Hodge Theory- Example Sheet 3

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1. Show that the set of holomorphic line bundles over a complex manifold X may be given a natural structure of commutative group, denote this group $\operatorname{Pic} X$. Show that $\operatorname{Pic} X$ is isomorphic to $\check{H}^1(X, \mathcal{O}_X^*)$.

2. Let X a topological space, $\mathcal{F}$ a sheaf of abelian groups on X and $\underline{\mathcal{U}}$ an open cover of X. Suppose that $H^q(U_i, \mathcal{F}) = (0)$ for all $q > o$ and all $U_i \in \underline{U}$. Give a counterexample to the assertion $\check{H}^q(\underline{U}, \mathcal{F}) \neq \check{H}^p(X, \mathcal{F})$ for all $p \geq 0$.

3. The $\overline{\partial}$-Poincaré lemma can be extended to show that for any (bounded or not) polydisc $D \subset \mathbb{C}^n$, $H^{p,q}_{\overline{\partial}}(D) = 0$ for all $q > 0$ (see Huybrechts, I,1.3.9). Assuming this for $n = 1$, show that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = (0)$. Compute $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))$ and $h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))$ for $k \in \mathbb{N}$ where $h^i = \dim_{\mathbb{C}} H^i$.

4. Show that any holomorphic line bundle on a disc $D \subset \mathbb{C}$ is trivial, and describe the Picard group of $\mathbb{P}^1$.

5. Let X be a complex manifold of dimension n and $D \subset X$ a smooth hypersurface, i.e. a disjoint union of complex submanifolds of codimension 1. Let $\{U_i, \phi_i\}_{i \in I}$ be a complex atlas, and denote $f_i \in \mathcal{O}_{U_i}$ local equations for D, i.e. $D \cap U_i = \{f_i = 0\}$ and $df_i(P) \neq 0$ for all $P \in U_i$. Define meromorphic functions on $U_i \cap U_j$:

$$g_{ij} = \frac{f_i}{f_j}$$

   and show that g_{ij} extends to a non vanishing holomorphic function on $U_i \cap U_j$, and that the $\{g_{ij}\}_{i,j \in I}$ define a Čech coccyle of $\mathcal{O}_X^*$; denote $\mathcal{O}_X(D) \in \operatorname{Pic} X$ the corresponding line bundle. Show that $\mathcal{O}_X(D)$ is well defined (does not depend of the choice of cover, or of local equations for D).

Define the *sheaf of ideals* $\mathcal{I}_D \subset \mathcal{O}_X$ of D as the sheaf such that for all $U \subset X$, $\mathcal{I}_D(U) = \{s \in \mathcal{O}_X(U) : s_{|D} = 0\}$. Show that $\mathcal{I}_D = \mathcal{O}_X(D)^*$, and deduce that the sequence of sheaves

$$0 \to \mathcal{O}_X(D)^* \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

is exact.

6. Lefschetz Theorem on $(1,1)$-classes. Let X be a complex manifold and $\omega \in \mathcal{A}^{1,1}_X(X)$ be a d-closed real form $(1,1)$-form such that $[\omega] \in H^2(X, \mathbb{Z})$. Using the explicit construction of the isomorphism in the De Rham theorem, show that there is a holomorphic line bundle such that $\frac{i}{2\pi}\Theta_{L,h} = \omega$.

7. Let X be a complex manifold and let

$$0 \to S \xrightarrow{\varphi} E \xrightarrow{\psi} Q \to 0$$

be an exact sequence of holomorphic vector bundles; E is an *extension* of Q by S. Two such extensions are equivalent when there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \mathrm{Id}_S} & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id}_Q} & & \\
0 & \longrightarrow & S & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & 0,
\end{array}$$

with F a holomorphic vector bundle. Show that the exact sequence of sheaves

$$0 \to \mathcal{H}\mathrm{om}(Q,S) \overset{\mathrm{Id}_{Q^*}\otimes\varphi}{\to} \mathcal{H}\mathrm{om}(Q,E) \overset{\mathrm{Id}_{Q^*}\otimes\psi}{\to} \mathcal{H}\mathrm{om}(Q,Q) \to 0$$

induces a long exact sequence in Čech cohomology

$$\cdots \check{H}^0(X, \mathcal{H}\mathrm{om}(Q,E)) \to \check{H}^0(X, \mathcal{H}\mathrm{om}(Q,Q)) \xrightarrow{\delta} \check{H}^1(X, \mathcal{H}\mathrm{om}(Q,S)) \to \cdots$$

Denote $[E] = \delta(\mathrm{Id}_Q) \in \check{H}^1(X, \mathcal{H}\mathrm{om}(Q,S))$. When does $[E] = 0$? Let $e \in \check{H}^1(X, \mathcal{H}\mathrm{om}(Q,S)))$. Show that there is an exact sequence

$$0 \to S \xrightarrow{\varphi} E \xrightarrow{\psi} Q \to 0$$

such that $[E] = e$. Deduce that $\check{H}^1(X, \mathcal{H}\mathrm{om}(Q,S))$ parametrises the classes of extensions of Q by S.

8. Show that $\omega = \frac{i}{2\pi}\partial\overline{\partial}\log(1 - ||z||^2)$ defines a Kähler metric on the unit disc $D \subset \mathbb{C}^n$.