Part III- Hodge Theory Lecture Notes

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These lecture notes will aim at presenting and explaining some special structures that exist on the cohomology of Kähler manifolds and to discuss some of the properties and consequences of these structures from the point of view of complex algebraic geometry. We will concentrate on nonsingular projective varieties, a special case of compact Kähler manifolds.

Overview

A *complex manifold* is a topological space that is locally modelled on open polydiscs of $\mathbb{C}^n$ and equipped with holomorphic transition functions. The natural isomorphism $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ endows X with the structure of a *differentiable manifold* of dimension $\dim_{\mathbb{R}} X = 2\dim_{\mathbb{C}} X$. These two structures on the underlying topological space turn out to behave quite differenttly in general; this reflects the fact that holomorphic function theory is in some sense much more "rigid" than differentiable function theory. If X is a complex manifold, the *tangent bundle* of the associated differentiable manifold $T_{X,\mathbb{R}}$ can be complexified $T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes \mathbb{C}$ and equipped with a *Hermitian metric* $h = g - i\omega$. This metric is inherited from the identification of $T_{x,X,\mathbb{C}}$ with a complex vector space at each point $x \in X$, and it varies smoothly with x. The complex manifold X is *Kähler* when the metric ω is closed for the exterior differential; this condition ensures compatibility between the complex structure of X and its differentiable structure. Nonsingular projective varieties are Kähler, and in fact Kähler manifolds can be thought of as a differential geomeric generalisation of these. However, this analogy should be taken with a pinch of salt; for instance, a Kähler manifold does not in general have any complex submanifold. Note that it is in general difficult to decide whether a given complex manifold is Kähler, or even to construct non projective Kähler manifolds.

Let X be a complex manifold and $T_{X,\mathbb{R}}$ its tangent space (when viewed as a differentiable manifold). At every point $x \in X$, $T_{x,X,\mathbb{R}}$ is equipped with a *complex structure* J_x, i.e. an endomorphism of $T_{x,X,\mathbb{R}}$ such that $J_x^2 = -id$, which can be extended to the complexified tangent space $T_{x,X,\mathbb{R}} \otimes \mathbb{C} = T_{x,X,\mathbb{C}}$, and induce a decomposition:

$$T_{x,X,\mathbb{R}} \otimes \mathbb{C} = T^{1,0}_{x,X} \oplus T^{0,1}_{x,X}$$

where $T^{1,0}_{x,X} = \{u \in T_{x,X,\mathbb{C}} | J_x u = iu\}$ and $T^{0,1}_{x,X} = \{u \in T_{x,X,\mathbb{C}} | J_x u = -iu\}$. This direct sum decomposition holds at the bundle level, and dualises to a similar decomposition on the *cotangent bundle*, the vector bundle of differential 1-forms $\Omega^1_{X,\mathbb{C}} = (T_{X,\mathbb{C}})^*$, and by extension, on the bundle of differtial k-forms Ω^k_X, namely:

$$\Omega^k_{X,\mathbb{C}} = \Omega^k_X \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}_X,$$

where we define $\Omega^{p,q}_X = \bigwedge^p \Omega^{1,0}_X \otimes \bigwedge^q \Omega^{0,1}_X$. This decomposition satisfies the *Hodge symmetry*, i.e. $\Omega^{p,q}_X = \overline{\Omega^{q,p}_X}$, where complex conjugation acts naturally on $\Omega^k_{X,\mathbb{C}} = \Omega^k_{X,\mathbb{R}} \otimes \mathbb{C}$.

Let $C^\infty(X, \Omega^k_{X,\mathbb{C}})$ denote the space of complex differential forms of degree of degree k on X– C^∞-sections of $\Omega^k_{X,\mathbb{C}}$– and denote $d\colon C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}})$ the exterior differential, with $d \circ d = 0$. Recall that the kth *de Rham cohomology group* of X are then defined as:

$$H^k(X,\mathbb{C}) = \frac{\ker(d\colon C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}})}{\mathrm{im}(d\colon C^\infty(X, \Omega^{k-1}_{X,\mathbb{C}}) \to C^\infty(X, \Omega^k_{X,\mathbb{C}})}.$$

Theorem 0.1 (Hodge Decomposition)**.** *Let X be a compact Kähler manifold. If $H^{p,q}(X) \subset H^k(X,\mathbb{C})$ is the set of De Rham cohomology classes that can be represented by a closed form α of type (p,q) at every point $x \in X$, then we have a decomposition:*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

and the summands satisfy Hodge symmetry $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

Remark 0.2. The Hodge decomposition theorem states that on a Kähler manifold, the decomposition of degree k differential forms into forms of type (p,q) with $p+q=k$ descends to the De Rham cohomology.

Remark 0.3. Note that in the statement of the Hodge Decomposition, we have written the $=$ sign between various cohomology rather than $\simeq$; this is because the isomorphisms in question will be shown to be canonical.

Remark 0.4. Even though the Kähler condition is crucial in order to prove Theorem 0.1, we will see that the Decomposition itself does not depend on the choice of Kähler metric, it only depends on the complex structure.

The principle behind the proof of the Hodge Decomposition Theorem is that each De Rham cohomology class has a unique representative that is a *harmonic form* for an elliptic differential operator, the Laplacian Δ_d. The Kähler hypothesis is crucial in the proof of this identification and also provides a decomposition of harmonic forms into forms of type (p, q).

Another consequence of this principle of representation of cohomology classes by harmonic forms is the Lefschetz Decomposition. As has been mentioned above, if X is a complex manifold of dimension n, X can be endowed with a Hermitian form that arises from its differentiable structure $h = g - i\omega$. When X is Kähler, the form ω is a representative of a (cohomology) class $[\omega]$ in $H^2(X, \mathbb{R})$. The exterior product with the class $[\omega]$ defines an operator $L\colon H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R})$.

Theorem 0.5 (Hard Lefschetz Theorem)**.** *Let X be a compact Kähler manifold. For every $k \leq n = \dim_{\mathbb{C}} X$, the map*

$$L^{n-k}\colon H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism. In particular, $L\colon H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R})$ is injective for $k < n$.

The *primitive cohomology* is then

$$H^k(X, \mathbb{R})_{prim} = \ker(L^{n-k+1}\colon H^k(X, \mathbb{R}) \to H^{2n-k+2}(X, \mathbb{R})).$$

Theorem 0.6 (Lefschetz Decomposition)**.** *Let X be a compact Kähler manifold. The natural map*

$$\begin{aligned} i\colon \bigoplus_{k-2r\geq 0} H^{k-2r}(X, \mathbb{R})_{\text{prim}} &\to H^k(X, \mathbb{R}) \\ (\alpha_r) &\to \Sigma_r L^r \alpha_r \end{aligned}$$

is an isomorphism for $k \leq n$

The existence of these decompositions on the De Rham cohomology groups of a compact Kähler manifold has important consequences. For instance, define the *Betti numbers* by $b_k(X) = \dim_{\mathbb{C}} H^k(X, \mathbb{C})$ and the *Hodge numbers* by $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$. By Theorem 0.5, the odd (resp even)

Betti numbers $b_{2k-1}(X)$ (resp. $b_{2k}(X)$) increase with k for $2k-1 \leq n$ (resp. $2k \leq n$). Theorem 0.1 shows that

$$b_k(X) = \Sigma_{p+q=k} h^{p,q}(X),$$

and by Theorems 0.1 and 0.6, we have:

$$h^{p,q}(X) = h^{q,p}(X) = h^{n-p,n-q}(X) = h^{n-q,n-p}(X),$$

so that, in particular, the odd Betti numbers are even.

The Hodge Decomposition has further reaching consequences when it is combined with the integral structure on the cohomology $H^k(X, \mathbb{Z})$. It would be nonsensical to consider the De Rham complex (i.e. degree k differential forms) to compute $H^k(X, \mathbb{Z})$- we will use the methods of *sheaf cohomology* to compute $H^k(X, \mathbb{Z})$. Using the language of sheaves, we will identify the summand $H^{p,q}(X)$ in Theorem 0.1 as the *Dolbeault cohomology groups* $H^q(X, \Omega^p_X)$–the q-th cohomology groups of X with values in the sheaf of holomorphic differential forms of degree p (this identification is possible because of the Kähler hypothesis).

We will formalize these results on the cohomology of compact Kähler manifolds by introducing the notion of *Hodge Structures.* An *integral Hodge structure of weight* k is an abelian group of finite type $H_{\mathbb{Z}}$ and a Hodge decomposition

$$H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q},$$

with $H^{p,q} = \overline{H^{q,p}}$. We have seen that a Hodge structure exists on the degree k cohomology of a Kähler manifold.

Here is an example of application of these Hodge Structures. If X is a nonsingular projective variety, $(H^1(X, \mathbb{Z}), H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X))$ is a weight 1 Hodge structure. To this Hodge structure, we may associate a complex torus $T = \mathbb{C}^k/\Gamma = H^{0,1}(X)^*/H^1(X, \mathbb{Z})$, where $k = b_1(X)$ and Γ is a lattice of rank $2k$. This complex torus is the *Picard variety* of X and we will see that it parametrises the holomorphic line bundles L on X which have trivial Chern class $c_1(L)$.

As I mentioned in Remark 0.4, the Hodge structure only depends on the complex structure and not on the differentiable structure, that is on the choice of a Kähler metric. We can ask how these Hodge Structures vary with the complex structure. These questions amount to studying varying decompositions on a fixed vector space. Indeed, the De Rham cohomology groups are invariant under diffeomorphism (differentiable isomorphism), they are even topological invariants. In particular, when the complex structure varies and the differentiable structure is "fixed", these groups will not change, but the decomposition into direct summands will. We will show how to construct a *period domain* $\mathcal{D}$ that "parametrises" deformations of Hodge Structures when the complex structure varies (that is when $H_{\mathbb{Z}} = \Gamma_0$ and $H_{\mathbb{C}} = \Gamma_0 \otimes \mathbb{C}$ are fixed, but the decomposition on $H_{\mathbb{C}}$ varies).

In fact, we will see that these questions are related to the study of "families" of compact complex manifolds. If $\mathcal{X} \to B$ is a "family of compact complex manifolds" over a contractible base B, then the fibres $\mathcal{X}_t$, $t \in B$ are all diffeomorphic to the central fibre $\mathcal{X}_0$. We will see that in some cases, there is a "universal" family of deformations $\mathcal{X} \to B$ of the central fibre $\mathcal{X}_0$. The *period map* $\mathcal{P} : B \to \mathcal{D}$ associates to $t \in B$ the Hodge structure on $H^k(X_t, \mathbb{C}) \simeq H^k(X_0, \mathbb{C})$. A fundamental result due to Griffiths is:

Theorem 0.7. *The period map is holomorphic.*

In fact, in nice cases, this period map is even an embedding or a submersion, and understanding how the Hodge Structure varies locally gives much information about the deformations of the manifold itself.

1 Complex manifolds

In these notes, I will assume some familiarity with basic notions of differential and algebraic geometry. Let $U \subset \mathbb{C}^n$ be an open subset and $f : U \to \mathbb{C}$ a complex valued function. The function f is *differentiable* if after some identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ and $\mathbb{C} \simeq \mathbb{R}^2$, the induced function $f : \mathbb{R}^{2n} \to \mathbb{R}^2$ is differentiable. This is independent of the choice of identification. In these notes, I always use the term *differentiable* for C^∞, but most of the statements will hold under weaker assumptions.

1.1 Local Theory: holomorphic functions of several variables

The reader who is not familiar with these notions should read [Voi02, Ch.1] or [Huy05, Section I.1].

Fix a standard system of coordinates $(z_1, \cdots, z_n)$ on $U \subset \mathbb{C}^n$, and let $x_j = \Re z_j$ and $y_j = \Im z_j$ be the canonical linear coordinates of $\mathbb{R}^{2n}$.

Definition 1.1. Let $U \subset \mathbb{C}^n$ be an open set and $f : U \to \mathbb{C}$ a differentiable function. The function f is *holomorphic* at $\omega = (\omega_1, \cdots, \omega_n) \in U$ if for all $j = 1, \cdots, n$, the function

$$z_j \mapsto f(\omega_1, \cdots, \omega_{j-1}, z_j, \omega_{j+1}, \cdots, \omega_n)$$

is holomorphic at ω_j, that is if

$$\frac{\partial f}{\partial \overline{z_j}}(\omega) := \frac{1}{2}(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j})(\omega) = 0.$$

The function f is holomorphic on U if f is holomorphic at ω for all $\omega \in U$.

When $f\colon U \to \mathbb{C}^m$ is a differentiable function, f is holomorphic if each $f_i\colon U \to \mathbb{C}$ is holomorphic, where $f = (f_1, \cdots, f_m)$.

Lemma 1.2. *If $f\colon U \to \mathbb{C}$ is holomorphic and does not vanish on U, $1/f$ is holomorphic on U. If f, g are holomorphic maps $U \to \mathbb{C}$, $fg, f+g$ and $g \circ f$ (when it is defined) are holomorphic on U.*

Proof. □

Exercise 1.3. Show that f is holomorphic at $\omega \in U$ precisely when the $\mathbb{R}$-linear application $df_\omega\colon \mathbb{C}^n \to \mathbb{C}$ is $\mathbb{C}$-linear.

Let $u = \Re f\colon U \to \mathbb{R}$ and $v = \Im f\colon U \to \mathbb{R}$ be the real and imaginary parts of f. Show that f is holomorphic on U if and only if for all $j = 1, \cdots, n$, u and v satisfy the Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j} \text{ and } \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}.$$

Definition 1.4. Let $\omega = (\omega_1, \cdots, \omega_n) \in U$ be a point and $R = (R_1, \cdots, R_n) \in (\mathbb{R}_+^*)^n$. The *polydisc* around ω with multiradius R is:

$$D(\omega, R) = \{(z_1, \cdots, z_n) \in \mathbb{C}^n | |z_j - \omega_j| < R_j, j = 1, \cdots, n\}.$$

Theorem 1.5. *[Voi02, 1.17] Let $U \subset \mathbb{C}^n$ be an open subset and $f\colon U \to \mathbb{C}$ be a differentiable map. The following are equivalent:*

1. *f is holomorphic at ω for all $\omega \in U$,*
2. *For all $\omega \in U$ there is a polydisc $D(\omega, R) \subset U$ such that f admits a power series expansion $f(z + \omega) = \Sigma_I \alpha_I z^I$ for multi-indices $I = (i_1, \cdots, i_n) \in \mathbb{N}^n$ that converges absolutely for $\omega + z \in D$.*
3. *If $D = D(\omega, R)$ is a polydisc contained in U, for all $z \in D$,*

$$f(z) = (\frac{1}{2i\pi})^n \int_{|\zeta_j - \omega_j| = R_j} f(\zeta)\frac{d\zeta_1}{\zeta_1 - \omega_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - \omega_n},$$

where the integral is taken over a product of circles, with the orientation that is the product of the natural orientations.

