

Approximation of the Rayleigh wave speed

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Abstract

This brief note describes an asymptotic technique for accurate computation of the Rayleigh wave speed both in compressible isotropic and incompressible orthotropic half-spaces. As opposed to recent publications by Malischewsky and Ogden & Vinh, the Cardan's formula for solving cubic equations is not used. A variant of Graeffe's method for finding the moduli of zeros of polynomials is formulated instead. The resulting approximation rapidly converges to the physically relevant solution.

Key words: Rayleigh waves, Surface waves, Explicit wave speed, Graeffe's method

Introduction

Lord Rayleigh was the first to study propagation of waves localised near the free surface of an isotropic elastic half-space, see [1]. The velocity of these *Rayleigh waves* is given by the well-known secular equation

$$(2 - \bar{v}_R^2)^2 = 4\sqrt{1 - \bar{v}_R^2}\sqrt{1 - \chi\bar{v}_R^2}, \quad \bar{v}_R = \frac{v}{c_2}, \quad \chi = \frac{c_2^2}{c_1^2}, \quad (1)$$

where c_1 and c_2 are (dimensional) speeds of longitudinal and shear body waves, respectively. They may be written in terms of the material density ρ , shear modulus μ and the Poisson ratio ν as

$$c_1^2 = \frac{2\mu(1 - \nu)}{\rho(1 - 2\nu)}, \quad c_2^2 = \frac{\mu}{\rho}. \quad (2)$$

The range of physically admissible values for the Poisson ratio ν is $(-1, 1/2)$, therefore we expect that $\chi \equiv (1 - 2\nu)/2(1 - \nu) \in (0, 3/4)$. The relevant solution of (1) exists in the interval $(0, 1)$ and may be proven to be unique, see [2].

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The first explicit solution of (1) was presented in [3] and subsequently improved in [4] and [2]. All of these solutions used variants of Cardan's formula for cubic equation. A complex alternative solution using Riemann problem was derived in [5], but the final result appears to be in error. The goal of the present paper is to present an alternative asymptotic technique for computation of the Rayleigh wave speed. It is based on the so-called Graeffe method of finding the moduli of zeros of polynomials, see e.g. [6]. Complexity and accuracy of the resulting solution is comparable with the exact Cardan's formula. The physically relevant root is selected by the procedure automatically.

Analysis

By squaring both sides of (1) and omitting factor \bar{v}_R^2 the Rayleigh secular equation may be recast as the cubic equation in \bar{v}_R^2 :

$$\bar{v}_R^6 - 8\bar{v}_R^4 + 8(3 - 2\chi)\bar{v}_R^2 - 16(1 - \chi) = 0. \quad (3)$$

The squaring introduces two spurious solutions, however the physically relevant root of (3) still exists and is unique within $(0, 1)$. Let us now specialise the Graeffe method to cubic equations generally and equation (3) in particular. The method is based on the simple observation that if cubic equation¹

$$f(x) \equiv x^3 + px^2 + qx - 1 \equiv (x - x_1)(x - x_2)(x - x_3) = 0, \quad (4)$$

has a large positive root $x_1 \gg \max\{|x_2|, |x_3|, 1\}$ then $x_1 \approx -p \equiv x_1 + x_2 + x_3$. We may slightly improve this approximation by taking $x_1 = -p + \delta$ with δ small and plugging it into (4) to determine that

$$\delta \approx \frac{1 + pq}{p^2 + q}, \quad \text{therefore} \quad x_1 \approx \frac{(1 - p)(p^2 + p + 1)}{(p^2 + q)}. \quad (5)$$

Suppose now that x_1 is the *largest* root of (4) and $|x_1| > \max\{|x_2|, |x_3|, 1\}$. The largest root of the product

$$-f(x)f(-x) \equiv (x^2 - x_1^2)(x^2 - x_2^2)(x^2 - x_3^2) = 0, \quad (6)$$

would be even further separated from the rest of roots. The requirements that enabled approximation (5) will be satisfied after several iterations. If coefficients of the cubic at the $(n - 1)$ th step are given by p_{n-1} and q_{n-1} , then the next step coefficients satisfy

$$p_n = 2q_{n-1} - p_{n-1}^2, \quad q_n = 2p_{n-1} + q_{n-1}^2, \quad n \geq 1, \quad (7)$$

¹ Please note that any cubic equation may be transformed to the form (4).

which implies the following recurrent relation

$$2p_n = 8p_{n-2} - p_{n-1}^2 + 2p_{n-1}p_{n-2}^2 + p_{n-2}^4, \quad n \geq 2. \quad (8)$$

Thus, it is possible to compute p_n recursively and use (5), (7) and (8) to determine the n th approximation for x_1 in the form

$$x_n = 2^n \sqrt[4]{\frac{4(1-p_n)(p_n^2 + p_n + 1)}{5p_n^2 + 8p_{n-1} + 2p_n p_{n-1}^2 + p_{n-1}^4}}, \quad n \geq 2. \quad (9)$$

The rate of convergence of this technique is asymptotically quadratic (note that $x_1^{2^n} = 1/x_2^{2^n} x_3^{2^n}$). The resulting accuracy would only be limited by the numerical accuracy of the computer.

