Hodge Theory- Example Sheet 1

Anne-Sophie Kaloghiros

1. (The maximum principle)

   (i) Let $U \subset \mathbb{C}^n$ be an open set and $f\colon U \to \mathbb{C}$ a holomorphic function. If $|f|$ admits a maximum at a point $z_0 \in U$, there is a polydisc $D = D(z_0, R)$ such that $f_{|D}$ is constant.

   (ii) Show that if X is a compact complex manifold and $f\colon X \to \mathbb{C}$ is a holomorphic map, f is constant.

   (iii) Let X be a compact submanifold of $\mathbb{C}^n$, show that $\dim X = 0$.

2. Some results on holomorphic functions.

   (Analytic Continuation) Let $U \subset \mathbb{C}^n$ be a connected open and $f\colon U \to \mathbb{C}$ a holomorphic function. If f vanishes on an open subset $V \subset U$, then f is identically 0 on U.

   (Riemann's extension theorem) Let $U \subset \mathbb{C}^n$ be an open set and $f\colon U \subset \mathbb{C}^n \to \mathbb{C}$ a function that is holomorphic on $U - \{z | z_1 = 0\}$. If f is locally bounded on U, f extends to a holomorphic function on U.

   (Hartog's theorem) Let $U \subset \mathbb{C}^n$ be an open set and $f\colon U \subset \mathbb{C}^n \to \mathbb{C}$ a function that is holomorphic on $U - \{z | z_1 = z_2 = 0\}$, then f extends to a holomorphic function on U.

3. ($\overline{\partial}$- Poincaré Lemma in 1 variable) Let $U \subset \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ be a differentiable map. Show that, locally on the open set U, there is a differentiable function g such that $\frac{\partial g}{\partial \bar{z}} = f$. The function g is defined up to addition of a holomorphic function. (Hint: Show that on a disc $D = D(\omega, \varepsilon)$ such that $\overline{D} \subset U$, the function $g(z) = \frac{1}{2i\pi} \int_D \frac{f(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z}$ is a solution on D).

4. (Grassmannian variety and Plücker embedding) Let $n \in \mathbb{N}$ be an integer with $n \geq 2$ and $0 < r < n$. Define the Grassmannian variety:

$$\mathrm{Gr}(r, n) := \{V \subset \mathbb{C}^n \text{ a vector subspace with } \dim V = r\}.$$

Let $\mathrm{U}_n(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$ denote the unitary group of $\mathbb{C}^n$.

(i) Show that there exists a surjective map $\mathrm{U}_n(\mathbb{C}) \to \mathrm{Gr}(r, \mathbb{C}^n)$. Endow $\mathrm{Gr}(r, \mathbb{C}^n)$ with the quotient topology, and define charts on $\mathrm{Gr}(r, \mathbb{C}^n)$ as follows. For any $T_i \subset \mathbb{C}^n$ vector subspace of dimension $n - r$, define:

$$U_i = \{S \subset \mathbb{C}^n \text{ a vector subspace of dimension r } | S \cap T_i = \emptyset\},$$

fix $S_i \in U_i$ and let $\phi_i \colon U_i \to \mathrm{Hom}(S_i, T_i) \simeq \mathbb{C}^{r(n-r)}$ be the map that sends $S \in U_i$ to $f \in \mathrm{Hom}(S_i, T_i)$, the unique $\mathbb{C}$-linear application such that $S \subset S_i \oplus T_i$ is the graph of f.

(ii) Show that $\{U_i, \phi_i\}$ defines a complex atlas on $\mathrm{Gr}(r, \mathbb{C}^n)$.

Define a map:

$$\begin{aligned} \psi \colon \mathrm{Gr}(r, \mathbb{C}^n) &\to \mathbb{P}(\textstyle\bigwedge^r \mathbb{C}^n) \\ U &\mapsto u_1 \wedge \cdots \wedge u_r, \end{aligned}$$

where $\{u_1, \cdots, u_r\}$ is a basis of U.

(iii) Show that ψ is (well defined and) is an embedding (ψ is the Plücker embedding) and that $\mathrm{Gr}(r, \mathbb{C}^n)$ is a projective manifold.

