)$$

we seek the solutions of Eqns. (3) and (4) in the form of waves travelling in the plane of the layer along the direction **d** at an angle α to the Ox_1 axis. Then **d** = (cos α, 0, sin α) and the solutions may be written as

$$(u_1, u_2, u_3, p_t) = (U, V, W, kP)e^{kqx_2}e^{i(kx_1d_1 + kx_3d_3 - \omega t)}, \quad (5)$$

in which k is the wave number, ω is the frequency and q is to be determined. Substituting (5) into (3) and making use of (4) it is easy to establish that the non-trivial solutions of the form (5) will exist provided

$$\gamma_{21}\gamma_{23}q^6 + [(\gamma_{23} + \gamma_{21})\bar{v}^2 - \mu_1]q^4 + (\bar{v}^4 - \mu_2\bar{v}^2 + \mu_3)q^2 - (\bar{v}^2 - \mu_4)(\bar{v}^2 - \mu_5) = 0, \quad (6)$$

where $\bar{v}$ denotes the scaled wave in speed $\bar{v} = \sqrt{\rho}v = \sqrt{\rho}\omega / k$,

$$\mu_1 = (2\beta_{12}\gamma_{23} + \gamma_{21}\gamma_{13})d_1^2 + (2\beta_{23}\gamma_{21} + \gamma_{23}\gamma_{31})d_3^2, \qquad \mu_5 = \gamma_{13}d_1^4 + 2\beta_{13}d_1^2d_3^2 + \gamma_{31}d_3^4,$$
$$\mu_2 = (2\beta_{12}\gamma_{23} + \gamma_{13})d_1^2 + (2\beta_{23} + \gamma_{21} + \gamma_{31})d_3^2, \quad \mu_4 = \gamma_{12}d_1^2 + \gamma_{32}d_3^2,$$
$$\mu_3 = (2\beta_{12}\gamma_{13} + \gamma_{23}\gamma_{12})d_1^4 + (2\beta_{23}\gamma_{31} + \gamma_{21}\gamma_{32})d_3^4 + (4\beta_{12}\beta_{23} + \gamma_{12}\gamma_{21} + \gamma_{13}\gamma_{31} + \gamma_{23}\gamma_{32} - B_{13}^2)d_1^2d_3^2, \quad (7)$$

and $\gamma_{ij} = B_{ijij}$, $2\beta_{ij} = B_{iiii} + B_{jjjj} - 2B_{iijj} - 2B_{ijji}$, $B_{ij} = \beta_{ij} - \beta_{ik} - \beta_{jk}$, $(k \neq i, j)$. The roots of the secular Eqn. (6) are denoted as $\pm q_i$ and the general solutions for U, V, W, P may then be represented as the linear combinations of functions of the form $E_i^{\pm} = \exp(\pm kq_ix_2)$.

2. Dispersion Relation

The dispersion relation associated with the wave propagation in an elastic plate subject to the traction free boundary conditions on the surfaces of the plate is to be derived. This can be done by inserting the travelling wave solutions (5) into the relations for the traction increments (2), subsequently substituting the functions U, V, W, P with their representations in terms of the functions $E_i^{\pm}$ and, finally, applying the boundary conditions in the form $\tau_i = 0$, at $x_2 = \pm h$. The homogeneous system of 6 equations found in this way may then be split into two systems of 3 equations associated with the flexural and extensional waves, respectively. We will focus our attention on the flexural waves, remarking that the results for the extensional waves can be obtained in a very similar manner. For the reasons of brevity the derivation of the dispersion relation is omitted, however a detailed derivation, in slightly different notation, is given by Rogerson and Sandiford [3]. The dispersion relation associated with the flexural waves is given by

$$\Phi(q_2, q_3, \bar{v})\Psi(q_1, \bar{v})T_1q_1 - \Phi(q_1, q_3, \bar{v})\Psi(q_2, \bar{v})T_2q_2 + \Phi(q_1, q_2, \bar{v})\Psi(q_3, \bar{v})T_3q_3 = 0, \quad (8)$$

with $T_i = \tanh(kq_ih)$ and the functions $\Phi(q_i, q_j, \bar{v}), \Psi(q_i, \bar{v})$ defined as