Definition 1.6. Let $U \subset \mathbb{C}^n$ be an open set. A holomorphic map $f\colon U \to \mathbb{C}^n$ is locally *biholomorphic* at $\omega \in U$ if there is a neighbourhood V of ω with $V \subset U$ such that $f_{|V}$ is bijective onto $f(V)$ and $f_{|V}^{-1}$ is holomorphic.

Definition 1.7. Let $U \subset \mathbb{C}^n$ be an open subset, and $f\colon U \to \mathbb{C}^m$ be a holomorphic map. The (complex) *Jacobian* of f at $\omega \in U$ is the matrix

$$J_f(\omega) = \big(\frac{\partial f_k}{\partial z_j}(\omega)\big)_{1\le k\le m, 1\le j\le n}.$$

The point $\omega \in U$ is *regular* if $J_f(\omega)$ is surjective, $f(\omega)$ is a *regular value* if every point $z \in \{f^{-1}(f(\omega))\}$ is regular.

Exercise 1.8. Let $U \subset \mathbb{C}^n$ be an open subset and $f\colon U \to \mathbb{C}^n$ a holomorphic map. Show that f is locally biholomorphic at $\omega \in U$ if and only if $\det J_f(\omega) \neq 0$.

Finally, the following result can be extracted from [Huy05, 1.10,1.11], and shows that a holomorphic map whose Jacobian matrix has locally constant rank locally has a canonical representation.

Theorem 1.9. *Let $U \subset \mathbb{C}^n$ be an open subset and $f\colon U \to \mathbb{C}^m$ a holomorphic map. Let $\omega \in U$ be a point such that $\operatorname{rk} J_f(z) = k$ for all z in a neighbourhood of ω. There are open neighbourhoods V of $\omega \in U$ and W of $f(\omega)$ in $\mathbb{C}^m$, and biholomorphic maps $\varphi\colon D^n \to V$ and $\psi\colon W \to D^m$ such that the composition $\psi \circ f \circ \varphi\colon D^n \to D^m$ is given by*

$$(z_1, \cdots, z_n) \mapsto (z_1, \cdots, z_k, 0, \cdots, 0).$$

1.2 Complex manifolds: definitions and first examples

Definition 1.10. A *complex manifold* of dimension n is a connected Hausdorff topological space X equipped with a *complex atlas* $\{(U_i), \phi_i\}$, where $(U_i)_{i\in I}$ is a countable covering by open subsets, and each $\phi_i\colon U_i \to V_i \subset \mathbb{C}^n$ is a homeomorphism from U_i onto an open subset of $\mathbb{C}^n$, and for all $i, j \in I$, the transition functions

$$\phi_j \circ \phi_i^{-1}\colon \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

are biholomorphic.

Two complex atlases are equivalent if their union defines a complex atlas. There is a *maximal atlas* equivalent to any given complex atlas on X, we say that such a maximal atlas is a *complex structure* on X.

A complex manifold X is compact if its underlying topological space is compact.

Recall that a *smooth manifold* M of dimension m is a topological space M and a maximal *differentiable atlas* $\{(W_i)_{i\in I}, \psi\}_{i\in I}$, where $(U_i)_{i\in I}$ is a countable covering by open subsets, and each $\psi_i \colon W_i \to \mathbb{R}^m$ is a homeomorphism from W_i onto an open subset of $\mathbb{R}^m$, and for all $i, j \in I$, the transition functions

$$\psi_j \circ \psi_i^{-1} \colon \psi_i(W_i \cap W_j) \to \psi_j(W_i \cap W_j)$$

are diffeomorphic.

In particular, every complex manifold of dimension n has a natural structure of smooth manifold of dimension $2n$.

Remark 1.11. Even though the definitions of complex and smooth manifolds are very similar, they have crucial differences. For instance, while a differentiable manifold M^m can always be covered by open subsets that are diffeomorphic to $\mathbb{R}^m$, a complex manifold X^n cannot in general be covered by open subsets that are biholomorphic to $\mathbb{C}^n$. For instance, take X to be the unit disc $D \subset \mathbb{C}$, then Liouville's theorem shows that there is no non-constant holomorphic map $\mathbb{C} \to X$.

Examples

1. Let $U \subset \mathbb{C}^n$ be an open subset. Then U is a complex manifold, with complex atlas $\{U, id\}$. More generally, if $U \subset X$ is a connected open subset of a complex manifold X, then it has a complex structure induced by that of X.

2. Let V be a complex vector space of dimension $n+1$. Let $\mathbb{P}(V)$ denote the set of lines through $\{0\} \in V$, i.e.

   $$\mathbb{P}(V) = \{l \subset V | l \text{ is a subspace of dimension } 1\} = \mathrm{Gr}(1, V),$$

   then $\mathbb{P}(V)$ is a complex manifold of dimension n (when $V = \mathbb{C}^{n+1}$, $\mathbb{P}(V) = \mathbb{CP}^n$). For every point $v \in V \setminus \{0\}$, denote $[v] = \mathbb{C} \cdot v \subset V$ the corresponding point of $\mathbb{P}(V)$, conversely, for every point $l \in \mathbb{P}(V)$, there is an element $v \in V \setminus \{0\}$ that is unique up to multiplication by a constant $\lambda \in \mathbb{C}^*$ such that $[v] = l$. We have a surjective map:

   $$\pi \colon V \setminus \{0\} \to \mathbb{P}(V),$$

   which endows $\mathbb{P}(V)$ with the quotient topology defined by π and the standard topology on V. We endow $\mathbb{P}(V)$ with a standard complex structure as follows. Choose a $\mathbb{C}$-linear isomorphism $V \simeq \mathbb{C}^{n+1}$, and for every point $v = (v_0, \cdots, v_n)$, denote $[v] = [v_0{:}\cdots{:}v_n]$ the homogeneous coordinates of $[v] \in \mathbb{P}(V)$. For each $i = 0, \cdots, n$ define the open set $U_i = \{[v] = [v_0{:}\cdots{:}v_n] \in \mathbb{P}(V) | v_i \neq 0\} \subset \mathbb{P}(V)$ and the homeomorphism $\phi_i \colon U_i \to \mathbb{C}^n$

$$[v_0{:}\cdots{:}v_n] \mapsto (\frac{v_0}{v_i}, \cdots, \widehat{\frac{v_i}{v_i}}, \cdots, \frac{v_n}{v_i}).$$

The transition functions are

$$\phi_j \circ \phi_i^{-1} \colon \{(z_1, \cdots, z_n) \in \mathbb{C}^n | z_j \neq 0\} \to \{(u_1, \cdots, u_n) \in \mathbb{C}^n | u_i \neq 0\}$$
$$(z_1, \cdots, z_n) \mapsto (\frac{z_1}{z_j}, \cdots, \widehat{\frac{z_j}{z_j}}, \cdots, \frac{z_{i-1}}{z_j}, \frac{1}{z_j}, \cdots, \frac{z_n}{z_j}),$$

these are biholomorphic.

Remark 1.12. As a differentiable manifold, $\mathbb{CP}^n \simeq S^{2n+1}/S^1$ (see example sheet 1), in particular $\mathbb{CP}^n$ is compact.

3. Let $\Lambda \subset \mathbb{C}^n$ be a lattice of rank $2n$. Denote $\pi \colon \mathbb{C}^n \to \mathbb{C}^n/\Lambda$ the quotient map and $X = \mathbb{C}^n/\Lambda$ the quotient. Then X is a complex manifold. Endow X with the quotient topology of $\mathbb{C}^n$. If $U \subset \mathbb{C}^n$ is a small open subset such that $U \cap (U + (\Lambda \setminus 0)) = \emptyset$, then $U \to \pi(U)$ is bijective. Covering X by such open subsets gives a complex atlas, whose transition functions are just translations by elements in Λ.

   More generally, let X be a complex manifold and $\Gamma \subset \operatorname{Aut} X$ a subgroup of the group of automorphisms of X that acts properly discontinuously on X, i.e for any two compact subsets $K_1, K_2 \subset X$, $\gamma(K_1) \cap K_2 \neq \emptyset$ for at most finitely $\gamma \in \Gamma$. Assume further that Γ acts without fixed point–i.e. $\gamma \cdot x \neq x$ for all $x \in X$ and $1 \neq \gamma \in \Gamma$–then X is a complex manifold and $X \to X/\Gamma$ is a locally biholomorphic map.

4. (Affine hypersurfaces) Let $f \colon \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function such that 0 is a regular value. Consider

$$X = f^{-1}(0) = \{z \in \mathbb{C}^n | f(z) = 0\}.$$

   By Theorem 1.9, there is an open cover $X = \cup_i U_i$, open subsets $V_i \subset \mathbb{C}^{n-1}$ and holomorphic maps $V_i \to \mathbb{C}^n$ inducing bijective maps $\phi_i \colon U_i \to V_i$. The transition maps $\phi_j \circ \phi_i^{-1}$ are biholomorphic, and X is a complex manifold of dimension $n-1$.

5. (Projective Hypersurfaces) Let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ be a homogeneous polynomial, and assume that $0 \in \mathbb{C}$ is a regular value for the induced holomorphic map $f\colon \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{C}$. Then, the affine hypersurface $f^{-1}(0)$ is a complex manifold. The projective hypersurface

$$X = \{[z_0{:}\cdots{:}z_n] \in \mathbb{CP}^n | f(z_0, \cdots, z_n) = 0\} \subset \mathbb{CP}^n$$

is a complex manifold of dimension $n-1$. The open subsets $U_i \cap X$ form an open cover of X, where U_i are the standard charts of $\mathbb{CP}^n$ defined above. Using the isomorphism $U_i \simeq \mathbb{C}^n$ above, $X \cap U_i$ is identified with $f_i^{-1}(0)$, for $f_i(z_1, \cdots, z_n) = f(z_1, \cdots, z_{i-1}, 1, z_{i+1}, \cdots z_n)$ and by the previous example, we can find a complex atlas.

Definition 1.13. Let X be a complex manifold of dimension n, equipped with a complex atlas $\{U_i, \phi_i\colon U_i \to \mathbb{C}^n\}_{i\in I}$ and Y a complex manifold of dimension m equipped with a complex atlas $\{W_j, \psi_j\colon W_j \to \mathbb{C}^m\}_{j\in J}$. A holomorphic map $f\colon X \to Y$ is a continuous map such that for all $(i,j) \in I \times J$,

$$\psi_j \circ f \circ \phi_i^{-1}\colon \mathbb{C}^n \to \mathbb{C}^m$$

is holomorphic. If $Y = \mathbb{C}$, we say that f is a holomorphic function on X. Two complex manifolds X and Y are *biholomorphic* if there exists a holomorphic homeomorphism $f\colon X \to Y$.

Definition 1.14. Let X be a complex manifold, define the *structure sheaf* $\mathcal{O}_X$ as the (pre)sheaf (see Definition 1.50, and the discussion there):

$$\mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_X) = \{f\colon U \to \mathbb{C} | f \text{ is holomorphic}\},$$

where U is an open subset of X. The presheaf $\mathcal{O}_X$ is a sheaf of rings (the restriction morphisms are ring morphisms). If $x \in X$, define

$$\mathcal{O}_{X,x} = \lim_{x\in U} \mathcal{O}_X(U),$$

where the limit is taken over all open subsets that contain $x \in X$.

Remark 1.15. Let X be a complex manifold and $(U, \phi\colon U \to \mathbb{C}^n)$ be a holomorphic chart. By definition, if $x \in U$ is such that $\phi(x) = 0 \in \mathbb{C}^n$, there is a natural identification $\mathcal{O}_{\mathbb{C}^n,0} \simeq \mathcal{O}_{X,x}$.

Remark 1.16. When X is a differentiable manifold, we define the sheaf of differentiable functions $\mathcal{A}_X = C^\infty(X)$ on X as the (pre)sheaf $U \mapsto \mathcal{A}_X(U) = \{f\colon U \to \mathbb{R} | f \text{ is differentiable}\}$.

Proposition 1.17. *Let X be a compact connected complex manifold, then $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$, i.e. every global holomorphic function is constant.*

Proof. Let $f : X \to \mathbb{C}$ be a holomorphic function. Since X is compact and f is continuous, f admits a maximum at a point $x \in X$. If (U, ϕ) is a holomorphic chart with $x \in U$, $f \circ \phi^{-1}$ is locally constant by the maximum principle, and hence constant because X is connected. □

Remark 1.18. Using Hartog's Theorem (cf Example Sheet 1), if X is a complex manifold of dimension at least 2, $\Gamma(X \setminus \{x\}, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$ for all $x \in X$, so that if X is compact and connected $\Gamma(X \setminus \{x\}, \mathcal{O}_X) = \mathbb{C}$.

Remark 1.19. In a sense, complex manifolds define a much more "rigid" structure than smooth manifolds; this reflects the properties of holomorphic functions *vs.* differentiable functions. We may give an equivalent definition of complex (resp. differentiable) manifolds that is closer to the spirit of algebraic geometry: a complex (resp. differentiable) manifold is a ringed space $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is the structure sheaf, whose sections we define to be the holomorphic (resp. differentiable) ones.

Definition 1.20. Let X and Y be complex manifolds. A holomorphic map $f : X \to Y$ is a *submersion* (resp. an *immersion*) if for all $x \in X$, there is a neighbourhood $U(x)$ of x such that $\operatorname{rk} J_f(z) = \dim Y$ (resp $\operatorname{rk} J_f(z) = \dim X$) for all $z \in U(x)$. The map f is an *embedding* if it is an immersion and if f is an homeomorphism from X onto $f(X)$.

Remark 1.21. The rank of the Jacobian matrix does not depend on the choice of coordinate charts.

