# Some examples of Calabi-Yau pairs with maximal intersection and no toric model

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**Abstract** It is known that a maximal intersection log canonical Calabi-Yau surface pair is crepant birational to a toric pair. This does not hold in higher dimension: this article presents some examples of maximal intersection Calabi-Yau pairs that admit no toric model.

## 1 Introduction and motivation

A Calabi–Yau (CY) pair $(X, D_X)$ consists of a normal projective variety $X$ and a reduced sum of integral Weil divisors $D_X$ such that $K_X + D_X \sim_{\mathbb{Z}} 0$.

The class of CY pairs arises naturally in a number of problems and comprises examples with very different birational geometry. Indeed, on the one hand, a Gorenstein Calabi–Yau variety $X$ can be identified with the CY pair $(X, 0)$. On the other hand, if $X$ is a Fano variety, and if $D_X$ is an effective reduced anticanonical divisor, then $(X, D_X)$ is also a CY pair.

**Definition 1** (a) A pair $(X, D_X)$ is (t,dlt) (resp. (t,lc)) if $X$ is $\mathbb{Q}$-factorial, terminal and $(X, D_X)$ divisorially log terminal (resp. log canonical).

(b) A birational map $(X, D_X) \overset{\varphi}{\dashrightarrow} (Y, D_Y)$ is volume preserving if $a_E(K_X + D_X) = a_E(K_Y + D_Y)$ for every geometric valuation $E$ with centre on $X$ and on $Y$.

The dual complex of a dlt pair $(Z, D_Z = \sum D_i)$ is the regular cell complex obtained by attaching an $(|I| - 1)$-dimensional cell for every irreducible component of a non-empty intersection $\bigcap_{i \in I} D_i$.

The dual complex encodes the combinatorics of the lc centres of a dlt pair and [4] show that its PL homeomorphism class is a volume preserving birational invariant.

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By [3, Theorem 1.9], a (t,lc) CY pair $(X, D_X)$ has a volume preserving (t,dlt) modification $(\widetilde{X}, D_{\widetilde{X}}) \to (X, D_X)$, and the birational map between two such modifications is volume preserving.

Abusing notation, I call dual complex the following volume preserving birational invariant of a (t,lc) CY pair $(X, D_X)$.

**Definition 2** $\mathcal{D}(X, D_X)$ is the PL homeomorphism class of the dual complex of a volume preserving (t,dlt) modification of $(X, D_X)$.

As the underlying varieties of CY pairs range from CY to Fano varieties, they can have very different birational properties. However, $X$ being Fano is not a volume preserving birational invariant of the pair $(X, D_X)$. Following [13], I consider the following volume preserving birational invariant notion:

**Definition 3** A (t,lc) CY pair $(X, D_X)$ has maximal intersection if $\dim \mathcal{D}(X, D_X) = \dim X - 1$.

In other words, $(X, D_X)$ has maximal intersection if there is a volume preserving (t,dlt) modification of $(X, D_X)$ with a 0-dimensional log canonical centre. Maximal intersection CY pairs have some Fano-type properties; Kollár and Xu show the following:

**Theorem 1** *Let $(X, D_X)$ be a dlt maximal intersection CY pair, then:*

1. *[13, Proposition 19] $X$ is rationally connected,*
2. *[13, Theorem 21] there is a volume preserving map $(X, D_X) \overset{\varphi}{\dashrightarrow} (Z, D_Z)$ such that $D_Z$ fully supports a big and semiample divisor.*

*Remark 1* The expression "Fano-type" should be understood with a pinch of salt. Having maximal intersection is a degenerate condition: a general (t,lc) CY pair $(X, D_X)$ with $X$ Fano and $D_X$ a reduced anticanonical section needs not have maximal intersection.

**Definition 4** A toric pair $(X, D_X)$ is a (t,lc) CY pair formed by a toric variety and the reduced sum of toric invariant divisors.

A toric model is a volume preserving birational map to a toric pair.

**Example 1.**

A CY pair with a toric model has maximal intersection.

*Remark 2* In dimension 2, the converse holds: maximal intersection CY surface pairs are precisely those with a toric model [6].

The characterisation of CY pairs with a toric model is an open and difficult problem. A characterisation of toric pairs was conjectured by Shokurov and is proved in [1], but it is not clear how to refine it to get information on the existence of a toric model. A motivation to better understand the birational geometry of CY pairs and their relation to toric pairs comes from mirror symmetry.

The mirror conjecture extends from a duality between Calabi-Yau varieties to a correspondence between Fano varieties and Landau-Ginzburg models, i.e. non-compact Kähler manifolds endowed with a superpotential. Most known constructions of mirror partners rely on toric features such as the existence of a toric model or of a toric degeneration. In an exciting development, Gross, Hacking and Keel conjecture the following construction for mirrors of maximal intersection CY pairs.

*Conjecture 1* [6] Let $(Y, D_Y)$ be a simple normal crossings maximal intersection CY pair. Assume that $D_Y$ supports an ample divisor, let $R$ be the ring $k[\mathrm{Pic}(Y)^\times]$, $\Omega$ the canonical volume form on $U$ and

$$U^{\mathrm{trop}}(\mathbb{Z}) = \Big\{\text{divisorial valuations } v\colon k(U)\setminus\{0\}\to\mathbb{Z} \text{ with } v(\Omega)<0\Big\}\cup\{0\}.$$

Then, the free $R$-module $V$ with basis $U^{\mathrm{trop}}(\mathbb{Z})$ has a natural finitely generated $R$-algebra structure whose structure constants are non-negative integers determined by counts of rational curves on $U$.