(iv) Let $e_1, \cdots, e_4$ be the canonical basis of $\mathbb{C}^4$, show that if $[X_0 \colon \cdots \colon X_5]$ denotes the homogeneous coordinates of $\mathbb{P}(\bigwedge^2 \mathbb{C}^4) \simeq \mathbb{P}^5$ associated to the basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$ of $\bigwedge^2 \mathbb{C}^4$, the Plücker embedding of $\mathrm{Gr}(2, \mathbb{C}^4)$ is given by the equation

$$X_0 X_5 - X_1 X_4 + X_2 X_3 = 0.$$

5. (Hopf manifolds.) Let $\lambda \in \mathbb{R}$ such that $0 < \lambda < 1$, and consider the group action:

$$\begin{aligned} \mathbb{Z} \times (\mathbb{C}^n \setminus \{0\}) &\to \mathbb{C}^n \setminus \{0\} \\ (m, z) &\mapsto \lambda^m \cdot z. \end{aligned}$$

Show that $H = (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ is a complex manifold, and that H is diffeomorphic to $S^{2n-1} \times S^1$, and deduce that the Hodge decomposition does not hold for $H^1(H, \mathbb{C})$. .

1. Let X be a complex manifold and Γ a subgroup of $\operatorname{Aut} X$; Γ acts properly discontinuously on X if for any compact subsets K_1 and K_2 of X, $\gamma(K_1) \cap K_2 \neq \emptyset$ for at most finitely many $\gamma \in \Gamma$. Assume that the action of Γ is properly discontinuous and without fixed points–that is, $\gamma \cdot x \neq x$ for all $x \in X$ and $\gamma \in \Gamma \setminus \{1\}$– and denote X/Γ the set of equivalence classes of X under the action of Γ. Show that X admits a unique complex structure such that the natural map $\pi \colon X \to X/\Gamma$ is holomorphic and locally biholomorphic.

2. Show that the Riemann surface $S = \mathbb{C}^*/\{z \mapsto 2z\}$ is isomorphic to an elliptic curve $\mathbb{C}/\Lambda$ for some lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ with $\Im\tau > 0$.

3. (Complex tori) Show that if $\Gamma \subset \mathbb{C}^n$ is a lattice of rank $2n$, then $\mathbb{C}^n/\Gamma$ is diffeomorphic to $\simeq (S^1)^{2n}$. We will show that if Γ_1 and Γ_2 are two different lattices, $\mathbb{C}^n/\Gamma_1$ and $\mathbb{C}^n/\Gamma_2$ are not isomorphic as complex manifolds in general.

   Let $\varphi \colon \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ be a biholomorphic map such that $\varphi(0) = 0$. Show that there is a unique $\alpha \in \mathbb{C}^*$ such that $\alpha\Lambda = \Lambda'$ and the diagram

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{C}/\Lambda_1 & \xrightarrow{\varphi} & \mathbb{C}/\Lambda_2 \end{array}$$

   commutes. Show that X is biholomorphic to a lattice $\mathbb{Z} + \mathbb{Z}\tau$ with $\Im\tau > 0$. Deduce that the biholomorphic equivalence classes of complex tori in dimension 1 are in bijection with $\mathbb{H}/\operatorname{SL}(2,\mathbb{Z})$, where

$$\mathbb{H} = \{\tau \in \mathbb{C} | \Im\tau > 0\}$$

   is the *Poincaré half plane* and the action of $\operatorname{SL}(2,\mathbb{Z})$ is defined by:

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau) \mapsto \frac{a\tau + b}{c\tau + d}.$$

   The set $\mathbb{H}/\operatorname{SL}(2,\mathbb{Z})$ has a natural complex structure, and the j-invariant $j \colon \mathbb{H}/\operatorname{SL}(2,\mathbb{Z}) \to \mathbb{C}$ is a biholomorphic map.

4. Let $f = (f_1, \cdots, f_n) \colon \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map, and $(z_1, \cdots, z_n)$ the standard coordinates on $\mathbb{C}^n$. Define $x_k = \Re z_k$, $y_k = \Im z_k$ and $u_j = \Re f_j$, $v_j = \Im f_j$. The (real) Jacobian of f at $a \in \mathbb{C}^n$ is

$$J_{\mathbb{R}}(f)(a) = (\frac{\partial u_1, v_1, \cdots, u_n, v_n}{\partial x_1, y_1, \cdots, x_n, y_n})(a).$$

Show that $\det J_{\mathbb{R}}(f)(a) = |\det J(f)(a)|^2$, and deduce that any complex manifold is orientable.