$$\Phi(q_i, q_j, \bar{v}) = (q_i^2 - q_j^2)\left\{\gamma_{23}\gamma_{21}\Phi_1(\bar{v})q_i^2q_j^2 + (\bar{v}^2 - \mu_5)(\gamma_{23}\gamma_{21}(\gamma_{23} - \gamma_{21})(q_i^2 + q_j^2) - \Phi_2(\bar{v}))\right\},$$
$$\Phi_1(\bar{v}) = (\gamma_{23} - \gamma_{21})\bar{v} + (\gamma_{23}(B_{23} + \gamma_{23} - \gamma_{21}) + \gamma_{21}\gamma_{13})d_1^2 - (\gamma_{21}(B_{12} - \gamma_{23} + \gamma_{21}) + \gamma_{23}\gamma_{31})d_3^2,$$
$$\Phi_2(\bar{v}) = \gamma_{23}(\gamma_{21} - \sigma_2)(B_{23}d_1^2 - \gamma_{31}d_3^2 + \bar{v}^2) + \gamma_{21}(\gamma_{23} - \sigma_2) + (\gamma_{13}d_1^2 - B_{12}d_3^2 - \bar{v}^2),$$
$$\Psi(q_i, \bar{v}) = \left\{(\gamma_{21} + \gamma_{13} + 2\beta_{12} - \sigma_2)d_1^2 + (\gamma_{23} + \gamma_{31} + 2\beta_{23} - \sigma_2)d_3^2\right\}\bar{v}^2 - (\bar{v}^2 + \gamma_{21}q_i^2) + (\bar{v}^2 + \gamma_{23}q_i^2) +$$

$$\left\{\gamma_{21}(\gamma_{13}d_1^2+(\gamma_{23}+2\beta_{23}-\sigma_2)d_3^2)+\gamma_{23}((\gamma_{21}+2\beta_{12}-\sigma_2)d_1^2+\gamma_{31}d_3^2)\right\}q_i^2-\gamma_{13}(\gamma_{21}+2\beta_{12}-\sigma_2)d_1^4+$$

$$2\beta_{13}\sigma_2+\gamma_{23}B_{23}+\gamma_{21}B_{12}-\gamma_{31}\gamma_{13}+B_{13}^2-4\beta_{23}\beta_{12})d_1^2d_3^2-\gamma_{31}(\gamma_{23}+2\beta_{23}-\sigma_2)d_3^4.$$

The possible behaviour of the solutions of dispersion relations (8) is demonstrated in Fig. 1. The first plot shows phase speed $\bar{v}$ against the scaled wave number kh, the second shows frequency ω against scaled wave number kh for the same set of material paramaters. In this paper we consider the long wave high frequency motions, in other words we consider the motions connected with the harmonics at the low wave number. These motions are characterised by the cut-off frequencies and it is worth noting that these plots clearly show presence of two distinct sets of the cut-off frequencies for the dispersion relation (8), whereas for the analogous problem solved by Rogerson and Nolde [2] under the assumption of a plane strain, only one set of the cut-off frequencies is present.

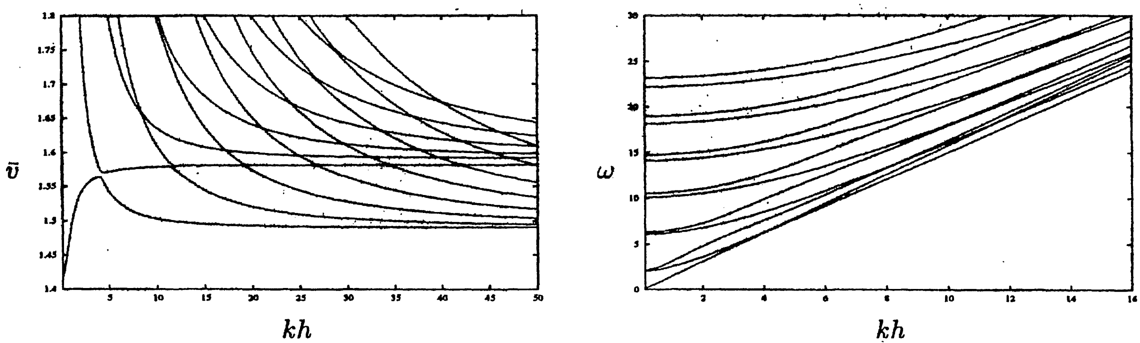

Figure 1. : Numerical solution of the dispersion relation (8). The fundamental mode and first 12 harmonics are shown for the propagation at angle $\alpha = 30^{\circ}$ to Ox_1 in a typical Mooney-Rivlin material.

