Michell structure for a uniform load over multiple spans*

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Abstract
A new half-plane Michell structure capable of carrying a uniformly distributed load of infinite horizontal extent over a series of equally-spaced pinned supports is presented. For the case of equal allowable stresses in tension and compression a full kinematic description of the structure is provided. Although formal proof of optimality of the solution presented is not yet available, the proposed analytical solution is supported by available numerical evidence. Numerical solutions for cases of unequal allowable stresses are also presented, and suggest the existence of a wider family of related, simple, and practically relevant structures, which range in form from an arch with vertical hangers to a cable-stayed bridge.

Keywords: Michell structure, uniformly distributed loading.

1. Introduction
In his groundbreaking contribution to the field of structural optimization [11], A. G. M. Michell formulated the criteria to be satisfied by all least-volume trusses with equal tensile and compressive yield stresses, see also [6, 12]. In trusses satisfying these criteria the magnitudes of the tensile and/or compressive stresses in load-carrying members must everywhere be at maximum allowable values and the virtual strains in such members must not exceed these limiting values. The displacement field must remain continuous throughout the design domain and satisfy the kinematic restrictions imposed on the solution. Michell’s criteria can be satisfied in several different ways, implying that every optimal structure can be split into one or several regions, distinguished by values of the member force components \( f' \) and \( f'' \):

\[
\begin{align*}
T : & \quad f' < 0, \quad f'' > 0, \quad \epsilon' = -\epsilon, \quad \epsilon'' = \epsilon; \\
S^C : & \quad f' < 0, \quad f'' < 0, \quad \epsilon' = -\epsilon, \quad \epsilon'' = -\epsilon; \\
S^T : & \quad f' > 0, \quad f'' > 0, \quad \epsilon' = \epsilon, \quad \epsilon'' = \epsilon; \\
R^C : & \quad f' = 0, \quad f'' < 0, \quad |\epsilon'| \leq \epsilon, \quad \epsilon'' = -\epsilon; \\
R^T : & \quad f' > 0, \quad f'' = 0, \quad \epsilon' = \epsilon, \quad |\epsilon''| \leq \epsilon.
\end{align*}
\]

Within (1)–(5), \( \epsilon' \) and \( \epsilon'' \) denote principal strains and \( \epsilon \) is the positive infinitesimal. Optimal trusses may also contain regions of uniform tension and/or uniform compression [15]. All trusses constructed by Michell [11], as well as the majority of optimal trusses identified in the early literature, only feature one or several regions of type \( T \); the term ‘Michell structure’ is therefore sometimes considered to be synonymous with (1). However in this paper we use this term to describe any structure that satisfies the Michell criteria, and any number of regions (1)–(5) can be present. The deceptive simplicity of the specified criteria should not obscure the fact that there is no known procedure for verifying whether a Michell structure exists for a given problem definition, or for determining its form. Unsurprisingly, the number of Michell structures to have been identified to date is not large, see e.g. [1, 2, 6, 8–11, 14]. Furthermore, whilst some notable exceptions exist [3, 7, 16, 17], the majority of known Michell structures are designed to support only a single external point load. In this paper, we present details of an apparently new Michell structure, for a problem which appears to have been hitherto overlooked. The problem involves a uniformly distributed vertical load applied along a horizontal line spanning across an infinite number of equally spaced pinned supports. The motivation for this configuration originates from the (still unsolved) classical problem of finding the optimal half-plane structure to transmit a uniformly distributed load along a line between two level pinned supports, to these supports [3, 7]. In the case of equal allowable stresses and an infinite number of supports, the

*This is an updated and corrected version of the paper that appeared in the conference proceedings.
resulting geometry of the Michell structure, and the mathematical solution for kinematic fields, all turn out to be comparatively simple. Importantly, the volume per single span of the resulting structure is approximately 11.0% lower than that of the parabolic arch with vertical hangers and 7.86% lower than that of the classical solution [7], which is known to be sub-optimal. We stop short of proving the optimality of the proposed structure for the half-plane; however, results from numerical simulations presented in the paper appear to support our claim. We also present a number of numerical solutions for similar problems with unequal allowable stresses, suggesting that a wider family of related, simple and practically relevant structures exists.