Denote by $K$ the torus $\mathrm{Ker}\{\mathrm{PicY}\to\mathrm{Pic}(U)\}$. The fibration

$$p\colon \mathrm{Spec}(V)\to\mathrm{Spec}(R) = T_{\mathrm{Pic}(Y)}$$

is a $T_K$-equivariant flat family of affine maximal intersection log CY varieties. The quotient

$$\mathrm{Spec}(V)/T_K\to T_{\mathrm{Pic}(U)}$$

only depends on $U$ and is the mirror family of $U$.

Versions of Conjecture 1 are proved for cluster varieties in [7], but relatively few examples are known.

The goal of this note is to present examples of maximal intersection CY pairs that do not admit a toric model and for which one can hope to construct the mirror partner proposed in Conjecture 1 (see Section 2 for a precise statement).

## 2 Auxiliary results on 3-fold CY pairs

The examples in Section 3 are 3-fold maximal intersection CY pairs whose underlying varieties are birationally rigid. In particular, such pairs admit no toric model; this shows that [6]'s results on maximal intersection surface CY pairs do not extend to higher dimensions. In this section, I first recall some results on birational rigidity of Fano 3-folds. Then, I introduce the (t,dlt) modifications suited to the construction outlined in Conjecture 1 and discuss the singularities of the boundary $D_X$.

### 2.1 Birational rigidity

Let $X$ be a terminal $\mathbb{Q}$-factorial Fano 3-fold. When $X$ has Picard rank 1, $X$ is a Mori fibre space, i.e. an end product of the classical MMP.

**Definition 5** A birational map $Y/S \overset{\varphi}{\dashrightarrow} Y'/S'$ between Mori fibre spaces $Y/S$ and $Y'/S'$ is square if it fits into a commutative square

$$\begin{array}{ccc} Y & \overset{\varphi}{\dashrightarrow} & Y' \\ \downarrow & & \downarrow \\ S & \overset{g}{\dashrightarrow} & S' \end{array}$$

where $g$ is birational and the restriction $Y_\eta \overset{\varphi_\eta}{\dashrightarrow} Y'_\eta$ is biregular, where $\eta$ is the function field of the base $k(S)$.

A Mori fibre space $Y/S$ is (birationally) rigid if for every birational map $Y/S \overset{\varphi}{\dashrightarrow} Y'/S'$ to another Mori fibre space, there is a birational self map $Y/S \overset{\alpha}{\dashrightarrow} X/S$ such that $\varphi \circ \alpha$ is square.

In particular, if $X$ is a rigid Mori fibre space, then $X$ is non-rational and no (t,lc) CY pair $(X, D_X)$ admits a toric model.

Non-singular quartic hypersurfaces $X_4 \subset \mathbb{P}^4$ are probably the most famous examples of birationally rigid 3-folds [9]. Some mildly singular quartic hypersurfaces are also known to be birationally rigid, in particular, we have:

**Proposition 1** *[2, 16]Let $X_4 \subset \mathbb{P}^4$ be a quartic hypersurface with no worse than ordinary double points. If $|\mathrm{Sing}(X)| \leq 8$, then $X$ is $\mathbb{Q}$-factorial (in particular, $X$ is a Mori fibre space) and is birationally rigid.*

### 2.2 Singularities of the boundary

I now state some results on the singularities of the boundary of a 3-fold (t,lc) CY pair. Let $(X, D_X)$ be a 3-fold (t,lc) CY pair and $(\widetilde{X}, D_{\tilde{X}})$ a (t,dlt) modification. A stratum of $(\widetilde{X}, D_{\tilde{X}})$ is an irreducible component of a non-empty intersection of components of $D_{\tilde{X}}$. Given a stratum $W$, there is a divisor $\mathrm{Diff}_W D_{\tilde{X}}$ on $W$ such that $(W, \mathrm{Diff}_W D_{\tilde{X}})$ is a lc CY pair and

$$K_W + \mathrm{Diff}_W D_{\tilde{X}} \sim_{\mathbb{Q}} (K_{\tilde{X}} + D_{\tilde{X}})_{|W}.$$

When $K_{\tilde{X}} + D_{\tilde{X}}$ is Cartier and $D_{\tilde{X}}$ reduced, $\mathrm{Diff}_W D_{\tilde{X}}$ is the sum of the restrictions of the components of $D_{\tilde{X}}$ that do not contain $W$.

In particular, for any irreducible component $S$ of $D_{\tilde{X}}$, the link of $[S]$ in $\mathcal{D}(X, D_X)$ is the dual complex $\mathcal{D}(S, \mathrm{Diff}_S D_{\tilde{X}})$. Therefore, if $(X, D_X)$ has maximal intersection, so does $(S, \mathrm{Diff}_S D_{\tilde{X}})$. By the results of [6], $(S, \mathrm{Diff}_S D_X)$ then has a toric model.

As $X$ has terminal singularities, $X$ is normal and Cohen-Macaulay. Any Cartier component $S$ of the boundary $D_X$ is Cohen-Macaulay and satisfies Serre's condition $S_2$. By [12, Proposition 16.9], $(S, \mathrm{Diff}_S D_X)$ is semi log canonical (slc). In particular, if $X$ is Gorenstein and $D_X$ irreducible, $D_X$ has slc singularities.

I am particularly interested in producing examples of (t,lc) CY pairs for which the mirror partners proposed in Conjecture 1 (see also [8]) can be constructed; this motivates the following definition:

**Definition 6** A (t,dlt) modification $(\widetilde{X}, D_{\tilde{X}}) \to (X, D_X)$ is called good if $(\widetilde{X}, D_{\tilde{X}})$ is log smooth in the sense of log geometry, that is if the components of $D_{\tilde{X}}$ are non-singular and if $\widetilde{X}$ has only cyclic quotient singularities.

