Research statement – Anne-Sophie Kaloghiros

My research is in higher dimensional birational geometry; I am interested in the explicit geometry of 3-folds and in more conceptual questions on Fano and Calabi–Yau geometries and in birational geometry in arbitrary dimension. Below, I outline some of my current projects.

Most of the techniques I use were developed for the Minimal Model Program (MMP), which aims to find a representative with *simple* geometry in each class of birational equivalence. Such a representative is made of building blocks of pure geometric type. There are three pure geometric type: Fano varieties have positive curvature, Calabi–Yau (CY) varieties are flat, and varieties of general type have negative curvature.

Running the MMP on a manifold X consists of performing a sequence of elementary operations directed by the canonical class K_X, each of which brings X closer to a variety with simple geometry. In good (conjecturally all) cases, after finitely many steps, we have constructed a map $X \dashrightarrow Y$ and Y has simple geometry. There are two *simple geometries*: Y is either a Mori fibre space or a minimal model. If X (the variety we started with) is covered by rational curves, Y is a Mori fibre space, a family $Y \to S$ of Fano varieties ($-K_{Y_s}$ ample) over a lower dimensional base. Otherwise, Y is a minimal model, i.e. a variety with non-positive curvature (K_Y nef). Conjecturally, a minimal model is a family $Y \to S$ of CY varieties (K_{Y_s} torsion) over a base with negative curvature (K_S ample).

Several varieties with simple geometry can lie in the same class of birational equivalence, but they are either all Mori fibre spaces or all minimal models. Birational maps between Mori fibre spaces are chains of so called Sarkisov links, while birational maps between minimal models are sequences of flops.

In many situations, it is useful to run a perturbed MMP (log MMP) on X that is directed by a pair (X, D), where D is a divisor on X. This amounts to replacing K_X by the perturbed divisor $K_X + D$ in the operations of the MMP or in the definitions of Fano, CY varieties and of minimal models and defines the log MMP, log Fano and log CY pairs, and log minimal models. Most results known in the classical MMP extend to the log MMP, but the relation between rational curves and the log pure geometries making up the end product of the log MMP does not hold.

I. Birational geometry of CY pairs and applications CY pairs blend features and properties of both simple geometries. A CY pair (X, D) consists of a normal projective variety and a reduced integral divisor D with $K_X + D \sim_{\mathbb{Z}} 0$. A CY pair is a log minimal model (hence the end product of a log MMP), and unless $D = 0$, it is also covered by rational curves. Running a classical MMP produces a CY pair $(\overline{X}, \overline{D})$ with $\overline{X}$ a Mori fibre space.

A birational map $(X, D_X) \overset{\varphi}{\dashrightarrow} (Y, D_Y)$ between CY pairs is volume preserving if pullbacks of $K_X + D_X$ and $K_Y + D_Y$ to high enough models coincide. If (X, D_X) and (Y, D_Y) are Mori fibred CY pairs, Corti and I show that φ can be decomposed in a finite sequence of volume preserving Sarkisov links. or (conjecturally) in finitely many $(K + D)$-flops.

I.A Explicit birational geometry. Volume preserving Sarkisov links are easy to describe in the surface case, but very little is known in higher dimensions. I am studying explicit constructions of volume preserving Sarkisov links in dimension three.

I.B Volume preserving maps of CY pairs with a toric model, cluster varieties. A toric pair (X_Δ, D_Δ) with $X_\Delta \setminus D_\Delta = (\mathbb{C}^*)^n$ is an example of CY pair, and mutations of algebraic tori $(\mathbb{C}^*)^n \dashrightarrow (\mathbb{C}^*)^n$ can be extended to volume preserving birational maps of toric pairs. Cluster varieties are obtained by gluing algebraic tori by volume preserving maps, so can be thought of as mutation equivalence classes of CY pairs with toric models. Corti conjectures that volume preserving birational maps between CY pairs with toric models can be decomposed into chains of mutations, and I have obtained partial results in dimension 3 by using the Sarkisov program. Specifically, I want to answer the following questions.

Question 1: Are all volume preserving birational maps of tori chains of mutations?