Definition 1.22. Let X be a complex manifold of dimension n and $Y \subset X$ be a closed subset. The subset Y is a closed *submanifold* of X of codimension k if for all $x \in Y$, there is an open neighbourhood $U \subset X$ of x and a holomorphic submersion $f : U \to D^k$ such that $U \cap Y = f^{-1}(0)$.

Example 1.23. Let X and Y be complex manifolds of dimension n and m, $f : X \to Y$ be a holomorphic map and $y \in Y$ such that $\operatorname{rk} J_f(z) = m$ for all $z \in f^{-1}(y)$; $f^{-1}(y)$ is a submanifold of dimension $n - m$. If $f : X \to Y$ is an embedding, then $f(X)$ is a submanifold of Y.

Definition 1.24. A *projective manifold* is a submanifold $X \subset \mathbb{CP}^N$ such that there exist homogeneous polynomials $f_1, \cdots, f_k \in \mathbb{C}[X_o, \cdots, X_N]$ of degrees $d_1, \cdots, d_k$ with

$$X = \{x \in \mathbb{CP}^N | f_1(x) = \cdots = f_k(x) = 0\}. \quad (1)$$

Remark 1.25. Note that if the Jacobian matrix

$$J = (\frac{\partial f_j}{\partial z_l})_{1\leq j\leq k, 0, \leq l\leq N}$$

has rank k everywhere, X is a submanifold of $\mathbb{CP}^N$ of codimension k. More generally, (1) defines a submanifold of $\mathbb{CP}^N$ if the Jacobian matrix has maximal rank. If the Jacobian matrix does not have maximal rank everywhere, (1) defines a *projective algebraic variety* and the points where the Jacobian has less than maximal rank are the *singular points* of the variety.

Definition 1.26. A projective manifold $X \subset \mathbb{CP}^n$ of dimension m that is defined by $m-n$ homogeneous polynomials such that the Jacobian has rank $n-m$ in every point is a *complete intersection*.

Exercise 1.27. Show that

$$C = \{[x_0:\cdots:x_3] \in \mathbb{P}^3 | x_0x_3 - x_1x_2 = x_1^2 - x_0x_2 = x_2^2 - x_1x_3 = 0\}$$

is a submanifold of dimension 1. Is it a complete intersection?

Exercise 1.28. Let $f = (f_1, \cdots, f_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map, and $(z_1, \cdots, z_n)$ the standard coordinates on $\mathbb{C}^n$. Define $x_k = \Re z_k$, $y_k = \Im z_k$ and $u_j = \Re f_j$, $v_j = \Im f_j$. The (real) Jacobian of f at $a \in \mathbb{C}^n$ is

$$J_{\mathbb{R}}(f)(a) = (\frac{\partial(u_1, v_1, \cdots, u_n, v_n)}{\partial(x_1, y_1, \cdots, x_n, y_n)})(a).$$

Show that $\det J_{\mathbb{R}}(f)(a) = |\det J(f)(a)|^2$, and deduce that any complex manifold is orientable.

1.3 Vector bundles

We will want to distinguish between two different notions:

- complex vector bundles are differentiable vector bundles that have values in $\mathbb{C}$, i.e. they have differentiable transition functions,
- holomorphic vector bundles have holomorphic transition functions.

Definition 1.29. Let X be a differentiable manifold. A *complex vector bundle* of rank r over $\mathbb{C}$ is a differentiable manifold E endowed with a surjective map $\pi\colon E \to X$ such that:

1. For all $x \in X$, the fibre $E_x = \pi^{-1}(x) \simeq \mathbb{C}^r$ has the structure of a $\mathbb{C}$-vector space of dimension r,

2. There is an open cover $X = \cup_i U_i$ and *local trivialisations* (U_i, h_i), where $h_i \colon \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ is a diffeomorphism with $\pi_{|\pi^{-1}(U_i)} = p_1 \circ h_i$ and $p_2 \circ h_i \colon E_x \to \mathbb{C}^r$ a $\mathbb{C}$-vector space isomorphism for all $x \in U_i$.

The manifold E is the *total space* of the vector bundle and X is its *base space*. Given two local trivialisations (U_i, h_i) and (U_j, h_j) of E,

$$h_i \circ h_j^{-1} \colon (U_i \cap U_j) \times \mathbb{C}^r \to (U_i \cap U_j) \times \mathbb{C}^r$$

induces a differentiable map $g_{i,j} \colon U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})$, where $g_{i,j}(x) = h_i^x \circ (h_j^x)^{-1}$ is a $\mathbb{C}$-linear automorphism. The $g_{i,j}$ are the *transition functions* of E.

Exercise 1.30. Check that the transition functions satisfy the cocycle conditions:

$$g_{i,j} \circ g_{j,k} \circ g_{k,i} = \mathrm{Id} \text{ on } U_i \cap U_j \cap U_k, \text{ and } g_{i,i} = \mathrm{Id} \text{ on } U_i.$$

Remark 1.31. A complex vector bundle E of rank r is determined uniquely by the differentiable cocycle $\{U_i, g_{i,j} \colon U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})\}$. Let $\widetilde{E} = \sqcup_i U_i \times \mathbb{C}^r$ and define $(x, v) \sim (y, w)$ if $x = y \in U_i \cap U_j$, and $w = g_{i,j}(x) \cdot v$, then $E = \widetilde{E}/\sim$.

Example 1.32. Let X be a differentiable manifold of dimension m; if $\{U_i, \phi_i\}$ is a differentiable atlas of X, the real tangent bundle $T_{X,\mathbb{R}}$ is the vector bundle associated to the cocycle $\{U_i, g_{i,j} = J_{\mathbb{R}}(\phi_i \circ \phi_j^{-1}) \circ \phi_j\}$, where $J_{\mathbb{R}}$ is the (real) Jacobian. Via the inclusion $\mathrm{GL}(m, \mathbb{R}) \subset \mathrm{GL}(m, \mathbb{C})$, the transition functions $g_{i,j} \colon U_i \circ U_j \to \mathrm{GL}(m, \mathbb{C})$ define a complex vector bundle $T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes \mathbb{C}$, the *complexified tangent bundle* of X.

Definition 1.33. Let X be a complex manifold and $\pi \colon E \to X$ a complex vector bundle associated to a cocycle $\{U_i, g_{i,j} \colon U_i \circ U_j \to \mathrm{GL}(r, \mathbb{C})\}$. The vector bundle E is *holomorphic* if E is a complex manifold and if the transition functions $g_{i,j}$ are holomorphic.

Remark 1.34. A natural question is to ask in how many (non-isomorphic) ways a given complex vector bundle can be seen as a holomorphic vector bundle. This is in general a non-trivial question; In some cases, no holomorphic structure exists, but there can also be several different holomorphic structures. This is already the case for line bundles: on a complex torus $\mathbb{C}^n/\Gamma$, the trivial complex bundle of rank 1 admits many holomorphic structures.

The first example of holomorphic vector bundle is of course that of the trivial vector bundle $E = X \times \mathbb{C}$.

Example 1.35. (The holomorphic tangent bundle) Let X be a complex manifold of dimension n endowed with a complex atlas $\{U_i, \phi_i \colon U_i \to V_i \subset \mathbb{C}^n\}$. Then, T_X is the holomorphic bundle of rank n associated to the cocycle $\{U_i, g_{i,j} = J(\phi_i \circ \phi_j^{-1}) \circ \phi_j \colon (U_i \cap U_j) \to \mathrm{GL}(r, \mathbb{C})\}$, where J is the complex (holomorphic) Jacobian. In other words, T_X is the holomorphic vector bundle which is trivial over each U_i, and whose transition functions correspond to the Jacobian of the change of (holomorphic) coordinates from those defined by the chart ϕ_i to those defined by the chart ϕ_j.

Remark 1.36. Recall from the previous section the definition of the two sheaves of algebras $\mathcal{A}_X$ and $\mathcal{O}_X$; the stalks of these sheaves at a point $x \in X$, $\mathcal{A}_{X,x}$ and $\mathcal{O}_{X,x}$, are the $\mathbb{C}$-algebras of germs of differentiable and holomorphic functions respectively. A *derivation* of the $\mathbb{C}$-algebra $\mathcal{A}_{X,x}$ (resp. $\mathcal{O}_{X,x}$) is a $\mathbb{C}$-linear map $D \colon \mathcal{A}_{X,x} \to \mathbb{C}$ (resp. $\mathcal{O}_{X,x} \to \mathbb{C}$) that satisfies the Leibniz rule, i.e. for any $f, g \in \mathcal{A}_{X,x}$ (resp. $f, g \in \mathcal{O}_{X,x}$), $D(fg) = D(f)g(x) + f(x)D(g)$. The complexified tangent space $T_{X,x,\mathbb{C}}$ is the space of derivations of $\mathcal{A}_{X,x}$, while the holomorphic tangent space T_X is the space of derivations of $\mathcal{O}_{X,x}$.

Exercise 1.37. Let $\mathcal{O}_{\mathbb{C}^n,0}$ be the $\mathbb{C}$-algebra of germs of holomorphic functions at $\{0\} \in \mathbb{C}^n$. Let $z_1, \cdots, z_n$ be standard coordinates on $\mathbb{C}^n$ and $\frac{\partial}{\partial z_i} := \frac{\partial}{\partial z_i}_{|0}$ be defined by $\frac{\partial}{\partial z_i} \colon f \in \mathcal{O}_{\mathbb{C}^n,0} \mapsto \frac{\partial f}{\partial z_i}_{|0} \in \mathbb{C}$. Show that the $\frac{\partial}{\partial z_i}$ are complex derivations of $\mathcal{O}_{\mathbb{C}^n,0}$ and form a basis of $T_{\mathbb{C}^n,0}$ over $\mathbb{C}$. Deduce a basis of $T_{X,x}$ for any complex manifold X.

Example 1.38. The *tautological vector bundle* $U(r, V) \to \mathrm{Gr}(r, V)$ over the Grassmannian is defined as the vector bundle with total space

$$U_r(V) = \{([U], x) \in \mathrm{Gr}(r, V) \times V | x \in U\} \subset \mathrm{Gr}(r, V) \times V,$$

and $\pi : U_r(V) \to \mathrm{Gr}(r, V)$ the projection onto the first factor. Using the complex atlas of $\mathrm{Gr}(r, V)$ determined in Example Sheet 1, show that $U_r(V)$ is holomorphic.

Examples As in the differentiable situation, any canonical construction in linear algebra gives rise to a geometric version for holomorphic vector bundles. Assume that E and F are holomorphic vector bundles over a complex manifold X, we can construct in this way:

$E \oplus F$, the direct sum,

$E \otimes F$, the tensor product,

$\bigwedge^i E$, the ith exterior product,

E^*, the dual bundle $\mathcal{H}\mathrm{om}(E, \mathbb{C})$ (i.e. fibre wise $\mathbb{C}$-linear maps $E \to \mathbb{C}$),

$\det E$ the determinant line bundle.

In all cases, a common local trivialisation of E and F will yield transition functions for these constructions. For example, if E is associated to the cocycle $\{U_i, e_{i,j}\}$ and F to the cocycle $\{U_i, f_{i,j}\}$, $E \oplus F$ is the vector bundle of rank $\operatorname{rk} E + \operatorname{rk} F$ associated to the cocycle $\{U_i, g_{i,j} = \begin{pmatrix} e_{i,j} & 0 \\ 0 & f_{i,j} \end{pmatrix}\}$; E^* is the vector bundle of rank $\operatorname{rk} E$ associated to the cocycle $\{U_i, g_{i,j} =^t e_{i,j}\}$ (Check that these are cocycles...).

Definition 1.39. Let X be a complex manifold and T_X its holomorphic tangent bundle. The *cotangent bundle* of X is the dual of the tangent bundle $\Omega_X = T_X^*$, the bundle of holomorphic k-forms is $\Omega_X^p = \bigwedge^p \Omega_X$, the *canonical bundle* is $K_X = \det \Omega_X$, and its dual is the *anticanonical bundle* $K_X^* = \det T_X$.

Remark 1.40. The definitions of T_X, Ω_X and K_X are independent of the choice of complex structure on X, they are invariant of the manifold X.

Definition 1.41. Let $E_1 \stackrel{\pi_1}{\to} X$ and $E_2 \stackrel{\pi_2}{\to} X$ be two complex (resp. holomorphic) vector bundles of rank r over $\mathbb{C}$. Then E_1 and E_2 are *isomorphic* if there is a diffeomorphism (resp. biholomorphism) $\phi\colon E_1 \to E_2$ such that $\pi_1 = \pi_2 \circ \phi$.

Let X be a complex manifold and $\{U_i, g_{i,j} \colon U_i \circ U_j \to \mathbb{C}^*\}$ be a collection of holomorphic functions that satisfy the cocycle condition. This defines a holomorphic line bundle $L \to X$. The holomorphic line bundle $L \to X$ is trivial (i.e. isomorphic to $X \times \mathbb{C}$) precisely when, possibly after refining the cover $\{U_i\}_{i \in I}$, there exist holomorphic functions $s_i \colon U_i \to \mathbb{C}^*$ such that $g_{i,j} = \frac{s_i}{s_j}$ on $U_i \cap U_j$.

Definition 1.42. Let $f\colon Y \to X$ be a holomorphic map between complex manifolds and E a holomorphic vector bundle over X. If $\{U_i, g_{i,j}\}$ is a cocycle for E, define f^*E as the holomorphic vector bundle over Y associated to the cocycle $\{f^{-1}(U_i), g_{i,j}\}$. For all $y \in Y$, there is a canonical isomorphism $(f^*E)_y \simeq E_{f(y)}$.

In particular, if Y is a submanifold of X, the restriction of E to Y is $E_{|Y} = i^*E$, where i is the inclusion map $i\colon Y \to X$.

Definition 1.43. Let $\pi\colon E \to X$ be a complex (resp. holomorphic) vector bundle over a differentiable (resp. complex) manifold X. A *global section* of E is a differentiable (resp holomorphic) map $s\colon X \to E$ such that $\pi \circ s = \mathrm{Id}$.

Remark 1.44. While a complex vector bundle always has many sections (because we may use local bump functions), this is not necessarily the case for a holomorphic vector bundle.

Using fibrewise addition, the set of sections $C^\infty(X, E)$ (resp. $\Gamma(X, \mathcal{O}_X(E))$) of a complex (resp. holomorphic) vector bundle has a natural $\mathbb{C}$-vector space structure. If $\{U_i, g_{i,j}\}$ is a cocycle for E, over an open set U_i, $C^\infty(U_i, E)$ (resp. $\Gamma(U_i, E)$) has dimension rank r.