An immediate consequence of the definition is that if $(\widetilde{X}, D_{\tilde{X}}) \xrightarrow{f} (X, D_X)$ is a good (t,dlt) modification and $D_X = \sum_i D_i$, then

$$D_{\tilde{X}} = \sum_i f_*^{-1} D_i + E,$$

where $E$ is reduced and $f$-exceptional, and the restriction of $f$ to $f_*^{-1}D_i$ is a resolution for all $i$.

**Normal singularities** Let $p \in \mathrm{Sing}(D_i)$ be an isolated singularity lying on a single component of the boundary. The restriction $f_i \colon \widetilde{D}_i \to D_i$ is a resolution and we have:

$$K_{\tilde{D}_i} = (K_{\tilde{X}} + D_i)_{|\tilde{D}_i} = (f_{|\tilde{D}_i})^* K_{D_i} - (E)_{|\tilde{D}_i}$$

where $E$ is defined by $K_{\tilde{X}} + f_*^{-1} D_{\tilde{X}} + E = f^*(K_X + D_X)$.

We now assume that $D_i$ is Cartier, as is the case when $X$ is Gorenstein and $D_X$ irreducible. Without loss of generality, assume that $\mathrm{Sing}(D_i) = p$. Then, $p$ is canonical if $E \cap \widetilde{D}_i = \emptyset$, and elliptic otherwise. Indeed, let

$$f_i \colon \widetilde{D}_i \xrightarrow{q} \overline{D}_i \xrightarrow{\mu} D_i$$

be the factorisation through the minimal resolution of $(p \in D_i)$. Then, $q$ is either an isomorphism or an isomorphism at the generic point of each component of $E_{|\tilde{D}_i}$ because $f$ is volume preserving. We have: $K_{\overline{D}_i} = \mu^* K_{D_i} - Z$, where the effective cycle $Z = q_*(E_{D_i})$ is either empty (and $p$ is canonical) or a reduced sum of $\mu$-exceptional curves (and $p$ is elliptic). In the second case, $Z \sim -K_{\overline{D}_i}$ is the fundamental cycle of $(p \in D_i)$. If $Z$ is irreducible, it is reduced and has genus 1; if not, every irreducible component of $Z$ is a smooth rational curve of self-intersection $-2$.

When $p$ is elliptic, $Z$ is reduced and $p$ is a Kodaira singularity [10, Theorem 2.9], i.e. a resolution is obtained by blowing up points of the singular fibre in a degeneration of elliptic curves; further, in Arnold's terminology, the singularity $p$ is uni or bimodal.

Further, $p \in D_i$ is a hypersurface singularity (resp. a codimension 2 complete intersection, resp. not a complete intersection) when $-3 \leq Z^2 \leq -1$ (resp. $Z^2 = -4$, resp. $Z^2 \leq -5$) [14]. When $-1 \leq Z^2 \leq -4$, normal forms are known for $p \in D_i$:

Table 1 lists normal forms of slc hypersurface singularities, while normal forms of codimension 2 complete intersections elliptic singularities are given in [19].

| type | name | symbol | equation $f \in \mathbb{C}[x,y,z]$ | | $\mathrm{mult}_0 f$ |
|---|---|---|---|---|---|
| terminal | smooth | $A_0$ | $x$ | | 1 |
| canonical | du Val | $A_n$ | $x^2+y^2+z^{n+1}$ | $n \geq 1$ | 2 |
| | | $D_n$ | $x^2+z(y^2+z^{n-2})$ | $n \geq 4$ | 2 |
| | | $E_6$ | $x^2+y^3+z^4$ | | 2 |
| | | $E_7$ | $x^2+y^3+yz^3$ | | 2 |
| | | $E_8$ | $x^2+y^3+z^5$ | | 2 |
| lc | simple elliptic | $X_{1,0}$ | $x^2+y^4+z^4+\lambda xyz$ | $\lambda^4 \neq 64$ | 2 |
| | | $J_{2,0}$ | $x^2+y^3+z^6+\lambda xyz$ | $\lambda^6 \neq 432$ | 2 |
| | | $T_{3,3,3}$ | $x^3+y^3+z^3+\lambda xyz$ | $\lambda^3 \neq -27$ | 3 |
| | cusp | $T_{p,q,r}$ | $x^p+y^q+z^r+xyz$ | $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ | 2 or 3 |
| slc | normal crossing | $A_\infty$ | $x^2+y^2$ | | 2 |
| | pinch point | $D_\infty$ | $x^2+y^2z$ | | 2 |
| | degenerate cusp | $T_{2,\infty,\infty}$ | $x^2+y^2+z^2$ | | 2 |
| | | $T_{2,q,\infty}$ | $x^2+y^2(z^2+y^{q-2})$ | $q \geq 3$ | 2 |
| | | $T_{\infty,\infty,\infty}$ | $xyz$ | | 3 |
| | | $T_{p,\infty,\infty}$ | $xyz+x^p$ | $p \geq 3$ | 3 |
| | | $T_{p,q,\infty}$ | $xyz+x^p+y^q$ | $q \geq p \geq 3$ | 3 |

**Table 1** Dimension 2 slc hypersurface singularities

## 3 Examples of rigid maximal intersection 3-fold CY pairs

All the examples below are $(t, lc)$ CY pairs $(X, D_X)$ which admit no toric model. Except for Example 5, all underlying varieties $X$ are birationally rigid quartic hypersurfaces by Proposition 1; the underlying variety in Example 5 is a smooth cubic 3-fold, and therefore non-rational.

### 3.1 Examples with normal boundary

**Example 2.**

Consider the CY pair $(X, D_X)$ where $X$ is the nonsingular quartic hypersurface

$$X = \{x_1^4 + x_2^4 + x_3^4 + x_0x_1x_2x_3 + x_4(x_0^3 + x_4^3) = 0\}$$

and $D_X$ is its hyperplane section $X \cap \{x_4 = 0\}$.