Even simple observation of Fig. 1 indicates that for the harmonics $\bar{v} \gg 1$ as $kh \to 0$. Further analysis of the secular Eqn. (6) allows us to conclude that $kh \to 0$ the two roots of (6) denoted by q_1^2 and q_3^2 behave like $O(\bar{v}^2)$ and the root denoted by q_2^2 behaves like $O(1)$ More specifically,

$$\gamma_{21}q_1^2=-\bar{v}^2+(2\beta_{12}-\gamma_{21})d_1^2+\gamma_{31}d_3^2+O(\bar{v}^{-2}),\quad q_2^2=1+O(\bar{v}^{-2}),\quad \gamma_{23}q_3^2=-\bar{v}^2+O(\bar{v}^0). \tag{9}$$

Now the leading order approximation of the dispersion relation (8) may be obtained by inserting approximations (9) into (8), this yields

$$D_1T_1+\bar{v}^3D_2T_2+D_3T_3\sim 0, \tag{10}$$

where D_i are $O(1)$ functions of the material parameters independent of $\bar{v}$. The approximation $(9)_2$ of q_2 indicates that $T_2=O(\bar{v}^2)$, then there are two possible cases in which terms on the left hand side of the relation (10) can be balanced asymptotic, these being $T_1=O(\bar{v}^2)$ or $T_3=O(\bar{v}^2)$. Each of these cases is associated with a particular set of cut-off frequencies, hence it supports the observation made before on the existence of two distinct sets of the cuto-off frequencies. For the sake of brevity only the first case will be further investigated, namely $T_1=O(\bar{v}^2)$. Since for the harmonics $\bar{v} \gg 1$ as $kh \to 0$ we may assume upon the use of $(9)_1$ that

$$k\hat{q}_1h=(\tfrac{1}{2}+n)\pi+\varsigma_1^{(2)}k^2h^2+O(k^4h^4)\Rightarrow \tan(k\hat{q}_1h)=(1/\varsigma_1^{(2)}k^2h^2)+O(1). \tag{11}$$

Invoking the relations (9) we may now find appropriate asymptotic representations for q_2, T_2, q_3, T_3 and then determine the value $\varsigma_1^{(2)}$, which can be done by considering the leading order of the dispersion relation (10). Finally we resort the equality $(9)_1$ to obtain the approximation of the scaled wave speed in the low wave number regime

$$\bar{v}^2=\left(\frac{\Lambda_{1n}^2}{k^2h^2}\right)+\bar{v}^{(0,1)}d_1+\bar{v}^{(0,3)}d_3+O(k^2h^2),\ \bar{v}^{(0,1)}=2\frac{(2\gamma_{21}-\sigma_2)^2}{\Lambda_{1n}^2}+2\beta_{12}-\gamma_{21},\ \bar{v}^{(0,3)}=\gamma_{31}. \tag{12}$$

where the values $\Lambda_{1n} = \sqrt{\gamma_{21}}(½+n)\pi$ from the first set of the cut-off frequencies and n indicates for the harmonic number. A plot of the frequency approximations derived from (12) are compared with the numerical solution for the first six harmonics in Fig. 2.

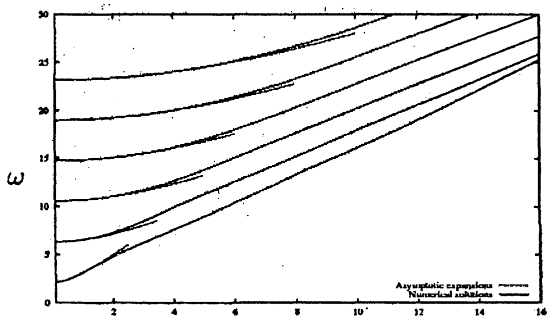

Figure 2 : The numerical and aymptotic solutions for the harmonics 1-6 associated with the first set of the cut-off frequencies for the same material as used in the Figure (1).