2. An auxiliary problem
Before analysing our main problem, featuring an infinite number of equally-spaced supports, it is instructive to examine a simpler set-up. Consider a uniformly distributed load \( w \) per unit length that is applied to a horizontal line segment of length \( L \), and needs to be transmitted to a pinned support at the centre of the segment. It is not difficult to verify that the suitable optimal solution for the upper half-plane is a ‘half-wheel’, the structure comprising concentric semicircles and orthogonal radii, as shown on Fig. 1.

![Figure 1: The Michell half-wheel subjected to a uniformly distributed load.](image)

Very similar structures for problems involving external point loads have been considered in [6, 11].

The structure is conveniently mapped by the orthogonal curvilinear system \((\alpha, \beta)\), such that
\[
\alpha = r, \quad \beta = \theta, \quad \phi = \beta + \frac{\pi}{2}, \quad A = 1, \quad B = \alpha,
\]
where \( r, 0 \leq r \leq L/2 \), is the linear distance from the support, \( \theta, -\pi/2 \leq \theta \leq \pi/2 \), the polar angle measured counter-clockwise from the vertical symmetry axis and \( \phi \) the angle measured from the horizontal line to the tangent of an \( \alpha \)-line. Functions \( A \) and \( B \) are the components of the metric tensor. A suitable strain field is given by
\[
u = -\epsilon \alpha, \quad v = 2\epsilon \alpha \beta, \quad \omega = 2\epsilon \beta,
\]
in which \( u \) and \( v \) denote displacements along \( \alpha \) and \( \beta \), respectively, and \( \omega \) denotes the rotation.

If \( T' \) and \( T'' \) denote the end loads per unit coordinate difference in the \( \alpha \) and \( \beta \) directions, then they must satisfy the standard equilibrium equations in curvilinear coordinates:
\[
\frac{\partial T'}{\partial \alpha} = T'' \frac{\partial \phi}{\partial \beta}, \quad \frac{\partial T''}{\partial \beta} = -T' \frac{\partial \phi}{\partial \alpha},
\]
see [6]. In our case \( \partial T''/\partial \beta = 0 \), and the equilibrium of vertical components of forces acting along the bottom of the structure requires that \( T'' = w \). Equation (8), can now be integrated, yielding \( T' = w\alpha + t'(\beta) \). One needs to add another boundary condition to fully specify the force field within the structure. For example, if \( T' \) is required to vanish along the outer rim of the structure, then \( t'(\beta) = -wL/2 \) and \( T' = w(\alpha - L/2) \), hence, completing the solution. The volume of the resulting structure is easily found from the virtual work of external forces, which in our particular case yields
\[
W_{aux} = \frac{2}{\sigma \epsilon} \int_{0}^{L/2} -wv|_{\beta=-\pi/2} \, d\alpha = \frac{\pi w L^2}{4 \sigma}.
\]
3. The virtual displacement field

Strain field (6) may be trivially extended to cover the entire half-plane, thus signalling the global optimality of the solution obtained in Section 2. Perhaps unsurprisingly it cannot be as easily applied to problems featuring multiple supports. Indeed, if we were to consider two level supports, and attempt to match two copies of field (7), expanding from each of the supports, then this would be found to be impossible due to the monotonic variation of each local \( u \) and \( v \) as functions of local \( \alpha \). Motivated by this observation, we consider an extension of the structure from Section 2, a half-span as shown in Fig. 2 (we assume that the other half is obtained by reflecting the structure about the vertical \( Oy \)). The original half-wheel, shaded in the new drawing, is expanded to the radius \( L/2 \) and then cut along the vertical lines originating from points \( x = \pm L/2 \) in the global Cartesian coordinate system \( Ox y \). Tensile circumferential members of the original half-wheel are then continued tangentially, as straight ties that connect the points along the cuts to horizontal compression members at \( L/2 \). These concentrated members are needed to rotate the tie forces and equilibrate the portions of external load that are uniformly distributed along \( L/2 \leq |x| \leq L \).