The quartic surface $D_X$ has a unique singular point $p = (1{:}0{:}0{:}0{:}0)$, and using the notation of Table 1, $p$ is locally analytically equivalent to a $T_{4,4,4}$ cusp

$$\{0\} \in \{x^4 + y^4 + z^4 + xyz = 0\}.$$

$D_X$ is easily seen to be rational: the projection from the triple point $p$ is

$$D_X \dashrightarrow \mathbb{P}^2_{x_1,x_2,x_3};$$

this map is the blowup of the 12 points $\{x_1^4 + x_2^4 + x_3^4 = x_1x_2x_3 = 0\}$, of which 4 lie on each coordinate line $L_i = \{x_i = 0\}$, for $i = 1, 2, 3$.

I treat this example in detail and construct explicitly a good (t,dlt) modification of the pair $(X, D_X)$.

Let $f \colon X_p \to X$ be the blowup of $p$, then $X_p$ is non-singular, the exceptional divisor $E$ satisfies $(E, \mathcal{O}_E(E)) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$, and if $D$ denotes the proper transform of $D_X$, we have:

$$K_{X_p} + D + E = f^*(K_X + D).$$

Explicitly, the blowup $\mathcal{F} \to \mathbb{P}^4$ of $\mathbb{P}^4$ at $p$ is the rank 2 toric variety $\mathrm{TV}(I, A)$, where $I = (u, x_0) \cap (x_1, \dots, x_4)$ is the irrelevant ideal of $\mathbb{C}[u, x_0, \dots, x_4]$ and $A$ is the action of $\mathbb{C}^* \times \mathbb{C}^*$ with weights:

$$\begin{pmatrix} u & x_0 & s_1 & s_2 & s_3 & s_4 \\ 1 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{1}$$

The equation of $X_p$ is

$$X_p = \{u^2(u(s_1^4 + s_2^4 + s_3^4 + (x_0s_1s_2s_3)) + s_4(x_0^3 + u^3s_4^3) = 0\},$$

while $E = \{u = 0\}$ and $D = \{u(s_1^4 + s_2^4 + s_3^4) + x_0s_1s_2s_3 = 0\}$. By construction, $E$ is the projective plane with coordinates $s_1, s_2, s_3$. Note that $(X_p, D + E)$ is not dlt because $D \cap E = \{x_0s_1s_2s_3 = 0\}$ consists of 3 concurrent lines $C_1, C_2, C_3$.

Consider $g_1 \colon X_1 \to X_p$ the blowup of the nonsingular curve

$$C_1 = \{u = s_1 = s_4 = 0\} \subset X_p.$$

The exceptional divisor of $g_1$ is a surface $E_1 \simeq \mathbb{P}(\mathcal{N}_{C_1/X_p})$, and since $C_1 \simeq \mathbb{P}^1$, the restriction sequence of normal bundles gives

$$\mathcal{N}_{C_1/X_p} \simeq \mathcal{N}_{C_1/E} \oplus (\mathcal{N}_{E/X_p})_{|C_1} \simeq \mathcal{O}_{C_1}(1) \oplus \mathcal{O}_E(-1)_{|C_1},$$

so that $E_1 = \mathbb{F}_2$. Further,

$$K_{X_1} + D + E + E_1 = g_1^*(K_{X_p} + D + E)$$

where, abusing notation, I denote by $D$ and $E$ the proper transforms of the divisors $D$ and $E$. The "restricted pair" on $E_1$ is a surface CY pair $(E_1, (D+E)_{|E_1})$ by adjunction. By construction, $E \cap E_1$ is the negative section $\sigma$. The curve $\Gamma = D \cap E_1$ is irreducible, and since $(D+E)_{|E_1}$ is anticanonical, we have

$$\Gamma \sim \sigma + 4f \text{ where } f \text{ is a fibre of } \mathbb{F}_2 \to \mathbb{P}^1, \text{ and } \Gamma^2 = 6, \Gamma \cdot E_{|E_1} = 2.$$

The divisors $D, E, E_1$ meet in two points, the dual complex $\mathcal{D}(X_1, D+E+E_1)$ is not simplicial it is a sphere $S^2$ whose triangulation is given by 3 vertices on an equator. While not strictly necessary, we consider a further blowup to obtain a (t,dlt) pair with simplicial dual complex.

Denote by $C_2$ the proper transform of the curve

$$\{u = s_2 = s_4 = 0\}.$$

Then $C_2 \subset E \cap D$ is rational, and as above

$$\mathcal{N}_{C_2/X_1} \simeq \mathcal{N}_{C_2/E} \oplus (\mathcal{N}_{E/X_2})_{|C_2} = \mathcal{O}_{C_2}(1) \oplus \mathcal{O}_{C_2}(-2).$$

Let $g_2\colon X_2 \to X_1$ be the blowup of $C_2$, then the exceptional divisor of $g_2$ is a Hirzebruch surface

$$E_2 \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{N}_{C_2/X_1}) \simeq \mathbb{F}_3.$$

Still denoting by $D, E, E_1$ the strict transforms of $D, E, E_1$, we have:

$$K_{X_2} + D + E + E_1 + E_2 = g_2^*(K_{X_1} + D + E + E_1).$$

The pair $(X_2, D+E+E_1+E_2)$ is dlt; the composition

$$g_2 \circ g_1 \circ f\colon (\widetilde{X}, D_{\tilde{X}}) = (X_2, D+E+E_1+E_2) \to (X, D_X)$$

is a good (t,dlt) modification.