A basis $s_{i_1}, \cdots, s_{i_r}$ of $C^\infty(U_i, E)$ (resp $\Gamma(U_i, E)$) is a *local frame* (resp. *local holomorphic frame*) of E. The data of an open set $U_i \subset X$ and a local frame $s_{i_1}, \cdots, s_{i_r}$ is equivalent to a local trivialisation of E. If $C^\infty(X, E)$, $\Gamma(X, E)$ have dimension r, then a basis $s_1, \cdots, s_r$ is a global (holomorphic frame) of E, and E is trivial, i.e. isomorphic to $X \times \mathbb{C}^r$.

Example 1.45. Exercise 1.37 shows that for any complex chart $\{U, \varphi\}$ of a complex manifold X, if $z_1, \cdots, z_n$ are local holomorphic coordinates at $u \in U$, $\frac{\partial}{\partial z_1}_{|u}, \cdots, \frac{\partial}{\partial z_n}_{|u}$ is a local holomorphic frame of the holomorphic tangent bundle T_X.

Example 1.46. Recall the identification $\mathbb{CP}^n = \mathbb{P}^n = \mathrm{Gr}(1, n+1)$ and define the tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1} \to \mathbb{P}^n$ as the tautological line bundle defined on $\mathrm{Gr}(1, n+1)$ in Example 1.38. Define $\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^n}(-1)^*$ and $\mathcal{O}_{\mathbb{P}^n}(k) = \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes k}$ for $k \in \mathbb{Z}^*$. Set $\mathcal{O}_{\mathbb{P}^n}(0)$ for the trivial line bundle.

If $U_i = \{[l_0 \colon \cdots \colon l_n] \in \mathbb{P}^n | l_i \neq 0\}$, define $s_i\colon U_i \to \mathcal{O}_{\mathbb{P}^n}(-1)$ by:

$$[l_0 \colon \cdots \colon l_n] \mapsto ([l]; (\frac{l_0}{l_i}, \cdots, \frac{l_n}{l_i})).$$

The section s_i does not vanish anywhere on U_i; the associated local trivialisation is:

$$\begin{aligned} h_i\colon \pi^{-1}(U_i) &\to U_i \times \mathbb{C} \\ ([l], x) &\mapsto ([l], x_i) \end{aligned}$$

where x_i is the unique complex number such that $x = x_i s_i([l])$. The transition functions $g_{i,j}\colon (U_i \cap U_j) \to (U_i \cap U_j) \times \mathbb{C}^*$ are $g_{i,j}\colon [l] \mapsto ([l], \frac{l_j}{l_i})$ (this is well defined on $U_i \cap U_j$). The sections of $\mathcal{O}_{\mathbb{P}^n}(k)$ for $k \in \mathbb{N}$ are homogeneous polynomials of degree k in the variables $l_0, \cdots, l_n$. The transition functions $g_{i,j}$ associated to the line bundle $\mathcal{O}_{\mathbb{P}^n}(k)$ are of the form $g^k_{i,j} = \frac{s_i}{s_j}$ for s_i, s_j homogeneous polynomials of degree k.

Exercise 1.47. Describe the global sections $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ for $k \in \mathbb{Z}$.

Definition 1.48. Let $E \to X$ be a complex (resp holomorphic) vector bundle of rank R over a complex manifold X. A submanifold $F \subset E$ is a *subbundle* of rank m if:

1. For all $x \in X$, $F \cap E_x$ is a subvector space of dimension m,
2. $\pi_{|F} \colon F \to X$ has the structure of a complex (resp holomorphic) vector bundle induced by that of E.

In other words, F is a subbundle of E if E and F are represented by cocycles $\{U_i, e_{i,j}\}$ and $\{U_i, f_{i,j}\}$ such that:

$$e_{i,j} = \begin{pmatrix} f_{i,j} & * \\ 0 & g_{i,j} \end{pmatrix}$$

Examples

1. The tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ is a subbundle of the trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$.
2. Let Y be a submanifold of X, T_Y is a subbundle of the restricted tangent bundle $T_{X|Y}$.

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Definition 1.49. Let $\phi \colon E \to F$ be a vector bundle homomorphism. There are well defined holomorphic vector bundles $\ker \phi$ and $\operatorname{Coker} \phi$.

If E and $\ker \phi$ are represented by cocycles $\{U_i, e_{i,j}\}$ and $\{U_i, k_{i,j}\}$ with

$$e_{i,j} = \begin{pmatrix} k_{i,j} & * \\ 0 & g_{i,j} \end{pmatrix},$$

$\operatorname{Coker} \phi$ is associated to the cocycle $\{U_i, g_{i,j}\}$.

Sheaf theory was introduced as a unified way of dealing with problems concerned with the passage from local data to global data; as such it is clear that sheaves are useful to the study of (topological, differentiable, complex) manifolds and of (topological, complex, holomorphic) vector bundles over these. Recall the definitions of sheaves:

Definition 1.50. Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups (vector spaces, rings, algebras, etc..) on X consists of an abelian group (vector space, ring, algebra, etc..) $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ for every open set $U \subset X$, and a group homomorphism (linear map, ring homomorphism..) $r_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ associated to each pair of nested open sets $V \subset U$, satisfying the compatibility conditions:

1. $r_{U,U} = \text{Id}$ for any $U \subset X$,
2. $r_{U,W} = r_{U,V} \circ r_{V,W}$ for any $W \subset V \subset U$.

The presheaf $\mathcal{F}$ is a *sheaf* if it satisfies the further two conditions. Denote $U = \cup_{i \in I} U_i$ an open cover.

3. If $s, t \in \mathcal{F}(U)$ are such that $r_{U,U_i}(s) = r_{U,U_i}(t)$ for all $i \in I$, then $s = t$,
4. If $s_i \in \mathcal{F}(U_i)$ is a collection of objects such that for all $i, j \in I$, $r_{U_i, U_i \cap U_j}(s_i) = r_{U_j, U_i \cap U_j}(s_j)$, there is an element $s \in \mathcal{F}(U)$ such that $r_{U,U_i}(s) = s_i$ for all $i \in I$.

The presheaves $U \mapsto C^\infty(U, E)$ and $U \mapsto \Gamma(U, E)$ are sheaves of abelian groups on X, in both cases (complex and holomorphic vector bundles), we denote $\mathcal{E}$ the sheaf of sections of $E \to X$. If E is a complex (resp holomorphic) vector bundle, $\mathcal{E}$ is a sheaf of $\mathcal{A}_X$-modules (resp $\mathcal{O}_X$-modules).

Remark 1.51. If $\mathcal{R}$ is a sheaf of rings over X, $\mathcal{F}$ is a sheaf of $\mathcal{R}$-modules over X if for every open $U \subset X$, $\mathcal{F}(U)$ has an $\mathcal{R}(U)$-module structure compatible with its group structure. The restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are morphisms of $\mathcal{R}(U)$ modules, where $\mathcal{F}(V)$ is equipped with an $\mathcal{R}(U)$-module structure via the restriction $\mathcal{R}(U) \to \mathcal{R}(V)$.

Definition 1.52. Let X be a connected topological space. A sheaf $\mathcal{F}$ of $\mathcal{R}$-modules over X is *locally free* of rank r if $\mathcal{F}$ is locally isomorphic to $\mathcal{R}^{\oplus r}$ as a sheaf of $\mathcal{R}$-modules.

Proposition 1.53. *Let X be a complex manifold. The map $E \to \mathcal{E}$ is a bijection between the set of holomorphic vector bundles $E \to X$ of rank r and locally free sheaves of rank r over X.*

Proof. We have seen that if $E \to X$ is a holomorphic vector bundle, $\mathcal{E}$ is a sheaf of $\mathcal{O}_X$-modules. For any local trivialisation, $E_{|U_i} \simeq U_i \times \mathbb{C}^r$ and hence $\mathcal{E}_{|U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$ and $\mathcal{E}$ is locally free of rank r. Conversely, let $X = \cup U_i$ be an open covering over which $\mathcal{E}_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$ is an isomorphism of sheaves. On each $U_i \cap U_j$, $\mathcal{O}^{\oplus r}_{U_i \cap U_j} \simeq \mathcal{E}_{U_i \cap U_j} \simeq \mathcal{O}^{\oplus r}_{U_i \cap U_j}$ is an isomorphism of sheaves that corresponds to an invertible $r \times r$ matrix M_{U_i, U_j} with holomorphic entries, i.e. to a holomorphic map $g_{i,j} \colon U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})$. The sheaf axioms for $\mathcal{E}$ ensure that the maps $g_{i,j}$ are cocycles. Define E as the vector bundle associated to the cocycle $\{U_i, g_{i,j}\}$ □

Remark 1.54. Beware that when working with this bijection between the category of locally free sheaves and that of holomorphic vector bundles, morphisms of vector bundles are required to have constant rank, but morphisms of sheaves are not. This bijection in fact defines an equivalence of categories between the category of holomorphic vector bundles over X and the category of *free* sheaves of $\mathcal{O}_X$-modules (this is one way to fix the rank of morphisms..).

Remark 1.55. An analogous statement holds for complex vector bundles and locally free sheaves of $\mathcal{A}_X$-modules.

1.4 The complexified tangent bundle

We now start to examine systematically the relationship between the complex and differentiable structures on a complex manifold X of dimension n.

Let V be an $\mathbb{R}$-vector space of dimension m, an *almost complex structure* on V is an endomorphism $I \in \mathrm{End}(V)$ such that $I^2 = -\mathrm{Id}$. An almost complex structure endows V with a structure of $\mathbb{C}$-vector space via:

$$(a + ib) \cdot v = a \cdot v + b \cdot I(v), \text{ for } v \in V, a \text{ and } b \in \mathbb{R}.$$

The dimension m has to be even, $m = 2n$. The $\mathbb{C}$-vector space is denoted $V_{\mathbb{C}} = V \otimes \mathbb{C}$. Define a complex conjugation on $V_{\mathbb{C}}$ induced by I by: $\overline{v \otimes \alpha} = v \otimes \overline{\alpha}$, where $v \in V$ and $\alpha \in \mathbb{C}$.

Conversely, a $\mathbb{C}$-vector space W of dimension n is an $\mathbb{R}$-vector space of dimension $2n$, endowed with an almost complex structure I given by multiplication by i.

If $z_1, \cdots, z_n$ are coordinates on $\mathbb{C}^n$, denote $x_i = \Re z_i$ and $y_i = \Im z_i$ the associated coordinates of $\mathbb{R}^{2n}$. The standard complex structure on $\mathbb{C}^n$ is the endomorphism of $\mathbb{R}^{2n}$ given by the matrix whose only nonzero entries are n blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal.

Let (V, I) be an almost complex structure on a $\mathbb{R}$-vector space V of dimension $2n$; I extends to an endomorphism of $V_{\mathbb{C}}$ by $I(v \otimes \lambda) = I(v) \otimes$

λ, and still satisfies $J^2 = -\operatorname{Id}$. The $\mathbb{C}$-vector space endomorpshism I is diagonalisable, and this gives a direct sum decomposition:

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1},$$

where $V^{1,0}$ is the eigenspace associated to the eigenvalue i, and $V^{1,0}$ to $-i$. Define a *conjugation* on $V_{\mathbb{C}}$ by $\overline{v \otimes \alpha} = v \otimes \alpha$, we have $V^{0,1} = \overline{V^{1,0}}$.

Definition 1.56. Let X be a differentiable manifold of (real) dimension m. An *almost complex structure* on X is a differentiable vector bundle isomorphism $J\colon T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ such that $J^2 = -\operatorname{Id}$.

Remark 1.57. In general, there does not exist an almost complex structure on a differentiable manifold, even when its dimension is even. In fact, any even dimensional vector space admits a linear complex structure. As a consequence, for an even dimensional differentiable manifold X, we may always define a linear transformation $I_p\colon T_{X,P,\mathbb{R}} \to T_{X,P,\mathbb{R}}$ such that $I_P^2 = -\operatorname{Id}$. The existence of an almost complex structure on X is equivalent to determining whether this local construction can be patched up to a vector bundle diffeomorphism (it is then uniquely determined by its action on each fibre). This becomes a question of reduction of the structure group of the tangent bundle from $\mathrm{GL}_{2n}(\mathbb{R})$ to $\mathrm{GL}_n(\mathbb{C})$ and is a purely algebraic topological question.The sphere S^4 is an example of a differentiable manifold which admits no almost complex structure.