![Figure 2: Half-span of the proposed structure.](image)

The description of this structure requires the use of several curvilinear coordinate systems. The part of the structure for \( -L/2 \leq x \leq L/2 \), termed region \( T_1 \), can be fully described using the same coordinate system as in Section 2. Thus, we assume that \( \alpha_1, \beta_1, \phi_1, A_1, B_1, u_1, v_1 \) and \( \omega_1 \) are defined precisely as in (6) and (8)\(^1\). The only difference concerns the ranges of variation of the coordinates; since the verticals \( x = \pm L/2 \) are described within region \( T_1 \) by equation \( \alpha_1 = L/2 \sin \beta_1 \), therefore, \(-\pi/2 \leq \beta_1 \leq \pi/2\) (as before) and \( 0 \leq \alpha_1 \leq \min\{L/\sqrt{2}, L/2\sin \beta_1\} \). In particular, the curvilinear displacements and rotation are given along the boundary with region \( R^T_2 \) by

\[
\begin{align*}
& u_1 = \frac{\varepsilon L}{2 \sin \beta_1}, \quad v_1 = -\varepsilon \beta_1 L \sin \beta_1, \quad \omega_1 = 2\varepsilon \beta_1, \quad \text{at} \quad x = \frac{L}{2}.
\end{align*}
\]

(10)

The curvilinear coordinate system appropriate for describing the strain field within region \( R^T_2 \) is harder to formulate. The systems of straight, non-intersecting ties are associated with regions described by (5); the mathematical formalism describing such regions is presented in [6, Sect. 4.2]. We begin by defining coordinate \( \beta_2 \) as the same polar angle as the one used within region \( T_1 \). The bottom left corner of \( R^T_2 \) corresponds to \( \beta_2 = -\pi/2 \), whereas the uppermost tie corresponds to \( \beta_2 = -\pi/4 \). More generally, all ties within \( R^T_2 \) belong to the family of straight lines parametrised by \( \beta_2 \):

\[
2x - 2y \cot \beta_2 - L(1 + \cot^2 \beta_2) = 0.
\]

(11)

It is possible to show that these lines envelop an evolute with the equation

\[
y^2 + 2Lx - L^2 = 0.
\]

(12)

\(^1\) The numeric subscripts indicate which specific region a given quantity relates to.
In an orthogonal coordinate system with $\alpha_2$ defined as the distance from a fixed involute, equation (12) may be alternatively written as $\alpha_2 + F(\beta_2) = 0$. Here $F(\beta_2)$ is an arc length measured along the evolute from the point where $\alpha_2 = 0$. Since evolute (12) touches the bottom left corner of $R_2^T$, it is convenient to use the involute passing through this point as the coordinate axis. We can now integrate along the evolute to obtain the full description of our curvilinear coordinates in the form

$$\phi_2 = \beta_2 + \frac{\pi}{2}, \quad A_2 = 1, \quad B_2 = \alpha_2 + F(\beta_2), \quad \text{where} \quad F(\beta_2) = \frac{L}{2} \left( \cot \beta_2 \csc \beta_2 - \ln(\cot \beta_2 - \csc \beta_2) \right), \quad (13)$$

see also [6]. The Cartesian description of coordinate lines in $(\alpha_2, \beta_2)$ is obtained by computing

$$x + iy = \frac{L}{2} + \alpha_2 e^{i\beta_2} + i \int_{-\pi/2}^{-\pi/4} e^{i\xi} F(\xi) \, d\xi; \quad (14)$$

which leads to the explicit formulae

$$x = (\alpha_2 - L \ln[\cot \beta_2 - \csc \beta_2]/2) \cos \beta_2 + L/2, \quad (15)$$

$$y = (\alpha_2 - L \ln[\cot \beta_2 - \csc \beta_2]/2) \sin \beta_2 - L \cot \beta_2/2. \quad (16)$$

An additional test of the validity of these equations may be performed by directly computing the metric tensor components from (15) and (16). The resulting expressions match equations (13) exactly. Table 1 presents some useful relationships between coordinates of various lines and points within the global Cartesian and the local curvilinear coordinate systems.

Table 1: Significant lines and points within the coordinate system $(\alpha_2, \beta_2)$.