The "restrictions" of $(\widetilde{X}, D_{\tilde{X}})$ to the component of the boundary are the following surface anticanonical pairs:

- On $D$: $(E+E_1+E_2)_{|D}$ is a cycle of $(-3)$-curves, the morphism $D \to D_X$ is the familiar resolution of the $T_{4,4,4}$ cusp singularity;
- On $E$: $(D+E_1+E_2)_E$ is the triangle of coordinate lines with self-intersections $(1,1,1)$;
- On $E_1$: $(D+E+E_2)_{E_1}$ is an anticanonical cycle with self-intersections $(5,-3,-1)$;
- On $E_2$: $(D+E+E_1)_{E_2}$ is an anticanonical cycle with self-intersections $(5,-3,0)$ (as above, $E_{|E_2} \sim \sigma$ is a negative section, $E_{1\,|E_2} \sim f$ a fibre of $\mathbb{F}_3 \to \mathbb{P}^1$, and $D_{|E_2} \sim 4f + \sigma$).

It follows that the dual complex $\mathcal{D}(X, D_X)$ is PL homeomorphic to a tetrahedron and $(X, D_X)$ has maximal intersection. Note that $(0 \in D_X)$ is a maximal intersection lc point, and since $D_X$ is a rational surface, it has a toric model.

**Example 3.**

Let $X$ be the hypersurface

$$X = \{x_3(x_0^3 + x_1^3) + x_2^4 + x_0x_1x_2x_3 + x_4(x_3^3 + x_4^3) = 0\},$$

and $D_X$ its hyperplane section $X \cap \{x_4 = 0\}$.

The quartic $X$ has 3 ordinary double points at the intersection points

$$L \cap \{x_0^3 + x_1^3 = 0\},$$

where $L$ is the line $\{x_2 = x_3 = x_4 = 0\}$. The singular locus of $D_X$ is $\mathrm{Sing}(X) \cup \{p\}$, where $p = (0{:}0{:}0{:}1{:}0)$ is a $T_{3,3,4}$ cusp, i.e. locally analytically equivalent to

$$\{0\} \in \{x^3 + y^3 + z^4 + xyz = 0\}.$$

The quartic surface $D_X$ is rational; the projection of $D_X$ from $p$ is

$$D_X \dashrightarrow \mathbb{P}^2_{x_0,x_1,x_2};$$

this map is defined outside of the 12 points (counted with multiplicity) defined by $\{x_2^4 = x_0^3 + x_1^3 + x_0x_1x_2 = 0\}$.

If $\tilde{X} \xrightarrow{f} X$ is the composition of the blowups at the ordinary double points and at $p$, $\widetilde{X}$ is smooth and $D_{\tilde{X}}$ is non-singular, so that $f$ is a good (t,dlt) modification.

The minimal resolution of $p \in D_X$ is a rational curve with self intersection $C^2 = -3$. Explicitly, taking the blowup of $X$ at $p$, the proper transform is a rational surface $D$. The exceptional curve is the preimage of a nodal cubic in $\mathbb{P}^2$ blown up at 12 points counted with multiplicities. Note that $(\widetilde{X}, D + E)$ is not dlt, but in order to obtain a (t,dlt) modification, we just need to blowup the node of $D \cap E$ which is a nonsingular point of $\widetilde{X}$, $D$ and $E$. The (t,dlt) modification of $(X, D_X)$ in a neighbourhood of $p$ is good and the associated dual complex is 2-dimensional.

The pair $(X, D_X)$ has maximal intersection; but as in the previous examples, $X$ is rigid, so that $(X, D_X)$ can have no toric model.

**Example 4.**

Let $X$ be the nonsingular quartic hypersurface

$$X = \{x_0^3x_3 + x_1^4 + x_2^4 + x_0x_1x_2x_3 + x_4(x_3^3 + x_4^3) = 0\} \subset \mathbb{P}^4$$

and $D_X$ its hyperplane section $X \cap \{x_4 = 0\}$.

The surface $D_X$ has a unique singular point $p = (0{:}0{:}0{:}1{:}0)$ of $D_X$, which is a cusp $T_{3,4,4}$, i.e. is locally analytically equivalent to

$$\{0\} \in \{x^3 + y^4 + z^4 + xyz = 0\}.$$

As in Example 2, $X$ is non-singular, and finding a good (t,dlt) modification of $(X, D_X)$ will amount to taking a minimal resolution of the singular point of $D_X$. Let $X_p \to X$ be the blowup of $X$ at $p$; $X_p$ is non-singular and if $D$ denotes the proper transform of $D_X$, and $E$ the exceptional divisor, $D \cap E$ consists of 2 rational curves of self intersection $-3$ and $-4$. These curves are the proper transforms of $\{x_0 = 0\}$ and of $\{x_0^2 + x_1x_2\}$ under the blow up of $\mathbb{P}^2_{x_0,x_1,x_2}$ at the points

$$\{x_1^4 + x_2^4 = x_0(x_1x_2 + x_0^2) = 0\}.$$

The dual complex consists of 3 vertices that are joined by edges and span 2 distinct faces: $\mathcal{D}(X, D_X)$ is PL homeomorphic to a sphere $S^2$ whose triangulation is given by 3 vertices on an equator. The CY pair $(X, D_X)$ has maximal intersection but no toric model.

### 3.2 Examples with non-normal boundary

**Example 5.**

This example is due to R. Svaldi. Consider the cubic 3-fold

$$X = \{x_0x_1x_2 + x_1^3 + x_2^3 + x_3q + x_4q' = 0\} \subset \mathbb{P}^4$$

where $q, q'$ are homogeneous polynomials of degree 2 in $x_0, \cdots, x_4$. If the quadrics $q$ and $q'$ are general and if

$$(q(1, 0, 0, 0, 0), q'(1, 0, 0, 0, 0)) \neq (0, 0),$$

then $X$ and $S = \{x_3 = 0\} \cap X$ and $T = \{x_4 = 0\} \cap X$ are nonsingular.