Proposition 1.58. *Let X be a complex manifold, then X induces an almost complex structure on its underlying differentiable manifold.*

Proof. If $\{U_i, \phi_i\colon U_i \to \mathbb{C}^n\}$ is a complex atlas of X, define a differentiable atlas on the underlying real manifold by $\{U_i, (u_i, v_i)\colon U_i \to \mathbb{R}^{2n}\}$, where $u_i = \Re\phi_i$ and $v_i = \Im\phi_i$. The real tangent bundle $T_{X,\mathbb{R}}$ is trivial over U_i, and $\{\frac{\partial}{\partial u_{i_1}}, \frac{\partial}{\partial v_{i_1}}, \cdots, \frac{\partial}{\partial u_{i_n}}, \frac{\partial}{\partial v_{i_n}}\}$ is a local frame. The holomorphic tangent bundle T_X is also trivial over U_i and a local holomorphic frame is $\{\frac{\partial}{\partial \phi_{i_1}}, \cdots, \frac{\partial}{\partial \phi_{i_n}}\}$. The identification

$$T_{X,\mathbb{R}|U} \simeq U \times \mathbb{R}^{2n} \simeq U \times \mathbb{C}^n \simeq T_{X|U}$$

endows $T_{X,\mathbb{R}}$ with a (complex) vector bundle diffeomorphism I such that $I^2 = -\operatorname{Id}$ (I is induced by the standard complex structure on $\mathbb{C}^n$). This definition is independent of the choice of chart in the complex atlas. Indeed, the transition functions $\phi_{i,j} = \phi_i \circ \phi_j^{-1}$ are holomorphic, so that the real Jacobian of $\psi_{i,j}$, the corresponding differentiable transition functions $\psi_{i,j}$,

$J_{\mathbb{R}}(\psi_{i,j})$ commutes with the matrix of I, the standard complex structure induced by $\mathbb{C}^n$. Recall that $T_{X,\mathbb{R}}$ is given by $\{U_i, J_{\mathbb{R}}(\psi_{i,j} \circ \psi_j^{-1}) \circ \psi_j\}$, and by the Cauchy Riemann equations, $J_{\mathbb{R}}(\psi_{i,j}) \circ \psi_j$ is an $n \times n$ matrix of 2×2 blocks of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. □

In fact, the same argument shows:

Lemma 1.59. *Let $f : U \subset \mathbb{C}^n \to V \subset \mathbb{C}^m$ be a holomorphic map and $x \in U$; then $df(T^{0,1}_{X,x}) \subset T^{0,1}_{f(x)}$ and $df(T^{1,0}_{X,x}) \subset T^{1,0}_{f(x)}$.*

Let $f : U \subset \mathbb{C}^n \to V \subset \mathbb{C}^m$ be a holomorphic map; we also denote by f the induced differentiable map $\mathbb{R}^{2n} \to \mathbb{R}^{2m}$. The *real Jacobian* of f is:

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \frac{\partial u_j}{\partial x_i} & \frac{\partial u_j}{\partial y_i} \\ \frac{\partial v_j}{\partial x_i} & \frac{\partial v_j}{\partial y_i} \end{pmatrix}$$

where the holomorphic coordinates on U are $z_i = x_i + iy_i$, with $x_i, y_i \in \mathbb{R}$ and $f_j = u_j + iv_j$, where $u_j, v_j : U \to \mathbb{R}$. Recall that $J_{\mathbb{R}}(f)$ viewed as a matrix with coefficients in $\mathbb{C}$ defines the transition functions of $T_{X,\mathbb{R}} \otimes \mathbb{C}$. When f is holomorphic, after an appropriate change of basis of $T_x\mathbb{R}^{2n} \otimes \mathbb{C}$ and of $T_{f(x)}\mathbb{R}^{2m} \otimes \mathbb{C}$

$$J_{\mathbb{R}}(f) = \begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix},$$

where $J(f)$ is the complex Jacobian of f.

Remark 1.60. This shows that every complex manifold has a natural orientation.

Proposition 1.61. *Let X be a complex manifold. The subbundle $T^{0,1}_X \subset T_{X,\mathbb{R}}$ is naturally diffeomorphic to the holomorphic tangent bundle.*

Proof. Let X be a complex manifold and $\{U_i, \phi_i : U_i \to \mathbb{C}^n\}$ a complex atlas. The complexified tangent bundle $T_{X,\mathbb{C}}$ is represented by the cocycle $\{U_i, J_{\mathbb{R}}(\phi_i \circ \phi_j^{-1}) \circ \phi_j\}$. When $\phi_i \circ \phi_j^{-1}$ is holomorphic, by what precedes, the subbundle $T^{0,1}_X$ is associated to the cocycle $\{U_i, J(\phi_i \circ \phi_j^{-1}) \circ \phi_j\}$. It follows that $T^{0,1}_X$ and T_X are naturally isomorphic as complex vector bundles. □

Remark 1.62. When X is a complex manifold, this shows that $T^{0,1}_X$ naturally has a holomorphic structure; however, we view it as a complex vector bundle. In other words, sections of $T^{1,0}_X$ are always differentiable sections.

We now go back to the general case; let (X, I) be an almost complex manifold. There is a direct sum decomposition

$$T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1},$$

where the summands are $\ker(I - i\,\mathrm{Id})$ and $\ker(I + i\,\mathrm{Id})$. This induces a dual direct sum decomposition on the (complexified) cotangent complex:

$$\Omega_{X,\mathbb{C}} = \Omega_X^{1,0} \oplus \Omega_X^{0,1},$$

and extends to a direct sum decomposition

$$\Omega^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega_X^{p,q},$$

where $\Omega_X^{p,q} = \bigwedge^p \Omega_X^{1,0} \otimes_{\mathbb{C}} \bigwedge^q \Omega_X^{0,1}$. Further, $\overline{\Omega_X^{1,0}} = \Omega_X^{0,1}$. These decompositions are obvious at the vector space level (i.e. over a point $x \in X$), and they are defined on the bundle via local trivialisations.

Definition 1.63. The sheaves $\mathcal{A}_X^k$ and $\mathcal{A}_X^{p,q}$ are the sheaves of sections of the complex vector bundles $\Omega^k_{X,\mathbb{C}}$ and $\Omega_X^{p,q}$. Consider $d\colon \mathcal{A}_X^k \to \mathcal{A}_X^{k+1}$ the $\mathbb{C}$-linear extension of the exterior differential. If $\Pi^k\colon \mathcal{A}_X^* \to \mathcal{A}_X^k$ and $\Pi^{p,q}\colon \mathcal{A}_X^k \to \mathcal{A}_X^{p,q}$ are the natural projection maps, define the operators ∂ and $\overline{\partial}$ as $\partial = \Pi^{p+1,q} \circ d\colon \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$, and $\overline{\partial} = \Pi^{p,q+1} \circ d\colon \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$.

Lemma 1.64. *The Leibniz rule holds for ∂ and $\overline{\partial}$, i.e. if for some open set $U \subset X$, $\alpha \in \mathcal{A}_X^k(U)$ and $\beta \in \mathcal{A}_X^{k'}(U)$,*

$$\overline{\partial}(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^k \alpha \wedge \partial\beta$$
$$\overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^k \alpha \wedge \overline{\partial}\beta.$$

Proof. Follows from the Leibniz rule for d. □

Definition-Lemma 1.65. Let (X, I) be an almost complex manifold, I is *integrable* if the following equivalent conditions hold:

1. $d = \partial + \overline{\partial}$,
2. $\Pi^{0,2} \circ d = 0$ on $\mathcal{A}_X^{0,1}$.

If X is a complex manifold, the almost complex structure induced by I is integrable.

Proof. We check that the conditions are indeed equivalent. If $d = \partial + \overline{\partial}$, since $d^2 = 0$, $\Pi^{0,2} \circ d = 0$. Conversely, assume that $\Pi^{0,2} \circ d = 0$ on $\mathcal{A}_X^{1,0}$ and let $\alpha \in \mathcal{A}_X^{p,q}$. We want to prove that $d\alpha \in \mathcal{A}^{p+1,q} \oplus \mathcal{A}^{p,q+1}$. Fix a common trivialising subset $U \subset X$ for $\Omega^{1,0}_{X\ |U}$ and $\Omega^{0,1}_{X\ |U}$ and associated local frames $(\omega_i)_{1 \leq i \leq n}$ and $(\omega'_i)_{1 \leq i \leq n}$. Note that for any i, $\overline{\omega'_i} \in \mathcal{A}_X^{1,0}(U)$. A section $\alpha \in \mathcal{A}_X^{p,q}(U)$ is of the form

$$\alpha = \Sigma_{I,J} f_{I,J} \omega_I \wedge \omega_J,$$

where $|I| = p$ and $|J| = q$, and where for each I, J, $f_{I,J}$ is a differentiable function on U, $\omega_I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$, for $I = \{i_1, \cdots, i_p\}$ and $\omega'_J = \omega'_{j_1} \wedge \cdots \wedge \omega'_{j_p}$, for $J = \{j_1, \cdots, j_p\}$. For each I, J, $df_{I,J} \in \mathcal{A}_X^{1,0}(U) \oplus \mathcal{A}_X^{0,1}(U)$ and $d\omega_i \in \mathcal{A}_X^{1,1}(U) \oplus \mathcal{A}_X^{2,0}(U)$ by assumption, so that $df_{I,J} \wedge \omega_I \wedge \omega_J$, and $f_{I_J} d\omega_I \wedge \omega_J$ are sections of $\mathcal{A}_X^{p+1,q}(U) \oplus \mathcal{A}_X^{p,q+1}(U)$. For any i, $\overline{d\overline{\omega'_i}} = d\omega'_i$, so that $d\omega'_i \in \overline{\mathcal{A}_X^{2,0}(U) \oplus \mathcal{A}_X^{1,1}(U)} = \mathcal{A}_X^{0,2}(U) \oplus \mathcal{A}_X^{1,1}(U)$ and the result follows. When X is a complex manifold, using trivialisations associated to any holomorphic chart, one sees that I is integrable. □

Lemma 1.66. *Let (X, I) be an integrable almost complex manifold. The operators ∂ and $\overline{\partial}$ satisfy $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$.*

Proof. This follows from $d = \partial + \overline{\partial}$ and $d^2 = 0$. □

Remark 1.67. In fact, one can show that if $\overline{\partial}^2 = 0$, I is integrable.

The notion of integrability is very important because of the following (hard) theorem, for which a proof in the real analytic case is given in [Voi02].

Theorem 1.68 (Newlander-Nirenberg Theorem)**.** *Any integrable almost complex structure is induced by a complex structure.*

Remark 1.69. It would be natural to ask how much of complex geometric methods can be applied to the setting of non-integrable almost complex manifolds. The proof of the Newlander-Nirenberg theorem relies on finding *holomorphic* coordinates near each point. If (X, I) is an almost complex manifold, a differentiable function $f \colon X \to \mathbb{C}$ is *I-holomorphic* if $Idf = idf$, that is if $du = Idv$ and $dv = -Idu$, where $u = \Re f$ and $v = \Im f$. When X has dimension $2n$, this is a set of 2 equations in $2n$ variables, and there may be few holomorphic functions. When I is integrable, the derivatives of holomorphic functions span $(T^{1,0}_{X,\mathbb{R},x})^*$ for all $x \in X$, and this yields a complex structure on X. When $\dim X = 2$, an almost complex structure is always integrable. On a general almost complex manifold however, there are in general few holomorphic functions $f\colon X \to \mathbb{C}$. In particular, there are few submanifolds $Y \subset X$ defined by holomorphic injections of dimension $2 \leq \dim Y \leq \dim X$. The complex submanifolds $Y \subset X$ of dimension 2 are *I-holomorphic curves*, and are in general well behaved, they are studied in Symplectic Geometry.

Proposition 1.70. *Let X, Y be complex manifolds and $f\colon X \to Y$ a holomorphic map. The pull back of differential forms respects the decomposition of k-forms into forms of type (p,q), i.e. $f^*\mathcal{A}_Y^{p,q} \subset \mathcal{A}_X^{p,q}$ for all p, q, and f^* is $\mathbb{C}$-linear and compatible with ∂ and $\overline{\partial}$.*

Proof. Since f is differentiable, the pull back map $f^*\colon \mathcal{A}_Y^k \to \mathcal{A}_X^k$ satisfies $f^* \circ d_Y = d_X \circ f^*$. The pullback is $\mathbb{C}$-linear, and as in Lemma 1.59, f^* respects the decomposition of k-forms into forms of type (p,q) for $p+q=k$ because f is holomorphic. The compatibility with ∂ and $\overline{\partial}$ follows from the compatibility with d. □

1.5 Sheaf cohomology and Dolbeault Cohomology groups

Let X be a complex manifold, we have seen that for integers p, q we may define vector bundles

$$\Omega_X^{p,q} = \bigwedge^p \Omega_X^{1,0} \otimes \bigwedge^q \Omega_X^{0,1}.$$

Let $\mathcal{A}_X^{p,q}$ denote the sheaf of (C^∞) sections of the bundle $\Omega_X^{p,q}$. The exterior diffential $d = \partial + \overline{\partial}$ induces differential operators

$$\mathcal{A}^{p,q} \xrightarrow{\partial} \mathcal{A}^{p+1,q} \text{ and } \mathcal{A}^{p,q} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q+1}$$

with $\partial^2 = \overline{\partial}^2 = 0$. These operators define cohomological complexes, and we now examine the cohomology of the Dolbeault complex, associated to $\overline{\partial}$. Denote $\mathcal{Z}_{\overline{\partial}}^{p,q}(U) = \{\alpha \in \mathcal{A}_X^{p,q}(U) : \overline{\partial}\alpha = 0\}$ and $B_{\overline{\partial}}^{p,q}(U) = \{\overline{\partial}\beta; \beta \in \mathcal{A}^{p,q-1}(U)\}$. The *Dolbeault cohomology groups* of X are:

$$H_{\overline{\partial}}^{p,q}(X) = \mathcal{Z}_{\overline{\partial}}^{p,q}(X)/B_{\overline{\partial}}^{p,q}(X).$$

We first examine the case $q = 0$.

Lemma 1.71. *For all $p \geq 0$:*

$$H_{\overline{\partial}}^{p,0}(X) = \Gamma(X, \Omega_X^p) := H^0(X, \Omega_X^p).$$

Proof. First note that since $q = 0$, $H^{p,0}_{\overline{\partial}}(X) = \mathcal{Z}^{p,0}_{\overline{\partial}}(X) = \ker(\overline{\partial}\colon \mathcal{A}^{p,0}(X) \to \mathcal{A}^{p,1}(X))$. Locally, if $z_1, \cdots, z_n$ is a set of holomorphic coordinates on X, $dz_1, \cdots, dz_n$ is a local frame for $\Omega^{1,0}_X$ and $d\overline{z_1}, \cdots, d\overline{z_n}$ is a frame for $\Omega^{0,1}_X$. The forms $dz_I \wedge d\overline{z}_J$ where I, J run through multiindices with $|I| = p$ and $|J| = q$ form a frame for $\Omega^{p,q}_X$. If $\alpha \in \mathcal{A}^{p,0}(X)$, locally, $\alpha = \Sigma_I \alpha_I dz_I$, where α_I is a differentiable function, and:

$$\overline{\partial}\alpha = \Sigma_{j=1}^n \Sigma_I \frac{\partial \alpha_I}{\partial \overline{z_j}} d\overline{z_j} \wedge dz_I.$$

The $\{d\overline{z_j} \wedge dz_I\}_{I,j}$ form a local frame of $\Omega^{p,1}_X$, so that if $\overline{\partial}\alpha = 0$, $\frac{\partial \alpha_I}{\partial \overline{z_j}} = 0$ for all I, j. Each α_I is a holomorphic function, and by the canonical identification between the holomorphic cotangent bundle Ω_X and $\Omega^{1,0}_X$, this shows that α is a holomorphic section of Ω^p_X. □

We will extend this statement to the higher cohomology groups of the holomorphic vector bundle Ω^p_X.

Proposition 1.72. *The Dolbeault complex* $\{\mathcal{A}^{p,\bullet}, \overline{\partial}\}$ *is an exact complex of sheaves and is a resolution of* Ω^p_X*, i.e. the sequence of sheaves*

$$0 \to \Omega^p_X \to \mathcal{A}^{p,0}_X \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1}_X \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{A}^{p,n}_X \to 0$$

is exact.