Question 2: Is a careful study of volume preserving Sarkisov links between 3-fold Mori fibre spaces with a toric model enough to conclude in dimension 3? (Partial results obtained so far.)

Question 3: Could tropical geometry yield a useful invariant to untwist chains of mutations?

If g_1, g_2 are volume preserving Cremona transformations, then $\mathrm{trop}(g_1)$ and $\mathrm{trop}(g_2)$ are naturally defined. Assume that $\mathrm{trop}(g_1) = \mathrm{trop}(g_2)$, then do g_1 and g_2 differ by mutations? In general $\mathrm{trop}(g_1.g_2) \neq \mathrm{trop}(g_1)\mathrm{trop}(g_2)$; could we have $g \in \mathrm{Aut}(\mathbb{C}^*)^n$ if $\mathrm{trop}(g) = \mathrm{Id}$? In that case, we could hope to get some information from "combinatorial mutations"?

I.C Application to Mirror Symmetry. In recent work, Gross, Hacking, and Keel construct mirror partners for some surface CY pairs in the framework of Homological Mirror Symmetry. The extent to which these specific ideas can be generalised to higher dimensions is still unclear, but mirror symmetry is expected to hold as a general duality between CY pairs. A special type of CY pairs are those whose underlying variety X is Fano. The mirror of X should be a Landau-Ginzburg model, i.e. a quasi-projective variety equipped with a regular function $X^\vee \to \mathbb{A}^1$. A basic form of the mirror conjecture predicts that the Gromov Witten invariants of X correspond to the periods of $X^\vee \to \mathbb{A}^1$. One can ask whether there is any relation between mirror partners for (X, D) and mirror partners for X. In dimensions two and three, mirror partners of Fano varieties with mild singularities can be constructed, and in dimension two, they appear to be related to the construction of Gross-Hacking-Keel. One of my goals is to understand the relation between the formulations of mirror symmetry for CY pairs and for Fano varieties, and to state more clearly the mirror conjectures for CY pairs and Fano varieties in dimensions two and three.

More precisely, if $X{=}X_\Delta$ is a toric variety defined by a polytope Δ, one associates to the big torus of X a torus $(\mathbb{C}^*)^n \subset X^\vee$ such that the restriction of w to $(\mathbb{C}^*)^n$ is a Laurent polynomial with Newton polytope Δ. When X is a complete intersection in a toric variety, Givental writes down a LG model $(X^\vee, w)$ which is a complete intersection in $(\mathbb{C}^*)^n$. More generally, if X is Fano and has a toric degeneration $X \rightsquigarrow X_\Delta$ to a (Fano) variety that is either toric or a complete intersection in a toric variety, then there is a birational map $\phi_\Delta \colon (\mathbb{C}^*)^n \dashrightarrow X^\vee$ associated to $(\mathbb{C}^*)^n \subset X_\Delta$ such that $\phi_\Delta^* w$ is a Laurent polynomial with Newton polytope Δ. In this case, I say that X has a Givental LG model and that ϕ_Δ is a torus chart on $X^\vee$.

Conjecturally, if $X \rightsquigarrow X_{\Delta_1}$ and $X \rightsquigarrow X_{\Delta_2}$ are toric degenerations and ϕ_{Δ_1} and ϕ_{Δ_2} are the associated torus charts, then $\phi_{\Delta_2}^{-1} \circ \phi_{\Delta_1}$ is a volume preserving map (this has been verified in a number of cases). There are reasons to believe that the mirror conjecture can be simplified in the case of Fano varieties with toric degenerations, and that it pairs objects in the following classes:

(i) a Fano variety X and its toric degenerations $X \rightsquigarrow X'$ with Givental LG models,

(ii) a quasi-projective variety with a regular map $w \colon X^\vee \to \mathbb{C}$ and torus charts $\phi \colon (\mathbb{C}^*)^n \dashrightarrow X^\vee$ related to each other by volume preserving maps.

I.C.(i) Characterisation of pairs with a toric model Of course, some Fano varieties have no toric degeneration so the formulation of the mirror conjecture above cannot replace the one in terms of periods. Conversely, one can ask what the mirror partner of a CY pair (X, D) which does not have a toric chart is. Examples of such pairs are obtained when one considers a Fano variety X that is birationally rigid.