3. Asymptotically Approximate Equations

In order to derive asymptotically approximate equations the parameter $\eta = h/l$ is introduced, where l is the wave length, therefore $\eta = kh$ indicating that parameter η is small when the wave number is small. The asymptotics for the scaled wave speed (12) obtained in the previous section can be used to determine the relative orders of the functions U, V, W, P and consequently the relative orders of the displacement components u_i and an incremental pressure p_t. An appropriate analysis reveals that for the first set of the cut-off frequencies at low wave number U, V, W, P have the following asymptotic orders : $U = O(1)$, $V = O(\eta)$, $W = O(\eta^2)$, $P = O(1)$. This suggests the introduction of the non-dimensional displacement and incremental pressure components in the form

$$u_\alpha(x_1,x_2,x_3,t) = l\eta^{\alpha-1}u_\alpha^*(\xi_1,\xi_2,\xi_3,\tau), \quad p_t(x_1,x_2,x_3,t) = B_{2121}p_t^*(\xi_1,\xi_2,\xi_3,\tau), \tag{13}$$

where superscript * indicates quantities of the same asymptotic order and no sum is assumed on α. An appropriate scaling of the space and time variables is defined by the formulas $x_i = l\xi_i$, $(i \in \{1,3\})$, $x_2 = \eta l\xi_2$, $t = l\eta\sqrt{\rho/B_{2121}}\,\tau$. The choice of scales is made with the purpose of balancing the time in which typical wave travels a distance of one wavelength. It is also noted for future references, that for the motions in the vicinity of cut-off frequencies

$$B_{2121}\ddot{u}_i^* + \Lambda_{1n}^2 u_i^* \sim \eta^2 u_i^*. \tag{14}$$

Substitution of the scaled space and time variables and scaled displacements (13) into the equations of motions (3) yields

$$B_{2121}u_{1,22}^* + \Lambda_{1n}^2 u_1^* + \eta^2\left(B_{1111}u_{1,11}^* + B_{3131}u_{1,33}^* + (B_{1122}+B_{2112})u_{2,12}^* - B_{2121}p_{t,1}^*\right)$$
$$-\left(B_{2121}\ddot{u}_{1,22}^* + \Lambda_{1n}^2 u_1^*\right) + \eta^4(B_{1122}+B_{2112})u_{3,13}^* = 0,$$
$$B_{2222}u_{2,22}^* + \Lambda_{1n}^2 u_2^* - \left(B_{2121}u_2^* + \Lambda_{1n}^2 u_2^*\right) + (B_{2211}+B_{1221})u_{1,12}^* - B_{2121}p_{t,2}^*$$
$$+\eta^2\left(B_{1212}u_{2,11}^* + B_{3232}u_{2,33}^* + (B_{2233}+B_{3223})u_{3,23}^*\right) = 0,$$
$$B_{2323}u_{3,22}^* + \Lambda_{1n}^2 u_3^* - \left(B_{2121}u_3^* + \Lambda_{1n}^2 u_3^*\right) + (B_{3311}+B_{1331})u_{1,13}^*$$
$$+(B_{3322}+B_{2332})u_{2,23}^* - B_{2121}p_{t,3}^* + \eta^2\left(B_{1313}u_{3,11}^* + B_{3333}u_{3,33}^*\right) = 0, \tag{15}$$

where the comma notation is preserved in a sense that the comma denotes the differentiation with respect to the implied *scaled* spatial coordinates. Eqns. (15) must be solved in conjunction with the scaled incompressibility constraints $u^*_{1,1}+u^*_{2,2}+\eta^2 u^*_{3,3}=0$, subjected to the boundary conditions

$$B_{2121}u^*_{1,2}+\left(B_{2112}+\bar{p}\right)u^*_{2,1}=0,\qquad \left(B_{2332}+\bar{p}\right)u^*_{2,3}+B_{2323}u^*_{3,2}=0,\qquad \text{at } \xi_2=\pm 1,$$

$$B_{2211}u^*_{1,1}+\left(B_{2222}+\bar{p}\right)u^*_{2,2}+\eta^2 B_{2233}u^*_{3,3}-B_{2121}p^*_t=0,\qquad \text{at } \xi_2=\pm 1. \tag{16}$$

