Hodge Theory- Example Sheet 5

Anne-Sophie Kaloghiros

1. Let $E \to X$ be a holomorphic vector bundle over a complex manifold X and $L \to X$ a holomorphic line bundle. Show that $\mathbb{P}(E \otimes L) \simeq \mathbb{P}(E)$, but that $\mathcal{O}_{\mathbb{P}(E\otimes L)}(1) \simeq \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* L^*$, where $\pi \colon \mathbb{P}(E) \to X$.

2. Let X be a complex manifold. Show that the *Bott-Chern cohomology* groups
$$H^{p,q}_{BC}(X) = \frac{\{\alpha \in \mathcal{A}^{p,q}(X) | d\alpha = 0\}}{\partial\overline{\partial}\mathcal{A}^{p-1,q-1}(X)}$$
are well defined, and that there are natural maps
$$H^{p,q}_{BC}(X) \to H^{p,q}(X) \text{ and } H^{p,q}_{BC}(X) \to H^{p+q}(X, \mathbb{C}).$$
Show that if X is Kähler, the first map is bijective. Use this to show again that the bidegree decomposition in the Hodge Decomposition Theorem is independent of the Kähler structure.

3. Show that any complex curve admits a Kähler structure.

4. Let $A \in \mathrm{GL}(n+1, \mathbb{C})$ and consider the natural induced isomorphism $F_A \colon \mathbb{P}^n \to \mathbb{P}^n$. Denote ω_{FS} the Fubini-Study form introduced in lectures. When is $F_A^* \omega_{FS} = \omega_{FS}$?

5. Let (X, h) be a connected Kähler manifold of dimension $\dim X \geq 2$, and let ω be a Kähler form. Let $\varphi \colon X \to \mathbb{R}^*_+$ be a differentiable function and assume that $\varphi\omega$ is Kähler. Show that φ is constant.

6. Let X be a compact Kähler manifold and consider $H^{1,1}(X)$ as a subspace of $H^2(X, \mathbb{C})$. Show that $\mathcal{K}_X = \{\alpha \in H^{1,1}(X) | d\alpha = 0\}$ is an open convex cone in $H^{1,1}(X, \mathbb{R}) = H^{1,1}(X) \cap H^2(X, \mathbb{R})$ and that $\mathcal{K}_X$ does not contain any line $\{\alpha + t\beta; t \in \mathbb{R}\}$ for any $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ with $\beta \neq 0$. Show that if $\alpha \in \mathcal{K}_X$, $\alpha + t\beta \in \mathcal{K}_X$ for any $\beta \in H^{1,1}(X, \mathbb{R})$ and $t > 0$ sufficiently large.

7. Let (X, h) be a Kähler manifold. Show that the Kähler class $\omega = -i\Im h$ is harmonic.

8. Let (X, h) be a compact Hermitian manifold of dimension n. Show that d-harmonic forms of degree 0 and $2n$ coincide with ∂ and $\overline{\partial}$ harmonic forms.

   Show that any d-harmonic form of type (p, q) is $\overline{\partial}$-harmonic.

9. (The residue formula) Let X be a compact complex curve, and let μ be a volume form on X (μ is a closed form of type $(1, 1)$).

   (i) Show that μ is not $\overline{\partial}$-exact and deduce that $H^1(X, K_X) \neq (0)$ and admits a surjective map $H^1(X, K_X) \to \mathbb{C}$ such that $\omega \mapsto \int_X \omega$.

   Let $D = \Sigma P_i$ be a *boundary* divisor of X (all the multiplicities are 1), and denote $K_X(D)$ the holomorphic line bundle defined as follows. If $U \subset X$ is an open set that contains no x_i, $\Gamma(U, K_X(D)) = \Gamma(U, K_X)$, and if U_i is a neighbourhood of x_i and z_i is a local holomorphic coordinate near x_i,

$$\Gamma(U_i, K_X(D)) = \{s \colon U_i \to \mathbb{C} : s(z_i) = \varphi(z_i)\frac{dz_i}{z_i}, \varphi \in \mathcal{O}_X(U_i)\}$$

   (ii) Show that there is an exact sequence

$$0 \to K_X \to K_X(D) \stackrel{Res}{\to} \Sigma_i \mathbb{C}_{x_i} \to 0, \tag{1}$$

   where $\mathbb{C}_{x_i}$ is the *skyscraper sheaf* supported at x_i (i.e. $\Gamma(U, \mathbb{C}_{x_i}) = \{0\}$ if $x_i \notin U$ and $\mathbb{C}$ otherwise). The map $Res_i \colon K_X(D) \to \mathbb{C}_{x_i}$ is defined by:

$$Res_i(\omega) = \frac{1}{2\pi}\int_{\partial D_i} \omega,$$

   for a disk D_i centered at x_i and containing no $x_j \neq x_i$.

   (iii) Let $\delta \colon H^0(X, \Sigma \mathbb{C}_{x_i}) \to H^1(X, K_X)$ be the map in the long exact sequence associated to (1). Show that $\delta(1_{x_i})$ is the class in $H^1(X, K_X)$ of $\overline{\partial}\mu_i$, where μ_i is a $(1, 0)$-form which is differentiable away from x_i and equal to $\frac{dz_i}{z_i}$ on a neighbourhood U_i of x_i.

   (iv) Show that $\int_X \delta\mu_i = -2i\pi$, and deduce that if ω is a meromorphic 1-form having poles of order at most 1 at each x_i and holomorphic otherwise, then

$$\Sigma Res_i \omega = 0.$$