Hodge Theory- Example Sheet 2

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1. Let $f = (f_1, \cdots, f_n) \colon \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map, and $(z_1, \cdots, z_n)$ the standard coordinates on $\mathbb{C}^n$. Define $x_k = \Re z_k$, $y_k = \Im z_k$ and $u_j = \Re f_j$, $v_j = \Im f_j$. The (real) Jacobian of f at $a \in \mathbb{C}^n$ is

   $$J_{\mathbb{R}}(f)(a) = \left(\frac{\partial u_1, v_1, \cdots, u_n, v_n}{\partial x_1, y_1, \cdots, x_n, y_n}\right)(a).$$

   Show that $\det J_{\mathbb{R}}(f)(a) = |\det J(f)(a)|^2$, and deduce that any complex manifold is orientable.

2. Let X be a complex manifold, P a point of X and $U \subset X$ a connected open subset with $P \in U$. Show that $\mathcal{A}_X(U) \to \mathcal{A}_{X,P}$ is always surjective but not, in general, injective. Show that $\mathcal{O}_X(U) \to \mathcal{O}_{X,P}$ is always injective but not, in general, surjective.

3. Show that if $\Gamma \subset \mathbb{C}^n$ is a lattice of rank $2n$, then $\mathbb{C}^n/\Gamma$ is diffeomorphic to $\simeq (S^1)^{2n}$, however, if Γ_1 and Γ_2 are two different lattices, $\mathbb{C}^n/\Gamma_1$ and $\mathbb{C}^n/\Gamma_2$ are not isomorphic as complex manifolds in general. Let $\varphi \colon \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ be a biholomorphic map such that $\varphi(0) = 0$. Show that there is a unique $\alpha \in \mathbb{C}^*$ such that $\alpha\Lambda = \Lambda'$ and such that

   $$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{C}/\Lambda_1 & \xrightarrow{\varphi} & \mathbb{C}/\Lambda_2 \end{array}$$

   commutes. Show that X is biholomorphic to a lattice $\mathbb{Z} + \mathbb{Z}\tau$ with $\Im\tau > 0$. Deduce that the biholomorphic equivalence classes of complex tori in dimension 1 are in bijection with $\mathbb{H}/\operatorname{SL}(2,\mathbb{Z})$, where

   $$\mathbb{H} = \{\tau \in \mathbb{C} | \Im\tau > 0\}$$

   is the *Poincaré half plane* and the action of $\operatorname{SL}(2,\mathbb{Z})$ is defined by:

   $$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau) \mapsto \frac{a\tau + b}{c\tau + d}.$$

The set $\mathbb{H}/\operatorname{SL}(2,\mathbb{Z})$ has a natural complex structure, and the j-invariant $j\colon \mathbb{H}/\operatorname{SL}(2,\mathbb{Z}) \to \mathbb{C}$ is a biholomorphic map.

4. Let E, F, G be holomorphic vector bundles. Recall that $E \stackrel{\phi}{\to} F \stackrel{\psi}{\to} G$ is an *exact sequence* if $\ker\psi = \operatorname{im}\phi$. A *short exact sequence*

$$0 \to E \stackrel{\phi}{\to} F \stackrel{\psi}{\to} G \to 0$$

is such that $\ker\phi = 0$ and $\operatorname{Coker}\psi = G$.

   (i) Show that if $0 \to E \stackrel{\phi}{\to} F \stackrel{\psi}{\to} G \to 0$ is a short exact sequence, then $\det F \simeq \det E \otimes \det G$.

   (ii) Let $L \to X$ is a holomorphic line bundle and let $\sigma \in \Gamma(X, L)$ be a nonzero section such that $\{\sigma(x) = 0\} = D$ is smooth. Show that there is an exact sequence

$$0 \to T_D \to T_{X|D} \to L_{|D} \to 0$$

   and deduce the *adjunction formula* $K_D \simeq (K_X \otimes L)_{|D}$.

   (iii) Let $Y \subset X$ be a submanifold of a complex manifold X, and define the *normal bundle* $\mathcal{N}_{Y/X}$ as the cokernel of the inclusion $T_Y \subset T_{X|Y}$. Show that $K_Y \simeq K_{X|Y} \otimes \det \mathcal{N}_{Y/X}$.

5. Show that the *Euler sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0$$

is exact and deduce that $K^*_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(n+1)$. Deduce that $K^*_H \simeq \mathcal{O}_{\mathbb{P}^n}(n+1-d)_{|H}$ for a submanifold $fH \subset \mathbb{P}^n$ defined by a homogeneous polynomial of degree d. Generalise this formula to $D = H_1 \cap \cdots \cap H_n \subset \mathbb{P}^n$ a complete intersection. Is the twisted cubic a complete intersection?

6. The *tautological vector bundle* $U(r, V) \to \operatorname{Gr}(r, V)$ over the Grassmannian is the vector bundle with total space

$$U_r(V) = \{([U], x) \in \operatorname{Gr}(r, V) \times V | x \in U\} \subset \operatorname{Gr}(r, V) \times V,$$

and $\pi : U_r(V) \to \operatorname{Gr}(r, V)$ the projection onto the first factor. Check that $U_r(V)$ is a holomorphic vector bundle. Let X be a complex manifold and $E \to X$ be a subbundle of rank r of $X \times \mathbb{C}^n$. Show that there is a unique holomorphic map $f\colon X \to \operatorname{Gr}(r, \mathbb{C}^n)$ such that $E = f^* U_r(\mathbb{C}^n)$, where $U_r(\mathbb{C}^n)$ is the tautological line bundle.