Let X be a topological space and $\mathcal{E}$,$\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups/ modules, etc.. over X (see definition 1.50). A *morphism of sheaves* $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms $\alpha(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U \subset X$ that are compatible with the restriction maps $r^{\mathcal{F}}$ and $r^{\mathcal{G}}$, i.e. for every pair of open sets $V \subset U \subset X$, $r^{\mathcal{G}}_{UV} \circ \alpha(U) = \alpha(V) \circ r^{\mathcal{F}}_{UV}$. If α is a morphism of sheaves, $\ker \alpha$ is the sheaf $U \mapsto \ker \alpha(U)$ and $\operatorname{Coker} \alpha$ and $\operatorname{im} \alpha$ are the sheaves associated to the presheaves $U \mapsto \operatorname{Coker} \alpha(U)$ and $U \mapsto \operatorname{im} \alpha(U)$ respectively.

Remark 1.73. These two last presheaves are not sheaves in general.. E.g., if $\exp(2i\pi\cdot)\colon \mathcal{O}_X \to \mathcal{O}^*_X$, the map $\{z \mapsto z\} \in \mathcal{O}^*_U$ is not in $\alpha(\mathcal{O}_U)$ for $U = \mathbb{C} \setminus \{0\}$, but $\{z \mapsto z\} \in \alpha(\mathcal{O}_V)$ for any contractible $V \subset U$.

Given a morphism of sheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$, a section $s \in \operatorname{Coker} \alpha$ is an open cover $X = \cup_{i \in I} U_i$ and a collection of sections $s_i \in \mathcal{G}(U_i)$ such that $s_{i|U_i \cap U_j} - s_{j|U_i \cap U_j} \in \alpha(U_i \cap U_j)(\mathcal{F}(U_i \cap U_j))$. Two such sections $\{U_i, s_i\}_{i \in I}$

and $\{V_j, t_j\}$ are identified if for any $P \in U_i \cap V_j$, there is a neighbourhood V of P such that $s_{i|V} - t_{j|V} \in \alpha(V)(\mathcal{F}(V))$.

A sequence of sheaf morphisms $0 \to \mathcal{E} \stackrel{\alpha}{\to} \mathcal{F} \stackrel{\beta}{\to} \mathcal{G} \to 0$ is *exact* if $\mathcal{E} = \ker \alpha$ and $\mathcal{G} = \operatorname{Coker} \beta$. A sequence of sheaves

$$\cdots \stackrel{\alpha_{n-1}}{\to} \mathcal{F}_n \stackrel{\alpha_n}{\to} \mathcal{F}_{n+1} \stackrel{\alpha_{n+1}}{\to} \cdots$$

is exact if for each $n \in \mathbb{N}$, $\ker \alpha_{n+1} = \operatorname{im} \alpha_n$, i.e. if for each $n \in \mathbb{N}$,

$$0 \to \ker \alpha_n \to \mathcal{F}_n \to \ker \alpha_{n+1} \to 0$$

is exact.

Remark 1.74. Let $0 \to \mathcal{E} \stackrel{\alpha}{\to} \mathcal{F} \stackrel{\beta}{\to} \mathcal{G} \to 0$ be a short exact sequence of sheaves. Let $U \subset X$ be an open set, $0 \to \mathcal{E}(U) \to \mathcal{F}(U) \to \mathcal{G}(U)$ is exact (i.e. $\alpha(U)$ is injective, and $\operatorname{im} \alpha(U) = \ker \beta(U)$) but $\beta(U)$ is not in general surjective. As we will see, the failure of surjectivity is "measured" by the cohomology of the sheaves $\mathcal{E}, \mathcal{F}, \mathcal{G}$. Note that the sequence of stalks $0 \to \mathcal{E}_P \stackrel{\alpha}{\to} \mathcal{F}_P \stackrel{\beta}{\to} \mathcal{G}_P \to 0$ is exact for each $P \in X$.

Proof of 1.72. Lemma 1.71 shows that $\Omega^p_X = \ker(\overline{\partial} \colon \mathcal{A}^{p,0}_X \to \mathcal{A}^{p,1}_X)$ so that:

$$0 \to \Omega^p_X \to \mathcal{A}^{p,0}_X \to \mathcal{A}^{p,1}_X$$

is exact. We want to show that for each p,

$$\operatorname{im}\left(\overline{\partial} : \mathcal{A}^{p,q-1}_X \to \mathcal{A}^{p,q}_X\right) = \ker\left(\overline{\partial} : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p,q+1}_X\right).$$

This follows from Lemma 1.75. ☐

Lemma 1.75 ($\overline{\partial}$-Poincaré lemma)**.** *Let $U \subset \mathbb{C}^n$ be an open neighbourhood of $\overline{D}(0, R)$, the closure of a bounded polydisc and let $\alpha \in \mathcal{A}^{p,q}_X(U)$ a $\overline{\partial}$-closed form. There is a polydisc $D' \subset D$ and $\beta \in \mathcal{A}^{p,q-1}_X(D')$ such that $\alpha_{|D'} = \overline{\partial}\beta$.*

Proof. We first reduce to the case $p = 0$.

Let α be a section of $\mathcal{A}^{p,q}_X(U)$. Locally, we may write

$$\alpha = \Sigma_{I,J} \alpha_{I,J} dz_I \wedge d\overline{z}_J = \Sigma_J \alpha_I dz_I,$$

where the first sum runs over multi-indices I and J of length p and q respectively, and where $\alpha_I = \Sigma \alpha_{I,J} d\overline{z_J} \in \mathcal{A}^{0,q}(U)$. Write:

$$\overline{\partial}\alpha = \Sigma^n_{l=1} \Sigma_{I,J} \frac{\partial \alpha_{IJ}}{\partial \overline{z_l}} d\overline{z}_l \wedge dz_I \wedge d\overline{z}_J = \Sigma_{j,l} \frac{\partial \alpha_J}{\partial \overline{z_l}} d\overline{z_l} \wedge dz_I.$$

Since $d\overline{z}_l \wedge dz_I \wedge d\overline{z}_J$ is a local frame for $\Omega_X^{p,q+1}$, $\overline{\partial}\alpha = 0$ precisely when $\overline{\partial}\alpha_J = \Sigma_l \frac{\partial \alpha_J}{\partial \overline{z_l}} d\overline{z_l} = 0$ for all J. Similarly, using local frames, wwe see that $\alpha = \overline{\partial}\beta$, precisely when $\alpha_I = \overline{\partial}\beta_I$ for all I.

We now assume that $\alpha \in \mathcal{A}^{0,q}(U)$ is such that $\overline{\partial}\alpha = 0$. Locally, write $\alpha = \Sigma_{|J|=q} f_J d\overline{z_J}$ and define:

$$k = \min\{l : \text{ no } d\overline{z_i} \text{ appears in } \alpha \text{ for } i > l\}.$$

We may write $\alpha = \alpha_1 \wedge d\overline{z_k} + \alpha_2$, where no $d\overline{z_i}$ for $i \geq k$ appears in $\alpha_1 \in \mathcal{A}^{0,q-1}(U)$ or $\alpha_2 \in \mathcal{A}^{0,q}(U)$. Define operators $\overline{\partial}_i$ by:

$$\overline{\partial} = \Sigma_{i=1}^n \frac{\partial}{\partial \overline{z_i}} d\overline{z_i} = \Sigma_{i=1}^n \overline{\partial}_i.$$

If $\overline{\partial}\alpha = 0$, we immediately see that $\overline{\partial}_i(\alpha_1) = \overline{\partial}_i(\alpha_2) = 0$ for all $i \geq k+1$, so that all f_J are holomorphic in the variables $z_{k+1}, \cdots, z_n$. By the $\overline{\partial}$-Poincaré lemma in 1-variable (see Example Sheet 1), for some $0 < \varepsilon_k < R_k$, if

$$g_J(z) = \frac{1}{2i\pi} \int_{D(0,\varepsilon_k)} \frac{f_J(z_1, \cdots, z_{k-1}, \omega, z_{k+1}, \cdots, z_n)}{\omega - z_k} d\omega \wedge d\overline{\omega},$$

then, on $D' = D(0, R')$ for $R' = (R_1, \cdots, R_{k-1}, \varepsilon_k, R_{k+1}, \cdots, R_n)$, $\frac{\partial g_J}{\partial \overline{z_k}} = f_J$ and g_J is C^∞ and holomorphic in the variables $z_{k+1}, \cdots, z_n$. Set $\gamma = (-1)^q \Sigma_{k \in J} g_J d\overline{z_{J-\{k\}}}$, then $\alpha + \overline{\partial}\gamma$ is $\overline{\partial}$-closed and does not involve any monomial $d\overline{z_l}$ for $l \geq k$. We conclude by induction on k. □

We have proved that the Dolbeault complex $\{\mathcal{A}^{p,\bullet}, \overline{\partial}\}$ is an exact complex of sheaves, the proof of Lemma 1.71 shows that this complex is a resolution of the sheaf Ω_X^p. We now show that the cohomology of the Dolbeault complex computes the cohomology of the sheaf Ω_X^p.

We first recall the setup of Čech cohomology. Let X be a (paracompact) topological space, $\mathcal{F}$ a sheaf on X and $\mathcal{U} = \{U_i\}_{i \in I}$ a locally finite open cover of X. Define the *Čech complex* $(C^\bullet(\mathcal{U}, \mathcal{F}), \delta)$ of $\mathcal{F}$ associated to the cover $\mathcal{U}$ as:

$$C^p(\mathcal{U}, \mathcal{F}) = \Pi_{|J|=p+1} \mathcal{F}(U_{J_p}),$$

where J runs through multi-indices of length $p+1$, and $U_J = U_{i_0} \cap \cdots \cap U_{i_p}$ for $J = \{i_0, \cdots, i_p\} \subset I$, and the boundary map

$$\delta \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}),$$

sends σ to the $p+1$-chain whose component on $\mathcal{F}(U_{i_0} \cap \cdots U_{i_{p+1}})$ is:

$$\delta(\sigma)_{i_0, \cdots, i_{p+1}} = \Sigma_{j=0}^{p+1} (-1)^j (\sigma_{i_0, \cdots, \hat{i_j}, \cdots, i_{p+1}})_{|U_{i_0} \cap \cdots \cap U_{i_p}}.$$

Note that this is a cohomological complex, i.e. $\delta^2 = 0$. An elements $\sigma \in C^p(\mathcal{U}, \mathcal{F})$ is a p-cochain, if $\delta\sigma = 0$, σ is a cocycle, and if $\sigma = \delta\tau$ for $\tau \in C^{p+1}(\mathcal{U}, \mathcal{F})$ is a boundary. Denote $\mathcal{Z}^p(\mathcal{U}, \mathcal{F}) = \{\sigma \in C^p(\mathcal{U}, \mathcal{F}) | \delta\sigma = 0\}$ and $B^p(\mathcal{U}, \mathcal{F}) = \{\delta\tau; \tau \in C^{p+1}(\mathcal{U}, \mathcal{F})\}$; the Čech cohomology groups of $\mathcal{U}$ are:

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = \mathcal{Z}^p(\mathcal{U}, \mathcal{F})/B^p(\mathcal{U}, \mathcal{F}).$$

Assume that $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ are two covers; $\mathcal{V}$ is *finer* than $\mathcal{U}$ if for any $j \in J$, there is $i \in I$ such that $V_j \subset U_i$. If $\mathcal{V}$ is finer than $\mathcal{U}$, define a map $\varphi \colon J \to I$ by sending $j \in J$ to an index $\varphi(j) \in I$ such that $V_j \subset U_{\varphi(j)}$; the map φ induces a map $\rho_\varphi \colon C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F})$ such that $\delta_\mathcal{V} \circ \rho_\varphi = \rho_\varphi \circ \delta_\mathcal{V}$ and hence a homomorphism $\rho : \check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{V}, \mathcal{F})$. This homomorphism is well defined because different choices $\varphi, \psi \colon I \to J$ induce chain homotopic maps ρ_φ and ρ_ψ. Define the Čech cohomology groups of $\mathcal{F}$ as the direct limit of $\check{H}^p(\mathcal{U}, \mathcal{F})$ as $\mathcal{U}$ gets finer:

$$\check{H}^p(X, \mathcal{F}) = \lim_{\rightarrow} \check{H}^p(\mathcal{U}, \mathcal{F}).$$

This definition is difficult to work with on an arbitrary topological space X. However, note that, for any cover $\mathcal{U}$ of X:

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}) := \check{H}^0(X, \mathcal{F}).$$

It is useful to know which open covers actually compute $\check{H}^p(X, \mathcal{F})$.

Theorem 1.76 (Leray theorem)**.** *If the cover $\mathcal{U}$ is* acyclic *for $\mathcal{F}$, i.e. for any intersection of open sets $U_J = U_{i_0} \cap \cdots \cap U_{i_p}$ with $p > 0$, and for all $q > 0$:*

$$\check{H}^q(U_J, \mathcal{F}) = (0),$$

then

$$\check{H}^\bullet(\mathcal{U}, \mathcal{F}) = \check{H}^\bullet(X, \mathcal{F}).$$

Proposition 1.77. *If X is separated and if $\mathcal{F}$ is a quasi-coherent sheaf, any cover by affine subschemes is a Leray cover.*

Remark 1.78. When X is a complex manifold and $\mathcal{F}$ is the sheaf of sections of a complex or holomorphic vector bundle, any cover by open affine sets is a Leray cover.

One way to think about Čech cohomology is as follows. If $\mathcal{U}$ is an open cover of X, then for any p, the $(p+2)$-fold intersections of open sets in $\mathcal{U}$ $\{U_J; |J| = p+2\}$ form an open cover of $\cup_{|J'|=p+1} U_{J'}$, the union of $(p+1)$-fold intersections of open sets in $\mathcal{U}$. A cochain $\sigma \in C^p(\mathcal{U},\mathcal{F})$ is a collection of sections $\{\sigma_J\} \in \mathcal{F}(U_J)$ for all $|J| = p+1$. If σ is a cocycle, definition 1.50 shows that σ defines a section of $\mathcal{F}(\cup_{|J|=p+1}U_J)$, if σ is a boundary, σ is the restriction of a section of $\mathcal{F}(\cup_{|J|=p}U_J)$. If the cover $\mathcal{U}$ is acyclic, the classes $\check{H}^p(X,\mathcal{F})$ corespond to the sections of $\mathcal{F}(\cup_{|J|=p+1}U_J)$ that are not restrictions of sections of $\mathcal{F}(\cup_{|J|=p}U_J)$.