CY pairs (X, D) with a volume preserving birational map to a toric pair are said to have a toric model. As explained above, such pairs form an interesting class of examples in mirror symmetry, that includes cluster varieties.

Putting together the mirror symmetry construction of Gross-Hacking-Keel for CY pairs and the mirror symmetry conjecture for Fano varieties indicates that CY pairs with toric models are the natural mirrors of Fano varieties with toric degenerations. These CY pairs have a so called "torus chart" and from the point of view of mirror symmetry, they should be studied along with Halphen pencils (a log Calabi–Yau fibration to $\mathbb{P}^1$) and volume preserving maps that preserve the pencil.

A characterisation of toric pairs was conjectured by Shokurov and only proved by Brown-Svaldi-McKernan and Zhong very recently. The invariants used to characterise toric pairs are difficult to keep track of in volume preserving birational maps.

Question 1: How to characterise CY pairs with toric models?

A maximal intersection CY pair is one which admits a volume preserving dlt blowup with a 0-dimensional non klt centre. It was conjectured that maximal intersection CY pairs were those admitting toric models. While this is true for surfaces, I gave examples of maximal intersection CY pairs (X, D) with X birationally rigid, hence with no toric model. I conjecture the following characterisation:

Conjecture A maximal intersection CY pair (X, D) has a toric model precisely when D supports an ample divisor.

I.C.(ii) Sarkisov type statements for CY pairs with an Halphen pencil. From the point of view of Mirror Symmetry, it is important to understand volume preserving birational maps between CY pairs endowed with Halphen pencils, i.e. that are relative CY over $\mathbb{P}^1$. I want to investigate whether an approach similar to Hacon-McKernan's treatment of the Sarkisov program could yield some results on the decomposition of volume preserving birational maps between CY pairs with an Halphen pencil.

II. On the topology of degenerations of Fano varieties with isolated Gorenstein singularities (with A. Petracci). Toric Fano 3-folds with isolated Gorenstein singularities correspond to 3-dimensional reflexive polytopes with unitary edges and which have at least one facet that is not a standard triangle. There are 137 such polytopes and a facet of such a polytope has at most one interior point. There are 6 possible local analytic types of singular points, and such a Fano 3-fold has at most one singular point that is not an ordinary double point. The functor of infinitesimal deformations is controlled by the types of singularities, and there are several smoothing components when X has a singular point that is a cone over a dP_6 surface. There are several ways of thinking about these different smoothings: these correspond to the two different Minkowski decompositions of a hexagon (the polytope associated to the singular point), or more geometrically to the two constructions of a del Pezzo 6 variety $X_6^n \subset \mathbb{P}^{n+4}$ by Fujita, or equivalently by unprojection from a member of one of the two families of maximal linear subspaces in the Plücker embedding $\mathrm{Gr}(2, 5)$ (Tom and Jerry setup - analysed by Reid).

The goal of this project is to analyse the topology of possible smoothings, and to find a full set of invariants of smoothings. More precisely, is it enough to consider as a full set of invariants the discriminant, degree, Picard rank and $h^{1,2}$. A crucial question to determine whether this is the case is the following:

Question 1 Do extremal rays deform in a smoothing when the central fibre is not $\mathbb{Q}$-factorial? Under what additional conditions do extremal rays deform?

III. Pliability of quartic 3-folds (with H. Ahmadinezhad). I give a brief description as this project is close to previous work. Let $X = X_4 \subset \mathbb{P}^4$ be a quartic hypersuface that is a Mori fibre space, i.e. X is factorial and has isolated hypersurface singularities. The local analytic description of these singularities is well understood, there are two infinite families (cAn and cDn) and three additional cases. Iskovskikh and Manin show that when X is smooth, it is birationally rigid: X is the only Mori fibre space in its class of birational equivalence. This holds if X has no worse than ordinary double points, but fails for more complicated singularities. Corti and Mella constructed an example of non-rigid quartic with a singular point of type $cA2$, and my collaborator Ahmadinezhad and I obtained examples of non-rigid quartics with a cAn point for any $n \geq 2$. Our current work shows that factorial quartics with no worse than $cA1$ points are rigid, as conjectured by Corti and Mella.