Hodge Theory- Example Sheet 6

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1. Let X be a Hopf surface. Show that the Jacobian $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$ is not a torus in a natural way.

2. Let X be a compact Kähler manifold. Show that the natural map $H^k(X, \mathbb{C}) \to H^k(X, \mathcal{O}_X)$ induced by the inclusion $\mathbb{C} \subset \mathcal{O}_X$ and the projection map $H^k(X, \mathbb{C}) \to H^{0,k}(X)$ induced by the Hodge decomposition coincide. Deduce that the image of the map $\operatorname{Pic} X \to H^2(X, \mathbb{C})$ induced by the exponential sequence is contained in

$$H^{1,1}(X, \mathbb{Z}) = \operatorname{im}(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})) \cap H^{1,1}(X).$$

   Use the Hodge decomposition to give a different proof of the Lefschetz decomposition on $(1,1)$-classes: prove that $\operatorname{Pic} X \to H^{1,1}(X, \mathbb{Z})$ is surjective.

3. Let X be a compact Kähler manifold of dimension n and $Y \subset X$ a smooth hypersurface such that $[Y] \in H^2(X, \mathbb{R})$ is a Kähler class. Show that the canonical restriction map $H^k(X, \mathbb{R}) \to H^k(Y, \mathbb{R})$ is injective for $k \leq n - 1$.

4. (The Hodge numbers are not topological invariants) Show that if X is a compact Riemann surface, the Hodge numbers are topological invariants.

   Show that is S and S' are two compact connected Kähler surfaces, the Hodge numbers of S and S' are equal if there is an orientation preserving homeomorphism between them. (Hint: Show that the Hodge numbers are)

   There are examples of simply connected projective surfaces S and S' that are homeomorphic but have different Hodge numbers. There are examples of diffeomorphic simply connected complex projective varieties of dimension ≥ 3 with different Hodge numbers.

   Give a criterion for diffeomorphic Kähler manifolds to have the same Hodge numbers.

5. Let $V_{\mathbb{Z}}$ be a lattice equipped with a symmetric intersection form $\mathbb{Q}\colon V_{\mathbb{Z}}\times V_{\mathbb{Z}} \to \mathbb{Z}$. Show that a Hodge decomposition of weight 2 polarised by Q is determined by the complex subspace $V^{2,0} \subset V_{\mathbb{C}}$ of rank $h^{2,0}$ (give the conditions on $V^{2,0}$).

   Show that if the signature of Q is $(2, \dim V - 2)$, weight 2 Hodge structures on V with $h^{2,0} = 1$ polarised by Q are parametrised by an open set in a projective quadric:

$$\mathcal{D} = \{\omega \in \mathbb{P}(V_{\mathbb{C}}) | Q(\omega, \omega) = 0; Q(\omega, \overline{\omega}) > 0\}.$$

6. (The Hodge decomposition on curves) Let X be a compact connected complex curve, denote $d\colon \mathcal{O}_X \to \Omega_X$ the differential between the sheaf of holomorphic functions and that of holomorphic differentials.

   Show that d fits in an exact sequence

$$0 \to \mathbb{C} \to \mathcal{O}_X \xrightarrow{d} \Omega_X \to 0.$$

   Show that $H^1(X, \Omega_X) \simeq \mathbb{C}$ and $H^2(X, \mathbb{C}) \simeq \mathbb{C}$; deduce that the exact sequence above induces a short exact sequence

$$0 \to H^0(X, \Omega_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to 0.$$

   Show that the map $\alpha \mapsto [\alpha]$ that sends a holomorphic form to its class in $H^1(X, \mathcal{O}_X)$ is injective. Deduce from Serre Duality that it is also surjective and that:

$$H^1(X, \mathbb{C}) = H^0(X, \Omega_X) \oplus \overline{H^0(X, \Omega_X)}, \text{ with } \overline{H^0(X, \Omega_X)} = H^1(X, \mathcal{O}_X).$$

7. Let $H_{\mathbb{R}}$ be a $\mathbb{R}$-vector space and $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C}$.

   Show that a decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$ such that $H^{p,q} = \overline{H^{q,p}}$ determines a real representation of $\mathbb{C}^*$, i.e. a group homomorphism

$$\rho\colon \mathbb{C}^* \to \mathrm{GL}(H_{\mathbb{R}}).$$

   Define $\rho\colon \mathbb{C}^* \to \mathrm{GL}(H_{\mathbb{C}})$ by $\rho(z)(\alpha^{p,q}) = z^p \overline{z}^q \alpha^{p,q}$ for $\alpha^{p,q} \in H^{p,q}$ and show that this representation is real, i.e. that $\rho(z)(\alpha) \in H_{\mathbb{R}}$ for any $\alpha \in H_{\mathbb{R}}$. Note that $\rho(t) = t^k \,\mathrm{Id}$ for all $t \in \mathbb{R}$.

   Conversely, let $\rho\colon \mathbb{C}^* \to \mathrm{GL}(H_{\mathbb{C}})$ be a continuous action of $\mathbb{C}^*$ on $H_{\mathbb{C}}$ such that $\rho(t) = t^k \,\mathrm{Id}$ and $\rho(\overline{z}) = \overline{\rho(z)}$ for $z \in \mathbb{C}^*$.

   Show that there is a decomposition $H = \bigoplus_{\chi} H_{\chi}$, where χ belongs to the set of characters of $\mathbb{C}^*$, and $\mathbb{C}^*$ acts by $z \mapsto \chi(z) \,\mathrm{Id}$ on H_{χ}. Show that only the characters $\chi_{p,q}\colon z \mapsto z^p \overline{z}^q$ appear in this decomposition. If $H^{p,q} = H_{\chi_{p,q}}$, show that $H^{p,q} = \overline{H^{q,p}}$.