Let $D_X$ be the anticanonical divisor $S + T$. The curve $C = S \cap T = \Pi \cap X$ for $\Pi = \{x_3 = x_4 = 0\}$ is a nodal cubic. It follows that both $(S, C)$ and $(T, C)$ are log canonical, and therefore so is $(X, D_X)$.

Since $S$ and $T$ are smooth, $\mathrm{Sing}(D_X) = S \cap T = C$, and if $p$ is the node of $C$, we have:

$$(p \in D_X) \sim \{0\} \in \{(xy + x^3 + y^3 + z)(xy + x^3 + y^3 + t) = 0\}$$
$$\sim \{0\} \in \{(xy + z)(xy + t) = 0\} \sim \{0\} \in \{(xy + z)(xy - z) = 0\}.$$

Thus, $p \in D_X$ is a double pinch point, i.e. $p$ is locally analytically equivalent to $\{0\} \in \{x^2y^2 - z^2 = 0\}$.

We now construct a good (t,dlt) modification of $(X, D_X)$. Let $f : X_C \to X$ be the blowup of $X$ along $C$; $\mathrm{Sing}(X_C)$ is an ordinary double point.

Indeed, let $\Pi = \{x_3 = x_4 = 0\}$, then $f$ is the restriction to $X$ of the blowup $\mathcal{F} \to \mathbb{P}^4$, where $\mathcal{F}$ is the rank 2 toric variety $\mathrm{TV}(I, A)$, where $I = (u, x_0, x_1, x_2) \cap (x_3, x_4)$ is the irrelevant ideal of $\mathbb{C}[u, x_0, \dots, x_4]$ and $A$ is the action of $\mathbb{C}^* \times \mathbb{C}^*$ with weights:

$$\begin{pmatrix} u & x_0 & x_1 & x_2 & x_3 & x_4 \\ 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

The equation of $X_C$ is

$$\{x_0x_1x_2 + x_1^3 + x_2^3 + u(x_3q + x_4q') = 0\},$$

so that $X_C$ has a unique singular point at

$$x_0 - 1 = u = x_1 = x_2 = x_3q(1,0,0,0,0) + x_4q'(1,0,0,0,0) = 0,$$

and this is a 3-fold ordinary double point. In addition, denoting by $E_f = \{u = 0\} \cap X_C$ the exceptional divisor, we have

$$K_{X_C} + \tilde{S} + \tilde{T} + E_f = K_X + S + T,$$

so that the pair $(X_C, \tilde{S} + \tilde{T} + E_f)$ is a (t,lc) CY pair.

The pair $(X_C, \tilde{S} + \tilde{T} + E_f)$ is not dlt as the boundary has multiplicity 3 along the fibre $F$ over the node of $S \cap T$. The blowup of $F$ is not $\mathbb{Q}$-factorial, therefore in order to obtain a good (t,dlt) modification, we consider the divisorial contraction $g\colon \tilde{X} \to X_C$ centred along $F$. This is obtained by (a) blowing up the node, (b) then blowing up the proper transform of $F$, (c) flopping a pair of lines with normal bundle $(-1,-1)$ and (d) contracting the proper transform of the $\mathbb{P}^1 \times \mathbb{P}^1$ above the node to a point $\frac{1}{2}(1,1,1)$. The exceptional divisor of $g$ is denoted by $E_g$.

The pair $(\tilde{X}, \tilde{S} + \tilde{T} + \tilde{E}_f + E_g)$ is the desired (t,dlt) modification of $(X, D_X)$, and it has maximal intersection. The dual complex is PL homeomorphic to a tetrahedron.

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**Example 6.**

Let $X$ be the quartic hypersurface

$$X = \{x_1^2x_2^2 + x_1x_2x_3l + x_3^2q + x_4f_3 = 0\} \subset \mathbb{P}^4,$$

where $l$ (resp. $q$) is a general linear (resp. quadratic) form in $x_0, \cdots, x_3$, and $f_3$ a general homogeneous form of degree 3 in $x_0, \cdots, x_4$. Let $D_X$ be the hyperplane section $X \cap \{x_4 = 0\}$.

As $l, q$ and $f_3$ are general, $X$ has 6 ordinary double points. Indeed, denote by $L = \{x_1 = x_3 = x = 4 = 0\}$ and $L' = \{x_2 = x_3 = x = 4 = 0\}$, then

$$\mathrm{Sing}(X) = \{L \cap \{f_3 = 0\}\} \cup \{L' \cap \{f_3 = 0\}\} = \{q_1, q_2, q_3\} \cup \{q_1', q_2', q_3'\}$$

which consists of 3 points on each of the lines. In the neighbourhood of each point $q_i$ (resp. $q_i'$) for $i = 1, 2, 3$, the equation of $X$ is of the form

$$\{0\} \in \{xy + zt = 0\}$$

(and $D_X = \{t = 0\}$) so that all singular points of $X$ are ordinary double points. The quartic hypersurface $X$ is birationally rigid by Proposition 1.

The surface $D_X$ is non-normal as it has multiplicity 2 along $L$ and $L'$. The point $p = L \cap L'$ is locally analytically equivalent to

$$\{0\} \in \{x^2y^2 + z^2 = 0\},$$

so that $p \in D_X$ is a double pinch point. We conclude that the surface $D_X$ has slc singularities, and hence $(X, D_X)$ is a (t,lc) CY pair.

We construct a good (t,dlt) modification as follows.