The important fact about Čech cohomology–which shows that this definition of sheaf cohomology agrees with that of Grothendieck, using right derived functors of the functor of global sections– is that, for any short exact sequence of sheaves

$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$$

the sequence

$$\begin{aligned} 0 \to H^0(X,\mathcal{E}) \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^1(X,\mathcal{E}) \to \cdots \\ \to \cdots H^{p-1}(X,\mathcal{G}) \to H^p(X,\mathcal{E}) \to H^p(X,\mathcal{F}) \to H^p(X,\mathcal{G}) \to \cdots \end{aligned}$$

is a long exact sequence of cohomology groups.

Assume that $\mathcal{U}$ is an acyclic cover for $\mathcal{E},\mathcal{F}$ and $\mathcal{G}$. The definition of cohomology groups is *functorial*; if $\alpha\colon \mathcal{E} \to \mathcal{F}$ is a sheaf homorphism, there is a well defined map $\alpha^*\colon C^p(\mathcal{U},\mathcal{E}) \to C^p(\mathcal{U},\mathcal{F})$, which commutes with the boundary maps δ of the Čech complexes. In particular, α,β induce maps

$$H^p(\mathcal{U},\mathcal{E}) \xrightarrow{\alpha^*} H^p(\mathcal{U},\mathcal{F}) \text{ and } H^p(\mathcal{U},\mathcal{F}) \xrightarrow{\beta^*} H^p(\mathcal{U},\mathcal{G})$$

and hence maps $H^p(X,\mathcal{E}) \xrightarrow{\alpha^*} H^p(X,\mathcal{F})$ and $H^p(X,\mathcal{F}) \xrightarrow{\beta^*} H^p(X,\mathcal{G})$.

For every $p \in \mathbb{N}$, the coboundary map $H^p(X,\mathcal{G}) \xrightarrow{\delta^*} H^{p+1}(X,\mathcal{E})$ is defined as follows. Let $[\sigma] \in H^p(X,\mathcal{G})$, and take a representative $\sigma \in Z^p(\mathcal{U},\mathcal{G})$ of $[\sigma]$ for some cover $\mathcal{U}$. Since $\mathcal{G} = \operatorname{im}\beta$, there is a refinement $\mathcal{V}$ of $\mathcal{U}$ –because $U \mapsto \operatorname{im}\beta(U)$ is a presheaf and not a priori a sheaf– and a chain $\tau \in C^p(\mathcal{V},\mathcal{F})$ with $\beta\tau = \rho_{\mathcal{V}}\sigma$. The cochain $\delta\tau$ is a section of $\ker\beta$ because $\beta\delta\tau = \delta\beta\tau = \delta\rho\sigma = 0$, and since $\ker\beta = \operatorname{im}\alpha$, there is a refinement $\mathcal{W}$ of $\mathcal{V}$ and a cochain $\mu \in C^{p+1}(\mathcal{W},\mathcal{E})$ such that $\alpha\mu = \rho_{\mathcal{W}}\delta\tau$. The cochain μ is a cocycle because $\alpha\delta\mu = \delta\alpha\mu = \delta\rho_{\mathcal{W}}\delta\tau = \rho_{\mathcal{W}}\delta^2\tau = 0$ and α is injective, so that $\delta\mu = 0$. Define $\delta^*[\sigma] = [\mu] \in H^{p+1}(X,\mathcal{E})$.

In this course, we will consider three types of sheaves, each type will carry information of a specific nature on a complex manifold X.

- Holomorphic sheaves such as $\mathcal{O}_X, T_X$ or Ω^p_X carry much information about the geometry, complex structure and deformations of the manifold. Studying the cohomology of holomorphic sheaves will emphasize properties that are particular to the complex structure, rather than depending only on the differentiable structure. Holomorphic sheaves have few sections (see Example sheets 1,2), but by analytic continuation, when a section exists, knowing its restriction to a small open set determines it entirely.

- Differentiable sheaves such as $\mathcal{A}^p_X, \mathcal{A}^{p,q}_X, T_{X,\mathbb{R}}, T_{X,\mathbb{C}}$ will mainly be auxiliary tools from the point of view of complex geometry. This idea is already at work in the statements of Lemmata 1.71 and 1.75. On a complex compact manifold, differentiable sheaves are useful because they have many sections. They are sheaves with "partitions of unity", so it is straightforward to extend sections.

- Topological sheaves such as $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$. The cohomology of these sheaves exhibits topological properties of the underlying space. A major idea of Hodge theory is to relate the topology of X to its complex structure.

We now show that sheaves with partition of unity (such as $\mathcal{A}^{p,q}$) are acyclic.

Let X be a paracompact complex manifold. Let $\mathcal{F}$ be a differentiable sheaf (a sheaf of $\mathcal{A}^0_X$-modules). For any locally finite open cover $\mathcal{U} = \{U_i\}_{i\in I}$, there is a partition of unity, that is a collection of differentiable functions $\rho_i \colon X \to \mathbb{R}$ such that $\operatorname{Supp}\rho_i \subset U_i$ for all $i \in I$ and such that $\Sigma\rho_i \equiv 1$. Given $\sigma \in Z^p(\mathcal{U}, \mathcal{F})$, then $\sigma = \delta\tau$, for $\tau \in C^{p-1}(\mathcal{U}, \mathcal{F})$ defined by $\tau_{i_0,\cdots i_{p-1}} = \Sigma_{k\in I}\rho_k\sigma_{k,i_0,\cdots,i_{p-1}}$. We have just shown:

Lemma 1.79. *If X is a complex manifold, any differentiable sheaf $\mathcal{F}$ (such as $\mathcal{A}^{p,q}_X, \mathcal{A}^k_X$..) is acyclic, i.e. for all $i > 0$:*

$$H^i(X, \mathcal{F}) = (0).$$

Lemma 1.80. *If $0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}_1 \to \cdots$ is the resolution of a sheaf $\mathcal{F}$ by an exact complex $\{\mathcal{F}^\bullet, \delta\}$ of acyclic sheaves, then $H^p(X\mathcal{F}) \simeq H^p_\delta(X, \mathcal{F}^\bullet)$, where*

$$H^p_\delta(X, \mathcal{F}^\bullet) = \ker(\delta_p : \mathcal{F}^p(X) \to \mathcal{F}^{p+1}(X))/\operatorname{im}(\delta_{p-1} : \mathcal{F}^{p-1}(X) \to \mathcal{F}^p(X)).$$

Proof. Denote $Z^p(\delta)(X, \mathcal{F}^\bullet) = \ker(\delta_p : \mathcal{F}^p(X) \to \mathcal{F}^{p+1}(X))$ and $B^p_\delta(X, \mathcal{F}^\bullet) = \operatorname{im}(\delta_{p-1} : \mathcal{F}^{p-1}(X) \to \mathcal{F}^p(X))$. If $0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}_1 \to \cdots$ is a resolution of $\mathcal{F}$, then by definition, the sequences

$$\begin{aligned} 0 \to \mathcal{F}(X) \to \quad & \mathcal{F}^0(X) \to Z^1_\delta(X) & \to 0 \\ 0 \to Z^p_\delta(X) \to \quad & \mathcal{F}^p(X) \to Z^{p+1}_\delta(X) & \to 0 \end{aligned}$$

for all $p \in \mathbb{N}$. Since $H^i(X, \mathcal{F}^p) = (0)$ for all $i > 0$ and $p \geq 0$, we obtain:

$$\begin{aligned} H^q(X, \mathcal{F}) \quad &\simeq H^{q-1}(X, Z^1_\delta(X)) \simeq \cdots \simeq H^1(X, Z^{p-1}_\delta(X)) \\ &\simeq H^0(X, Z^p_\delta(X))/\delta H^0(X, \mathcal{F}^{p-1}_\delta(X)) \simeq H^p_\delta(X, \mathcal{F}^\bullet). \end{aligned}$$

□

Recall that Lemmata 1.71 and 1.75 show that the Dolbeault complex $\{\mathcal{A}^{p,\bullet}_X, \overline{\partial}\}$ is a resolution of Ω^p_X, and by Lemma 1.79, the sheaves $\mathcal{A}^{p,q}_X$ are acyclic for all p, q. We have proved:

Theorem 1.81. *For all $p, q \in \mathbb{N}$:*

$$H^q(X, \Omega^p_X) \simeq H^{p,q}_{\overline{\partial}}(X).$$

Example 1.82. Let $X = \mathbb{C}^n$, $H^k(X, \mathcal{O}_X) = H^{0,k}_{\overline{\partial}}(X) = 0$ for all $k \in \mathbb{N}$.

Example 1.83. Let $E \to X$ be a holomorphic vector bundle, and $\mathcal{A}^{p,q}_X(E)$ the sheaf of differentiable sections of the vector bundle $\Omega^{p,q}_X \otimes E$. Define a sheaf homomorphism $\overline{\partial}_E \colon \mathcal{A}^{p,q}_X(E) \to \mathcal{A}^{p,q+1}_X(E)$ as follows. Let U be an open set and $E_{|U} \simeq U \times \mathbb{C}^r$ be a local trivialisation of E. If $\alpha \otimes s$ is a section of $\mathcal{A}^{p,q}_X(E)(U)$, $\overline{\partial}_E(\alpha \otimes s) = \overline{\partial}(\alpha) \otimes s$. The kernel of $\overline{\partial}_E \colon \mathcal{A}^{0,0}_X(E) \to \mathcal{A}^{0,1}_X(E)$ is $\mathcal{E}$, the sheaf of holomorphic sections of E (this is entirely similar to the proof of Lemma 1.71). The operator $\overline{\partial}_E$ satisfies the Leibniz rule and $\overline{\partial}^2_E = 0$. Local exactness of the complex $\{\mathcal{A}^{p,\bullet}, \overline{\partial}_E\}$ follows from that of the Dolbeault complex (Lemma 1.75). In particular,

$$0 \to \mathcal{E} \to \mathcal{A}^{0,0}_X(E) \xrightarrow{\overline{\partial}_E} \mathcal{A}^{0,1}_X(E) \xrightarrow{\overline{\partial}_E} \cdots$$

is a resolution of the holomorphic sheaf of sections $\mathcal{E}$. This resolution is acyclic, and $H^p(X, \mathcal{E}) \simeq H^{0,p}_{\overline{\partial}_E}(X)$.

We will now look at another application of sheaf cohomology: the De Rham theorem. We denote $\mathcal{A}^k_X$ the sheaf of real or complex k-forms; this will not lead to any (major) confusion. The ordinary Poincaré lemma states that the (real) De Rham complex $\{\mathcal{A}^\bullet_X, d\}$ is a resolution of the sheaf $\underline{\mathbb{R}}$ of locally constant functions:

$$0 \to \mathbb{R} \to \mathcal{A}^0_X \xrightarrow{d} \mathcal{A}^1_X \to \cdots$$

By Lemma 1.79, the De Rham complex is an acyclic complex of sheaves, and therefore the *Betti cohomology of X*, that is the cohomology of the sheaf $\underline{\mathbb{R}}$

$$H^k(X, \underline{\mathbb{R}}) \simeq H^k_{DR}(X).$$

In fact, more is true. If X is a differentiable manifold, $X = |K|$ is the geometric realisation of a simplicial complex K. Recall that the singular cohomology of X is then

$$H^k_{sing}(X, \mathbb{R}) = H^k_{sing}(X, \mathbb{Z}) \otimes \mathbb{R} = \mathrm{Hom}(H_k^{sing}(X, \mathbb{Z}), \mathbb{R}),$$

where $H_k^{sing}(X, \mathbb{Z})$ the homology of the complex $\{C_\bullet^{sing}(X, \mathbb{Z}), \partial\}$ of singular chains. Recall from Algebraic Topology that $C_p^{sing}(X, \mathbb{Z})$ is the free abelian group generated by continuous maps $\varphi\colon \Delta_p \to X$, where Δ_p is the standard p-simplex. The boundary map is $\partial\varphi = \Sigma(-1)^i \varphi_{|\Delta_k^i}$, where Δ_k^i is the ith face of Δ_k. If X is the geometric realisation of an abstract simplex K, $H^k_{sing}(X, \mathbb{Z}) \simeq H^k(X, \mathbb{Z})$. If K is a simplicial complex with vertices $\{v_{i_o}, \cdots, v_{i_n}\}$, we can associate to K an open cover $\mathcal{U}$ of X as follows. For each vertex $v_i \in K$, define U_i as the Star of v_i, i.e. the union of the interiors of all simplices of which v_i is a vertex. Then $v_{i_0}, \cdots, v_{i_p}$ span a p-simplex precisely when $U_{i_0} \cap \cdots \cap U_{i_p} \neq \emptyset$. A Čech cochain $\sigma \in C^p(\mathcal{U}, \mathbb{Z})$ is then a collection of sections of $\underline{Z}(U_{i_0, \cdots, i_p})$ (this is $\mathbb{Z}$ if $v_{i_0}, \cdots, v_{i_p}$ span a p-simplex, and 0 otherwise). To σ, we associate the cochain $\sigma' \in C^p_{sing}(X, \mathbb{Z})$, the map $\sigma'\colon \Delta =< v_{i_0}, \cdots, v_{i_p} >\mapsto \sigma_{i_0, \cdots, i_p}$. This correspondence $\sigma \to \sigma'$ is an isomorphism of abelian groups, and since $\partial\sigma' = (\delta\sigma)'$, we have an isomorphism of chain complexes $C^\bullet(\mathcal{U}, \mathcal{F}) \to C_\bullet^{sing}(X, \mathbb{Z})$, and hence:

$$H^k(\mathcal{U}, \underline{Z}) \simeq H^k_{sing}(X, \mathbb{Z}).$$

Note that refining the cover $\mathcal{U}$ corresponds to subdividing the simplicial complex K, and since this does not change the singular cohomology, $\mathcal{U}$ is a Leray cover. We have proved:

Theorem 1.84. *For a differentiable manifold X and for all $k \in \mathbb{N}$:*

$$\check{H}^k(X, \mathbb{Z}) \simeq H^k_{sing}(X, \mathbb{Z}) \text{ and } H^k_{DR}(X) \simeq \check{H}^k(X, \mathbb{R}) \simeq H^k_{sing}(X, \mathbb{R}).$$

Remark 1.85. The sheaf $\underline{\mathbb{R}}$ may be replaced by $\underline{\mathbb{C}}$-provided the De Rham complex is taken to be the complex of $\mathbb{C}$-valued forms. As I have mentioned above, in the cases we consider, Čech cohomology always agrees with sheaf cohomology. I have already written H instead of $\check{H}$ several times. From now on, I always write H.