<table>
<thead>
<tr>
<th><strong>Cartesian</strong></th>
<th><strong>Curvilinear</strong></th>
<th><strong>Significance</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = L/2$</td>
<td>$\alpha_2 = L \ln[\cot \beta_2 - \csc \beta_2]/2$</td>
<td>the boundary between $T_1$ and $R_2^T$</td>
</tr>
<tr>
<td>$y = 0$</td>
<td>$\alpha_2 = L (\cot \beta_2 \csc \beta_2 + \ln[\cot \beta_2 - \csc \beta_2])/2$</td>
<td>the bottom of $R_2^T$</td>
</tr>
<tr>
<td>$y = L - x$</td>
<td>$\beta_2 = -\pi/4$</td>
<td>the top tie of $R_2^T$</td>
</tr>
<tr>
<td>$(L/2,0)$</td>
<td>$\beta_2 = -\pi/2$</td>
<td>the bottom left corner of $R_2^T$</td>
</tr>
<tr>
<td>$(L/2, L/2)$</td>
<td>$(L \ln(\sqrt{2} - 1)/2, -\pi/4)$</td>
<td>the top left corner of $R_2^T$</td>
</tr>
<tr>
<td>$(L,0)$</td>
<td>$(L(\sqrt{2} + \ln(\sqrt{2} - 1))/2, -\pi/4)$</td>
<td>the right corner of $R_2^T$</td>
</tr>
</tbody>
</table>

Given orthogonal coordinates (13), we can formulate the system of partial differential equations describing principal and shear strains, as well as the rotation, in the form:

$$\frac{\partial u_2}{\partial \alpha_2} = \epsilon, \quad \omega_2 = \omega_2(\alpha_2 + F(\beta_2)) + \frac{\partial u_2}{\partial \beta_2}, \quad (17)$$

$$\omega_2 = \frac{\partial v_2}{\partial \alpha_2}, \quad \epsilon_2'' = (\alpha_2 + F(\beta_2))^{-1} \left( \frac{\partial v_2}{\partial \beta_2} + u_2 \right). \quad (18)$$

Equation (17) implies that $u_2 = \epsilon(\alpha_2 + G(\beta_2))$, with $G(\beta_2)$ chosen to ensure the continuity along the line $x = L/2$. Since $\beta_1$ and $\beta_2$ denote the same angle, the continuity with circumferential displacements requires $u_2|_{\beta_2 = \beta_1} = -v_1|_{\beta_1 = \beta_2}$, so that a reference to (10)$_2$ leads to the full definition

$$u_2 = \epsilon(\alpha_2 + G(\beta_2)), \quad G(\beta_2) = F(\beta_2) + L(2\beta_2 - \cot \beta_2) \csc \beta_2/2. \quad (19)$$

The rotation is fixed along $\alpha_2$-lines within $R_2^T$; therefore, the continuity of rotation along $x = L/2$ and equation (10)$_3$ give $\omega_2 = 2\epsilon \beta_2$. This enables us to compute $v_2$ directly from (17)$_2$, with the result

$$v_2 = \epsilon(2\beta_2[2\alpha_2 + F(\beta_2)] - L[2\beta_2 \cot \beta_2 - 1] \csc \beta_2/2). \quad (20)$$

The substitution of displacement (20) into (18)$_2$ again gives $\omega_2 = 2\epsilon \beta_2$, as it should. By substituting the value for $v_2$ associated with $x = L/2$ from Table 1 into (20), it is also possible to verify that $v_2|_{\beta_2 = \beta_1} = u_1|_{\beta_1 = \beta_2}$. The only remaining equation (18)$_2$ provides a direct mean for computing the strain along $\beta_2$-lines, which is found to be

$$\epsilon_2'' = 2\epsilon \left( 2 - \frac{L \cot \beta_2 \csc \beta_2}{\alpha_2 + F(\beta_2)} \right) - \epsilon. \quad (21)$$
For the field within $R^T$ to satisfy the Michell criteria (5), we must ensure that $|\varepsilon''_2| < \epsilon$. It is worth reminding ourselves that the denominator within (21) is an equation of the evolute. It can only vanish in a single point of region $R^T$, where the evolute touches the bottom left corner, see (12). However, due to the cancellation of terms, one has everywhere along the bottom boundary of $R^T$:

$$\varepsilon''_2|_{y=0} = \epsilon,$$

(22)

see (21) and Table 1. Simultaneously, everywhere along the boundary between regions $T_1$ and $R^T$,

$$\varepsilon''_2|_{x=L/2} = -\epsilon.$$  

(23)

Keeping in mind that, for every fixed $\beta_2$, $\varepsilon''_2$ is a monotonously increasing function of $\alpha_2$, see (21), that changes from $-\epsilon$ at $x = L/2$ to $\epsilon$ at $y = 0$, we come to the sought-for conclusion that $R^T$ is a valid Michell region of type $R^T$, see (5).