First, since $\mathrm{Sing}(X) \cap L$ (resp. $\mathrm{Sing}(X) \cap L'$) is non-empty, the blowup of $X$ along $L$ (resp. along $L'$) is not $\mathbb{Q}$-factorial. In order to remain in the (t,dlt) category, we consider the divisorial extraction $f\colon X_L \to X$ centered on $L$ (resp. $L'$).This is obtained by (a) blowing up the 3 nodes lying on $L$, (b) blowing up the proper transform of $L$, (c) flopping 3 pairs of lines with normal bundle $(-1, -1)$ and (d) contracting the proper transforms of the three exceptional divisors $\mathbb{P}^1 \times \mathbb{P}^1$ lying above the nodes to points $\frac{1}{2}(1, 1, 1)$. The exceptional divisor of $f$ is denoted by $E$. Let $p\colon \widetilde{X} \to X$ denote the morphism obtained by composing the divisorial extraction centered on $L$ with that centered on $L'$ (in any order), and let $E, E'$ denote the exceptional divisors of the divisorial extractions. Then

$$K_{\widetilde{X}} + \widetilde{D} + E + E' = p^*(K_X + D)$$

is a (t,dlt) modification of $(X, D_X)$ and it has maximal intersection. The dual complex $\mathcal{D}(X, D_X)$ is is PL homeomorphic to a sphere $S^2$ whose triangulation is given by 3 vertices on an equator.

## 4 Further results on quartic 3-fold CY pairs: beyond maximal intersection

This section concentrates on (t,lc) CY pairs $(X, D_X)$, where $X$ is a factorial quartic hypersurface in $\mathbb{P}^4$ and $D$ is an irreducible hyperplane section of $X$. I give some more detail on the possible dual complexes of such pairs.

As explained in Section 2.2, $D_X$ is slc because $(X, D_X)$ is lc. In order to study completely the dual complexes of such (t,lc) CY pairs, one needs a good understanding of the normal forms of slc singularities that can lie on $D$. In the case of a general Fano $X$, this step would require additional work, but here, $D_X$ is a quartic surface in $\mathbb{P}^3$ and the study of singularities of such surfaces has a rich history. I recall some results directly relevant to the construction of degenerate $CY$ pairs $(X, D_X)$. The classification of singular quartic surfaces in $\mathbb{P}^3$ can be broken in three independent cases.

(a) Quartic surfaces with no worse than rational double points: the minimal resolution is a $K3$ surface. Possible configurations of canonical singularities were studied by several authors using the moduli theory of $K3$ surfaces; there are several thousands possible configurations. The pair $(X, D_X)$ is (t,dlt) and the dual complex of $(X, D_X)$ is reduced to a point.
(b) Non-normal quartic surfaces were classified by Urabe [17]; there are a handful of cases recalled in Theorem 2.
(c) Non-canonical quartic surfaces with isolated singularities. These are studied by Wall [20] and Degtyarev [5] among others; their results are recalled in Theorem 3.

**Theorem 2** *[17]A non-normal quartic surface $D \subset \mathbb{P}^3$ is one of:*

1. *the cone over an irreducible plane quartic curve with a singular point of type $A_1$ or $A_2$.*
2. *a ruled surface over a smooth elliptic curve $G$, $D = \varphi_{\mathcal{L}}(Z)$, where:*

*(a)* $\mathcal{L} = \mathcal{O}_Z(C_1) \otimes \pi^* M$*, and* $Z = \mathbb{P}_G(\mathcal{O}_G \oplus N)$*, for*
*-$M$ a line bundle of degree* 2 *and*
*-$N$ a non-trivial line bundle of degree* 0*.*
*Denoting by $L_i$ the images by $\varphi_{\mathcal{L}}$ of the sections of $Z$ associated to $\mathcal{O}_G \oplus N \to \mathcal{O}_G$ and $\mathcal{O}_G \oplus N \to N$,* $\mathrm{Sing}(D) = L_1 \cup L_2$*.*

*(b)* $\mathcal{L} = \mathcal{O}_Z(C) \otimes \pi^* M$*, and* $Z = \mathbb{P}_G(E)$ *for a rank* 2 *vector bundle $E$ that fits in a non-splitting*

$$0 \to \mathcal{O}_G \to E \to \mathcal{O}_G \to 0.$$

*Denoting by $L$ the image by $\varphi_{\mathcal{L}}$ of a section $G \to Z$,* $\mathrm{Sing} D = L$*.*

3. *a rational surface $D \subset \mathbb{P}^3$ which is*

*(a) the image of a smooth $S \subset \mathbb{P}^5$ under the projection from a line disjoint from $S$; $D$ has no isolated singular point and*
*-$S = v_2(\mathbb{P}^2)$, where $v_2$ is the Veronese embedding; $D$ is the Steiner Roman surface and is homeomorphic to $\mathbb{RP}^2$;*
*-$S = \varphi(\mathbb{P}^1 \times \mathbb{P}^1)$, where $\varphi$ is the embedding defined by $|l_1 + 2l_2|$ for $l_{1,2}$ the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$;*
*-$S = \varphi(\mathbb{F}_2)$, where $\varphi$ is the embedding defined by $|\sigma + f|$ for $\sigma$ the negative section and $f$ the fibre of $\mathbb{F}_2$.*

*(b) the image of a surface $\hat{D} \subset \mathbb{P}^4$ with canonical singularities under the projection from a point not lying on it; $\hat{D}$ is a degenerate $dP4$ surface which is the blowup of $\mathbb{P}^2$ in 5 points in almost general position.*

*(c) a rational surface embedded by a complete linear system on its normalisation $\hat{D}$; the non-normal locus of $D$ is a line $L$ and $D$ may have isolated singularities outside $L$. The minimal resolution of the normalisation of $D$ is a blowup of $\mathbb{P}^2$ in* 9 *points. The normalisation of $D$ has at most two rational triple points lying on the inverse image of the non-normal locus; their images on $D$ are also triple points.*

*Remark 3* $D$ is not slc in case 1.