Now it would suffice to ensure that the Rayleigh wave speed \bar{v}_R is the largest solution of some cubic equation. For example, this may be achieved by considering (3) in terms of $x = \kappa^{1/3}/\bar{v}_R^2$ and taking

$$p_0 = -(8 + \kappa)\kappa^{-2/3}, \quad p_1 = 16\kappa^{-1/3} - (8 + \kappa)^2\kappa^{-4/3}, \quad (10)$$

where $\kappa = 16(1 - \chi)$. Then, by using (8) and (9) we may write the n th approximation to the Rayleigh wave velocity as

$$\bar{v}_R \approx \frac{\kappa^{1/6}}{\sqrt{x_n}}. \quad (11)$$

Numerical tests indicate that when $n = 5$ the relative error of formula (11) does not exceed 10^{-13} for all meaningful Poisson ratios. The numerical performance of the method is best verified by using several exact solutions of equation (1), provided in Table 1.

Discussion

If the attention can be restricted to positive Poisson ratios, the substitution $x = 1/(1 - \bar{v}_R^2)$ would deliver a faster convergence. In this case

$$p_0 = \kappa', \quad p_1 = -10 - \kappa'^2, \quad \kappa' = 16\chi - 11, \quad \text{and} \quad \bar{v}_R \approx \sqrt{\frac{x_n - 1}{x_n}}, \quad (12)$$

and the resulting relative error does not exceed 10^{-15} when $n = 4$.

It is also worth noting that there still is some interest to using inaccurate but simple approximations for Rayleigh wave speed. For example, the often

ν	$-\frac{15}{17}$	$-\frac{2}{7}$	$-\frac{5}{123}$	$\frac{3}{28}$	$\frac{77}{365}$	$\frac{20}{69}$	$\frac{55}{136}$	$\frac{114}{235}$
\bar{v}_R^2	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{8}{9}$	$\frac{10}{11}$

Table 1

Exact solutions of the Rayleigh secular equation (1).

quoted classical estimate

$$\bar{v}_R \approx \frac{0.87 + 1.12\nu}{1 + \nu}, \quad (13)$$

has the relative error about 0.46% and is only valid for $\nu > 0$, see [7]. A range of similar approximations is discussed in [8] and references therein. One of the best methods for deriving this kind of approximations involves fitting an interpolation polynomial at the family of Chebyshev nodes, see e.g. [9]. A couple of resulting interpolation polynomials may be given by

$$\bar{v}_R \approx \frac{125}{143} + \nu \left(\frac{29}{151} - \nu \left(\frac{8}{215} + \frac{7}{156}\nu \right) \right), \quad (14)$$

with the relative error below 0.14% and

$$\bar{v}_R \approx \frac{256}{293} + \nu \left(\frac{60}{307} - \nu \left(\frac{4}{125} + \nu \left(\frac{5}{84} + \frac{4}{237}\nu \right) \right) \right), \quad (15)$$

with the relative error below 0.05% for $\nu \in \{-1, 1/2\}$.

The reported technique is not limited to the case of isotropic elastic media. For example, the propagation of surface waves in incompressible orthotropic half-space is governed by the secular equation of the form

$$\eta^3 + \eta^2 + (\Delta - 1)\eta - 1 = 0, \quad (16)$$

in which $0 < \eta \equiv \sqrt{1 - \bar{v}_R^2} < 1$, $\bar{v}_R^2 = \rho v / c_{66}$, and $\Delta = (c_{11} + c_{22} - 2c_{12}) / c_{66}$, see [10]. If we introduce $x = \Delta^{2/3} / \bar{v}_R^2$ so that

$$p_0 = -(\Delta + 2)\Delta^{-1/3}, \quad p_1 = 4\Delta^{1/3} - (\Delta + 2)^2\Delta^{-2/3}, \quad \bar{v}_R \approx \frac{\Delta^{1/3}}{\sqrt{x_n}}, \quad (17)$$

the same procedure (8), (9) with $n = 4$ would approximate the relevant solution of (16) with the relative error below 10^{-14} for all $\Delta > 0$. In fact, the described technique is suitable for polynomials of higher degree and may prove invaluable when computing the surface wave speed with explicit secular equations, see e.g. [11], because it enables automatic selection of the physically relevant (usually lowest) root of the secular equation.

References

- [1] J. W. S. Rayleigh, On waves propagated along the plane surface of an elastic solid, *Proc. Lond. Math. Soc.* 17 (1885) 4–11.
- [2] P. C. Vinh, R. W. Ogden, On formulas for the Rayleigh wave speed, *Wave Motion* 39 (2004) 191–197.
- [3] M. Rahman, J. R. Barber, Exact expressions for the roots of the secular equation for Rayleigh waves, *ASME J. Appl. Mech.* 62 (1995) 250–252.
- [4] P. G. Malischewsky, Comment to “A new formula for the velocity of Rayleigh waves” by D. Nkemzi [*Wave Motion* 26 (1997) 199–205], *Wave Motion* 31 (2000) 93–96.
- [5] D. Nkemzi, A new formula for the velocity of Rayleigh waves, *Wave Motion* 26 (1997) 199–205.
- [6] E. T. Whittaker, G. Robinson, *The calculus of observations: a treatise on numerical mathematics*, Dover, New York, 1967.
- [7] L. Bergmann, *Ultrasonics and their scientific and technical applications*, Wiley, New York, 1948.
- [8] M. Rahman, T. Michelitsch, A note on the formula for the Rayleigh wave speed, *Wave Motion* 43 (2006) 272–276.
- [9] P. Dueflhard, A. Hohmann, *Numerical analysis in modern scientific computing*, Springer-Verlag, New York, 2003.
- [10] R. W. Ogden, P. C. Vinh, Rayleigh waves in orthotropic elastic solids, *J. Acoust. Soc. Am.* 115 (2004) 530–533.
- [11] M. Destrade, Rayleigh waves in symmetry planes of crystals: explicit secular equations and some explicit wave speeds, *Mech. Mater.* 35 (2003) 931–939.