Having constructed a consistent strain field for a single half-span does not yet solve the original problem, featuring infinite sequence of equally-spaced level supports. A full span of length $2L$ can be obtained by reflecting the constructed fields with respect to $Oy$. In addition, we can use equations (19), (20) and Table 1 to write $u^2$, the horizontal component of displacement along $y = 0$, in the form

$$u^2|_{y=0} \equiv u_2|_{y=0} \sin \phi_2 + v_2|_{y=0} \cos \phi_2 = -\epsilon \frac{L \cos 2\beta_2}{2 \sin^2 \beta_2}.$$  

(24)

Clearly, this vanishes when $\beta_2 = -\pi/4$, i.e. when $x = L$. This means that we can also reflect the resulting structure with respect to the vertical line $x = L$. Therefore, it is now possible to produce a structure that, via a series of simple reflections, spans across an infinite sequence of level supports placed $2L$ apart along $Ox$. An illustration of a single span of such a structure is given in Fig. 3.

It has already been mentioned that regions of type $T$, i.e. the regions that satisfy the Michell criteria and conditions (1), are often perceived to be synonymous with all Michell structures. Since these regions feature systems of mutually orthogonal members, the requirement of member orthogonality often presumed for general Michell structures. This requirement is, evidently, violated at the bottom boundary of region $R^T$, where ties join a concentrated tensile member. Interestingly, Rozvany [13] presents several examples showing how the orthogonality requirement can be relaxed along boundaries between $R^C$ and $R^T$ regions. The situation is simpler in our case; since both principal strains become equal to $\epsilon$ at the bottom boundary, see (22), one can interpret the concentrated member at the bottom of $R^T$ as a degenerate $S^T$ region (3), within which the orthogonality requirements do not hold.

It is also worth noting that Cartesian point $(L/2, 0)$ is a singular point of the strain field. Clearly, within region $T_1$ the horizontal component of the strain at $y = 0$ is always compressive, because $\varepsilon'_2 = -\epsilon$. This is also true at the boundary with $R^T$, in particular, when one approaches $(L/2, 0)$ along the vertical $x = L/2$. At the same time, when one approaches $(L/2, 0)$ along the involute $\alpha_2 = 0$ in $R^T$, i.e. takes the limit $\beta_2 \to -\pi/2$ of (21) with $\alpha_2 = 0$ fixed, or when one approaches $(L/2, 0)$ along the bottom of $R^T$, one obtains (22), i.e. maximum tensile strain. This observation does not affect the continuity of the displacement or stress fields.

Although we have now obtained a continuous virtual displacement field that satisfies all of our kinematic requirements, this does not constitute a proof of global optimality for our solution. Such a proof would
require constructing a continuous virtual displacement field that covers the entire half-space. The derivation of such a field is beyond the scope of the present paper.

4. The volume of the structure

The volume of a single span of the proposed structure can be computed by calculating the work done by the external forces and dividing it by $\epsilon \sigma$. The work $W_I$ done by the distributed load acting along $-L/2 \leq x \leq L/2$ has already been computed in (9): $W_I = \epsilon \sigma w L^2/4$. In order to determine the work $W_H$, done by the distributed load acting along $L/2 \leq |x| \leq L$, one needs to find the vertical displacement $u_2^y$ along the bottom boundary of $R_2^L$. Using equations (19), (20) and Table 1, we obtain

$$u_2^y|_{y=0} = -u_2|_{y=0} \cos \phi_2 + v_2|_{y=0} \sin \phi_2 = \epsilon L (\beta_2 \csc^2 \beta_2 + \cot \beta_2).$$

(25)

With the help of (25), the work integral is computed as

$$W_H = 2 \int_{L/2}^{L} -u_2^y|_{y=0} dx = 2wL \int_{-\pi/2}^{-\pi/4} u_2^y|_{y=0}(\cot^2 \beta_2 + 1) \cot \beta_2 d\beta_2 = \epsilon \left( \frac{4}{3} + \frac{\pi}{4} \right) wL^2.$$  

(26)

It is now self-evident that

$$W_{\min} = \frac{W_I + W_H}{\epsilon \sigma} = \left( \frac{4}{3} + \frac{\pi}{2} \right) \frac{wL^2}{\sigma} \approx 2.90413 \frac{wL^2}{\sigma}.$$ 

(27)

Therefore, the volume of a single span of the described structure is 11.0% lower than the volume of a simple parabolic arch with vertical hangers, and 7.86% lower than that of the classical solution obtained by Hemp [7], which is known to be sub-optimal (see also [3]).