**Corollary 1** *Let $(X, D_X)$ be a (t,lc) quartic CY pair with non-normal boundary. Then, $(X, D_X)$ has maximal intersection except in the cases described in 2.(a) and (b) of Theorem 2.*

**Example 7.**

Consider the pair $(X, D_X)$ where:

$$X = \{x_0^2 x_3^2 + x_1^2 x_2 x_3 + x_2^2 q(x_0, x_1) + x_4 f_3 = 0\}, D_X = X \cap \{x_4 = 0\},$$

where $q$ is a general quadratic form in $(x_1, x_2)$ and $f_3$ a general cubic in $x_0, \cdots, x_4$.

When $q$ and $f_3$ are general, the quartic hypersurface $X$ has 3 ordinary double points. Indeed, denote by $L = \{x_0 = x_1 = x_4 = 0\}$, then $\mathrm{Sing}(X)$ consists of points of intersection of $L$ with $\{f_3 = 0\}$; there are 3 such points $\{q_1, q_2, q_3\}$ when $f_3$ is general. In the neighbourhood of each point $q_i$ for $i = 1, 2, 3$, the equation of $X$ is of the form

$$\{0\} \in \{xy + zt = 0\}$$

(and $D_X = \{t = 0\}$) so that all singular points of $X$ are ordinary double points. The nodal quartic $X$ is terminal and $\mathbb{Q}$-factorial because it has less than 9 ordinary double points; $X$ is birationally rigid by [2, 16].

Taking the divisorial extraction of the line $L$ is enough to produce a dlt modification $(\widetilde{X}, D_{\tilde{X}} + E)$ of $(X, D_X)$; this shows that $(X, D_X)$ does not have maximal intersection. The dual complex has a single 1-stratum, the elliptic curve $D_{\tilde{X}} \cap E$, which is a $(2, 2)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$. The quartic surface $D_X$ is a ruled surface over an elliptic curve isomorphic to $D_{\tilde{X}} \cap E$; it is an example of case 2.(b) in Theorem 2.

**Theorem 3** *[20] A normal quartic surface $D \subset \mathbb{P}^3$ with at least one non-canonical singular point is one of:*

1. *D has a single elliptic singularity and D is rational, or*
2. *D is a cone, or*
3. *D is elliptically ruled and*

*(a) D has a double point p with tangent cone $z^2$, the projection away from p is the double cover of $\mathbb{P}^2$ branched over a sextic curve $\Gamma$. The curve $\Gamma$ is the union of 3 conics in a pencil that also contains a double line. When this line is a common chord, D has two $T_{2,3,6}$ singularities, when this line is a common tangent, D has one singularity of type $E_{4,0}$. In the first case, D may have an additional $A_1$ singular point.*

*(b) D is $\{(x_0x_3 + q(x_1, x_2))^2 + f_4(x_1, x_2, x_3) = 0\}$ and $\{f_4 = 0\}$ is four concurrent lines. Depending on whether $L = \{x_3 = 0\}$ is one of these lines or not and on whether the point of concurrence lies on L, D has either two $T_{2,4,4}$ singular points or one trimodal elliptic singularity. The surface may have additional canonical points $A_n$ for $n = 1, 2, 3$ or $2A_1$.*

**Example 8.**

Let $X$ be the nonsingular quartic hypersurface

$$X = \{x_0^2x_3^2 + x_0x_1^3 + x_3x_2^3 + x_0x_1x_2x_3 + x_4(x_0^3 + x_3^3 + x_4^3) = 0\}$$

and $D_X$ its hyperplane section $X \cap \{x_4 = 0\}$. The surface $D_X$ is normal,

$$\operatorname{Sing}(D_X) = \{p, p'\} = \{(1{:}0{:}0{:}0{:}0), (0{:}0{:}0{:}1{:}0)\},$$

and each singular point is simple elliptic $J_{2,0} = T_{2,3,6}$, i.e. is locally analytically equivalent to $\{0\} \in \{x^2 + y^3 + z^6 + xyz = 0\}$.

Here $X$ is nonsingular and $D_X$ is irreducible and normal, and as I explain below, finding a good (t,dlt) modification amounts to constructing a minimal resolution of $D_X$. Let $\widetilde{X} \to X$ be the composition of the weighted blowups at $p = (1{:}0{:}0{:}0{:}0)$ with weights $(0, 2, 1, 3, 1)$ and at $p' = (0{:}0{:}0{:}1{:}0)$ with weights $(3, 1, 2, 0, 1)$, and denote by $E$ and $E'$ the corresponding exceptional divisors. Note that $\widetilde{X}$ is terminal and $\mathbb{Q}$-factorial by [11, Theorem 3.5] and has no worse than cyclic quotient singularities. The morphism

$$(\widetilde{X}, D + E + E') \xrightarrow{f} (X, D)$$

is volume preserving and the intersection of $D$ with each exceptional divisor is a smooth elliptic curve $C_6 \subset \mathbb{P}(1, 1, 2, 3)$ not passing through the singular points of $E$ and $E'$; $f$ is a good (t,dlt) modification.

The dual complex $\mathcal{D}(X, D_X)$ is 1-dimensional, it has 3 vertices and 2 edges; $(X, D_X)$ does not have maximal intersection. The quartic surface $D_X$ is an example of case 3.(a) in Theorem 3.

**Corollary 2** *Let $(X, D_X)$ be a (t,lc) quartic CY pair. Assume that $D_X$ is normal, has non-canonical singularities but is not a cone. Then $(X, D_X)$ has maximal intersection except in cases 3.(a) and (b) of Theorem 3.*

*Remark 4* When $\dim \mathcal{D}(X, D_X) = 1$, $D_X$ either has two $T_{2,3,6}$ or two $T_{2,4,4}$ singularities. Indeed, as is explained in Section 2.2, singular points $p \in D$ are Kodaira singularities, and in particular are at worst bimodal. The description of cases 3.(a) and (b) of Theorem 3 immediately implies the result, because a surface singularity of type $E_{4,0}$ is trimodal.

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