We search the solutions for u^*_i, p^*_t in the form of the power series expansions

$$u^*_i=u^{(0)}_i+\eta^2 u^{(2)}_i+O(\eta^4),\qquad p^*_t=p^{(0)}_t+\eta^2 p^{(2)}_t+O(\eta^4), \tag{17}$$

which form (15), (16) together with the relations (14) gives the leading order problem

$$B_{2121}u^{(0)}_{1,22}+\Lambda^2_{1n}u^{(0)}_1=0,\quad B_{2222}u^{(0)}_{2,22}+\Lambda^2_{1n}u^{(0)}_2+\left(B_{2211}+B_{1221}\right)u^{(0)}_{1,12}-B_{2121}p^{(0)}_{t,2}=0, \tag{18}$$

$$B_{2323}u^{(0)}_{3,22}+\Lambda^2_{1n}u^{(0)}_3+\left(B_{3311}+B_{1331}\right)u^{(0)}_{1,13}+\left(B_{3322}+B_{2332}\right)u^{(0)}_{2,23}-B_{2121}p^{(0)}_{t,3}=0, \tag{19}$$

$$u^{(0)}_{1,1}+u^{(0)}_{2,2}=0,\quad u^{(0)}_{1,2}=0,\quad B_{2211}u^{(0)}_{1,1}+\left(B_{2222}+\bar{p}\right)u^{(0)}_{2,2}-B_{2121}p^{(0)}_t=0,\quad \text{at } \xi_2=\pm 1, \tag{20}$$

$$\left(B_{2332}+\bar{p}\right)u^{(0)}_{2,3}+B_{2323}u^{(0)}_{3,2}=0,\quad \text{at } \xi_2=\pm 1. \tag{21}$$

The solutions of the boundary-value problem $(18)_1$, $(20)_2$ is

$$u^{(0)}_1=U^{(0,1)}_1(\xi_1,\xi_3,\tau)\sin\left(\frac{\Lambda_{1n}\xi_2}{\sqrt{B_{2121}}}\right). \tag{22}$$

Thereon the incompressibility constraint $(20)_1$ can be used to find the general solutions for $u^{(0)}_2$. The boundary-value problem $(18)_2$, (21) then delivers the solutions for $u^{(0)}_2$ and $p^{(0)}_t$

$$u^{(0)}_2=\left\{\frac{\sqrt{B_{2121}}}{\Lambda_{1n}}\cos\left(\frac{\Lambda_{1n}\xi_2}{\sqrt{B_{2121}}}\right)-\frac{\left(2B_{2121}-\sigma_2\right)}{\Lambda^2_{1n}}\sin\left(\frac{\Lambda_{1n}\xi_2}{\sqrt{B_{2121}}}\right)\right\}U^{(0,1)}_{1,1}(\xi_1,\xi_3,\tau),$$

$$B_{2121}p^{(0)}_t=\left\{\left(B_{2121}+B_{2211}+B_{1221}-B_{2222}\right)\sin\left(\frac{\Lambda_{1n}\xi_2}{\sqrt{B_{2121}}}\right)\right.$$

$$\left.-\left(2B_{2121}-\sigma_2\right)\sin\left(\frac{\Lambda_{1n}\xi_2}{\sqrt{B_{2121}}}\right)\xi_2\right\}U^{(0,1)}_{1,1}(\xi_1,\xi_3,\tau). \tag{23}$$