The solutions for force fields within regions $T_1$, $R_1^L$ and $R_2^L$ can be computed without much difficulty and are omitted here for the sake of brevity. We used these solutions to compute the volume of the structure directly, and to verify formula (27). The volumes obtained via primal and dual formulations matched, therefore providing further confirmation of the correctness of the reported result.

5. Numerical solutions

In order to verify the optimality of the structure described in previous sections, a numerical solution has also been obtained using an efficient numerical layout optimisation procedure [5]. The same procedure was recently used to provide compelling numerical evidence that the parabolic arch is not an optimal structure to transfer a uniformly distributed transmissible load to two pinned supports [4]; see also subsequent formal proof of this [17].

The numerical solutions presented in this paper were computed for several combinations of allowable stresses, using numerical discretizations comprising 61 nodal points in the $x$ direction and 41 nodal points in the $y$ direction, therefore optimising over 3,126,250 potential members. The computations, in each of the cases, took around 30 seconds of CPU time on a modern PC. The plots of resulting solutions are grouped together in Fig. 4. Note that the solutions are plotted using a perspective projection, which makes the upper parts of structures appear narrower than they are.

Let $\sigma_T$ and $\sigma_C$ be the tensile and compressive yield stresses, respectively. The structure obtained in the case when $\sigma_C = \sigma_T$ and shown (twice) at the top of Fig. 4, displays a remarkable similarity to the analytical solution shown in Fig. 3. However, a slight mismatch in the positions of nodes at the top of Region 2 leads to the appearance of an additional (feint) fan region, comprising straight lines and concentric circles. Supplementary runs were performed to ensure that this vanishes as the numerical discretization is refined.

Interestingly, the numerical solutions for unequal allowable stresses indicate that our solution, although seemingly unusual, is closely related to two well known classes of structure, widely used in engineering practice. In particular, the left hand side of Fig. 4 presents structures dominated by compression ($\sigma_C > \sigma_T$). As $\sigma_C/\sigma_T$ increases, the fans around the supports shrink in size, with the overall structure tending towards a simple arch with vertical hangers. In the case of structures dominated by tension ($\sigma_C < \sigma_T$), shown on the right hand side of Fig. 4, the solutions metamorphose into a cable stayed bridge structure, with the fans shrinking to become stocky, near-vertical, towers.

Closer inspection of the optimal structures dominated by tension enables fairly accurate determination of numerical values of the abscissas at which half-wheel fields are replaced by systems of straight tension members. This allows us to formulate a conjecture about the structures of this type. If $X$ denotes an
In particular, in the case when \( \sigma_C = \sigma_T \), (28) yields \( X = L/2 \), precisely the same as assumed in our earlier derivations. Expression (28) can also be reformulated in terms of slope \( \theta_0 \) of the top tie within region \( R^T_2 \), see Fig. 2. A simple geometric argument leads then to the following conclusion
\[
\theta_0 = \arctan \sqrt{\frac{\sigma_C}{\sigma_T}},
\]
which is precisely the same condition as the one previously obtained for the parabolic funicular loaded by a transmissible, uniformly distributed load, see [4, 18].

6. Conclusions
Details of a new half-plane Michell structure capable of carrying a uniformly distributed load of infinite horizontal extent over a series of equally-spaced pinned supports have been presented. Although formal proof of optimality of the structure has not yet been demonstrated, the proposed analytical solution is supported by available numerical evidence. Numerical solutions also suggest the existence of a wider family of related, simple, and practically relevant structures, which range in form from an arch with vertical hangers to a cable-stayed bridge, depending on the specified ratio of limiting compressive to tensile stress.
7. References