Finally, the solution for $u^{(0)}_3$ can be found from (19), $(20)_3$ and may be represented in its general form as

$$u^{(0)}_3=\left\{U^{(0,1)}_3(\xi_1,\xi_3,\tau)\sin\left(\frac{\Lambda_{1n}\xi_2}{\sqrt{B_{2323}}}\right)+U^{(0,2)}_3(\xi_1,\xi_3,\tau)\sin\left(\frac{\Lambda_{1n}\xi_2}{\sqrt{B_{2121}}}\right)\right.$$

$$\left.+U^{(0,3)}_3(\xi_1,\xi_3,\tau)\sin\left(\frac{\Lambda_{1n}\xi_2}{\sqrt{B_{2121}}}\right)\xi_2\right\}U^{(0,1)}_{1,13}(\xi_1,\xi_3,\tau). \tag{24}$$

It is worth noting that all of the displacement components and pressure increment are defined in terms of the function $U^{(0,1)}_1(\xi_1,\xi_3,\tau)$, which may be interpreted as the long wave amplitude.

It is impossible to determine the governing equation $U^{(0,1)}_1(\xi_1,\xi_3,\tau)$ without resorting to the higher order. At second order the equation of motion $(15)_1$ together with the boundary conditions $(16)_1$ give the boundary value problem for $u_1^{(2)}$

$$B_{2121}u_{1,22}^{(2)} + \Lambda_{1n}^2 u_1^{(2)} = \eta^{-2}\left(B_{2121}\ddot{u}_1^{(0)} + \Lambda_{1n}^2 u_1^{(0)}\right)$$
$$-B_{1111}u_{1,11}^{(0)} - B_{3131}u_{1,33}^{(0)} - \left(B_{1122} + B_{2112}\right)u_{2,12}^{(0)} + B_{2121}p_{t,1}^{(0)},$$

$$B_{2121}u_{1,2}^{(2)} + \left(B_{2112} + \bar{p}\right)u_{2,1}^{(0)} = 0, \quad \text{at } \xi_2 = \pm 1. \tag{25}$$

which, recalling that u_1^* must be odd function of ξ_2, is solved to yield

$$u_1^{(2)} = U_1^{(2,1)}(\xi_1,\xi_3,\tau)\sin\left(\frac{\Lambda_{1n}\xi_2}{\sqrt{B_{2121}}}\right)$$
$$-\xi_2\left\{\frac{\left(2B_{2121} - \sigma_2\right)^2}{\Lambda_{1n}^3\sqrt{B_{2121}}}\cos\left(\frac{\Lambda_{1n}\xi_2}{\sqrt{B_{2121}}}\right) + \frac{\left(2B_{2121} - \sigma_2\right)}{\Lambda_{1n}^2}\sin\left(\frac{\Lambda_{1n}}{\sqrt{B_{2121}}}\right)\right\}U_{1,11}^{(0,1)}(\xi_1,\xi_3,\tau). \tag{26}$$

This solution is only valid when the following equation for $U_1^{(0,1)}(\xi_1,\xi_3,\tau)$ is satisfied

$$\gamma_{21}\ddot{U}_1^{(0,1)}(\xi_1,\xi_3,\tau) + \Lambda_{1n}^2 U_1^{(0,1)}(\xi_1,\xi_3,\tau) - \eta^2\left(\bar{v}^{(0,1)}U_{1,11}^{(0,1)}(\xi_1,\xi_3,\tau) + \bar{v}^{(0,3)}U_{1,33}^{(0,1)}(\xi_1,\xi_3,\tau)\right) = 0, \tag{27}$$

in which use has been made of the previous parameter definitions $(12)_{2,3}$. Finally we may use (13) to recast this equation in terms of the intial dimensional variables, that is

$$\rho h^2\ddot{U}_1^{(0,1)}(\xi_1,\xi_3,\tau) + \Lambda_{1n}^2 U_1^{(0,1)}(\xi_1,\xi_3,\tau) - h^2\left(\bar{v}^{(0,1)}U_{1,11}^{(0,1)}(\xi_1,\xi_3,\tau) + \bar{v}^{(0,3)}U_{1,33}^{(0,1)}(\xi_1,\xi_3,\tau)\right) = 0. \tag{28}$$

Direct substitution of the solution in a form (5) into this equation immediately gives the dispersion relation of exactly the same form as the approximation (12). This demonstrates the asymptotic consistency of the described model.

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