A more detailed version of Theorem 1.84 is known as the De Rham Theorem. Results in Differentiable Topology imply that the cohomology of $\{C^\bullet_{sing}(X, \mathbb{R}), \partial\}$ coincides with that of the subcomplex of piecewise smooth cochains $\{(C^{p.s.})^\bullet_{sing}(X, \mathbb{R}), \partial\}$. Let $[\gamma] \in H^k_{sing}(X, \mathbb{Z})$ be represented by

$\gamma = \Sigma a_i f_i$, where $f_i \colon \Delta \to \mathbb{R}$ is piecewise smooth–i.e. extends to a C^∞ map in a neighbourhood of Δ– and $[\alpha] \in H^k_{DR}(X)$ be represented by a k-form $\alpha \in \mathcal{A}^k_X(X)$. Define a pairing:

$$< \alpha, \gamma >= \int_\gamma \alpha = \Sigma a_i \int_\Delta f_i^* \alpha$$

with values in $\mathbb{R}$. The Stokes formula implies that this pairing asociates to $[\alpha]$ a cohomology class in $H^k_{sing}(X, \mathbb{R})$. Indeed, for any $\beta \in \mathcal{A}^{k-1}(X)$,

$$\int_\gamma \alpha = \int_\gamma \alpha + d\beta,$$

and if $d\alpha = 0$,

$$\int_{\partial \epsilon} \alpha = \int_\epsilon d\alpha = 0$$

for any boundary of a $k+1$-chain ϵ.

Theorem 1.86 (De Rham Theorem)**.** *Let X be a differentiable manifold. The map*

$$\alpha \mapsto (\gamma \to \int_\gamma \alpha)$$

is an isomorphism and provides a canonical identification $H^k_{DR}(X, \mathbb{R}) \simeq H^k_{sing}(X, \mathbb{Z}) \otimes \mathbb{R}$.

2 Hermitiann and Kähler metrics on complex manifolds

2.1 Hermitian metrics and connections of vector bundles

The concepts of Hermitian metrics, connections and curvature for complex vector bundles are very close to the notions of Riemannian metrics on real differentiable manifolds.

Definition 2.1. Let $\pi \colon E \to X$ be a complex vector bundle over a differentiable manifold. Let $\mathcal{A}^k_X(E)$ be the sheaf of differentiable sections of the vector bundle $\Omega^k_{X,\mathbb{C}} \otimes E$. A *connection* on E is a $\mathbb{C}$-linear sheaf homomorphism $D \colon \mathcal{A}^0_X(E) \to \mathcal{A}^1_X(E)$ that satisfies the Leibniz rule $D(f \otimes s) = df \otimes s + f \otimes Ds$.

Remark 2.2. A connection induces sheaf homomorphisms $D\colon \mathcal{A}^k_X(E) \to \mathcal{A}^{k+1}_X(E)$, with $D(\tau \otimes s) = d\tau \otimes s + (-1)^k \tau \otimes Ds$ for $\tau \in \mathcal{A}^k_X(U)$ and $s \in \mathcal{A}^0_X(E)(U)$ over a trivisalising open set U for the vector bundle E.

Example 2.3. Let $E \simeq X \times \mathbb{C}^r$ be the trivial vector bundle; define $D(\alpha \otimes s) = d\alpha \otimes s$ for local sections $\alpha \in \mathcal{A}^k_X(U)$ and $s \in \mathcal{A}^0_X(U)$. The operator D is the *trivial connection*: it acts as the exterior differential d on the form part of sections of $\mathcal{A}^k_X(E)$.

If $E \to X$ is a holomorphic vector bundle over a complex manifold, the operator $\overline{\partial}_E$ in Example 1.83 is a connection.

If D and D' define two connections on E, $D - D'$ is $\mathcal{A}^0_X$-linear, that is $(D - D')(fs) = f(D - D')(s)$. The operator $D - D'$ hence is a global section of the sheaf $\mathcal{A}^1_X(\operatorname{End} E)$. We write $\mathcal{A}^k(X, F) = C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes F)$ for global sections of $\mathcal{A}^k_X(F)$, where F is a complex vector bundle. Let $a \in \mathcal{A}^1(X, \operatorname{End} E)$, a acts on a local section $\alpha \otimes s$ of $\mathcal{A}^k(E)$ by multiplication on the form part and evaluation $\operatorname{End}(E) \times E \to E$ on the bundle component.

If E is a trivial vector bundle, let d be the trivial connection defined in Example 2.3. Any connection D on E is of the form $d + a$, where $a \in \mathcal{A}^1(X, \operatorname{End} E) = \mathcal{A}^1_X \otimes \operatorname{End} E$.

We want to deduce from this the form of connections on an arbitrary vector bundle $E \to X$. Let $\{U_i, h_i \colon E_{|U_i} \simeq U_i \times \mathbb{C}^r\}$ be local trivialisations of E, and recall that we denote $g_{i,j}$ the induced transition map $U_i \cap U_j \to \mathrm{GL}_r(\mathbb{C})$. In a local frame, sections of $\operatorname{End} E$ are represented by $(r \times r)$ matrices with differentiable entries, and sections of $\mathcal{A}^1_X(\operatorname{End} E)$ by $(r \times r)$-matrices of 1-forms. If d is the trivial connection on $E_{|U_i}$, the restriction $D^i = D_{|U_i} \colon \mathcal{A}^0(E)(U_i) \to \mathcal{A}^1(E)(U_i)$ is of the form $d^i + A^i$, where A^i is a $(r \times r)$-matrix of 1-forms. On $U_i \cap U_j$, we have the transition relations:

$$D^j = h_j^{-1} \circ (d^j + A^j) \circ h_j, \text{ and } A^j = g_{ij}^{-1} \circ dg_{ij} + g_{ij}^{-1} A^i g_{ij}. \tag{2}$$

Definition-Lemma 2.4. The *curvature of D* is

$$\Theta_D = D^2 \colon \mathcal{A}^0(E) \to \mathcal{A}^2(E).$$

The curvature Θ_D is $\mathcal{A}^0$-linear, i.e. defines a global section of $\mathcal{A}^2(\operatorname{End} E)$.

Proof. If f is a differentiable function, $D^2(fs) = fD^2(s)$. □

Let D be a connection on E and $a \in \mathcal{A}^1(X, \operatorname{End} E)$, then

$$\Theta_{D+a} = \Theta_D + D(a) + a \wedge a,$$

where $a \wedge a \in \mathcal{A}^2(X, \operatorname{End} E)$ acts by exterior product on the form part and by composition in $\operatorname{End} E$.

Definition 2.5. A *Hermitian metric* h on $E \to X$ is a Hermitian inner product $h_x \colon E_x \times E_x \to \mathbb{C}$ on each fibre E_x such that for any open set $U \subset X$ and $s_1, s_2 \in \mathcal{A}^0(E)(U)$ local sections of E over U,

$$h(s_1, s_2)\colon \qquad U \to \mathbb{C}$$
$$x \to h_x(s_1(x), s_2(x))$$

is differentiable. A Hermitian vector bundle $(E, h) \to X$ is a complex vector bundle endowed with a Hermitian metric.

Proposition 2.6. *Every complex vector bundle $E \to X$ is Hermitian.*

Proof. Let $\{U_i, h_i \colon E_{|U_i} \simeq U_i \times \mathbb{C}^r\}_{i \in I}$ be local trivialisations of $E \to X$. Let $e_1^i, \cdots, e_r^i$ the local frame associated to h_i. In that local frame, a Hermitian metric h is represented by an $(r \times r)$-matrix of differentiable functions $H^i = (h_{k,l})_{1 \leq k,l \leq r}$, where $h_{k,l}(x) = h_x(e_k(x), e_l(x))$. If $s_1, s_2 \in \mathcal{A}^0(E)(U_i)$ are local sections of E, $s_1 = \Sigma s_{1,k}^i e_k^i$ and $s_2 = \Sigma s_{2,k}^i e_k^i$, then $h(s_1, s_2)_{|U_i} = \Sigma s_{1,k}^i h^i(k,l) \overline{s_{2,l}^i}$. Note that if $x \in U_i \cap U_j$, $H^j = g_{i,j}^t H^i \overline{g_{ij}}$.

Define a Hermitian metric h on $E \to X$ by setting

$$h^i(s_1^i, s_2^i) = \Sigma_{k=1}^r s_{1,k}^i \overline{s_{2,k}^i}$$

and $h = \Sigma \rho_i h^i$, where $\{\rho_i\}_{i \in I}$ is a partition of unity subordinate to the open cover $\{U_i\}_{i \in I}$ of X. □

Note that applying the Gram-Schmid orthonormalisation process, we may always choose local frames of $E \to X$ in which the Hermitian metric on each fibre E_x is the standard Hermitian metric of $\mathbb{C}^r$. Such frames are called *isometric*.

Definition 2.7. Let $(E, h) \to X$ be a Hermitian vector bundle and D a connection on E. The connection D is compatible with the metric h if for any two local sections $s_1, s_2 \in \mathcal{A}^0(E)(U)$ over an open set $U \subset X$,

$$dh(s_1, s_2) = h(Ds_1, s_2) + h(s_2, Ds_1),$$

where for any two 1-forms $\alpha, \beta \in \mathcal{A}_X^1(U)$, $h(\alpha \otimes s_1, s_2) = \alpha h(s_1, s_2)$ and $h(s_1, \beta \otimes s_2) = \overline{\beta} h(s_1, s_2)$.

Assume that in a local trivialisation $E_{|U} \simeq U \times \mathbb{C}^r$, $D = d + A$, where A is an $(r \times r)$-matrix of 1-forms. Then, D is Hermitian if:

$$dH = A^t \cdot H + H\overline{A}. \tag{3}$$

If the local frame is isometric, D is Hermitian precisely when $\overline{A}^t = -A$. We say that D is compatible with the metric.

If D is a connection given by $d + A$ in a local trivialisation of E, define the *adjoint connection* D^{adj} as the connection given by the matrix

$$A^{adj} = -\overline{A}^t$$

(check that this defines a connection: i.e. satisfies the transition formulae of (2)). The connection $D^h = \frac{1}{2}(D + D^{adj})$ is always a Hermitian connection.

Now assume that $E \to X$ is a holomorphic vector bundle over a complex manifold. We have seen that there is a direct sum decomposition

$$\mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E),$$

and this induces a decomposition $D = D^{1,0} + D^{0,1}$. Consider a local trivialisation of E, and let $f \in \mathcal{A}^0(U)$ and $s \in \mathcal{A}^0(E)(U)$, then:

$$D^{1,0}(fs) = \partial f \cdot s + f \cdot A^{1,0}s \text{ and } D^{0,1}(fs) = \overline{\partial} f \cdot s + f \cdot A^{0,1}s,$$

where $A^{1,0}$ (resp. $A^{0,1}$) is an $(r \times r)$-matrix of $(1,0)$-forms (resp. $(0,1)$-forms).

Definition 2.8. The connection D is compatible with the holomorphic structure if $D^{0,1}(fs) = \overline{\partial} f \cdot s$, i.e. if $D^{0,1} = \overline{\partial}_E$ is the Dolbeault connection of Example 1.83.

Proposition 2.9. *Let $(E, h) \to X$ be a holomorphic Hermitian vector bundle. There is a unique Hermitian connection D_E that is compatible with the holomorphic structure.This connection is the* Chern connection *of $E \to X$; its curvature Θ_E is the* Chern curvature*.*

Proof. We first prove that if such a connection exists, it is uniquely determined. A connection is a sheaf homomorphism, hence uniqueness is a local question. Let $E_{|U} \simeq U \times \mathbb{C}^r$ be a local trivialisation and assume that in the associated local frame, the connection D is given by $d + A$, where $A = A^{1,0} + A^{0,1}$ is an $(r \times r)$-matrix of 1-forms, and h is given by an $(r \times r)$-Hermitian matrix of differentiable functions. Since D is Hermitian, $dH = A^t H + H\overline{A}$. If D is compatible with the holomorphic structure, $A = A^{1,0}$ and $\partial H = A^t H$ and $\overline{\partial} H = H\overline{A}$. In particular,

$$A = \overline{H}^{-1} \partial H, \tag{4}$$

and D is uniquely determined by H.

In order to prove the existence of the connection D_E, it is enough to prove that (4) defines a connection. We want to prove that if $\{U_i, h_i\colon E_{|U_i} \simeq U_i \times \mathbb{C}^r\}$ is a system of local trivialisations for E, the D_E^i defined by (4) define a connection globally, i.e. satisfy the transition relations (2). This is left as an exercise. □

Example 2.10. 4.1 Let $L \to X$ be a holomorphic line bundle. A Hermitian metric on L is a positive real valued function $h\colon X \to \mathbb{R}_+^*$ given by $h(x) = |e_1(x)|^2$, where e_1 is a local holomorphic frame of L. The Chern connection of (L, h) is then $D_E = d + \log h$.

Corollary 2.11. *Let $(E, h) \to X$ be a holomorphic Hermitian vector bundle over a complex manifold X and D_E, Θ_E be its Chern connection and curvature. If, in a local holomorphic frame, $D_E = d + A$, then*

1. *A is of type $(1, 0)$ and $\partial A = A \wedge A$,*
2. *$\Theta_E = \overline{\partial} A$ is of type $(1, 1)$, so that $\Theta_E \in \mathcal{A}^{1,1}(X, \operatorname{End} E)$,*
3. *$\overline{\partial}\Theta_E = 0$, the 2-form Θ_E is $\overline{\partial}$-closed.*

Proof. This all follows easily from the relation (4) and from the local form of the curvature $\Theta_E = dA + A \wedge A$. □

Remark 2.12. We have seen that if $(E, h) \to X$ is a Hermitian (complex) vector bundle over a differentiable manifold, we may always pick isometric local (differentiable) frames for E. In such a frame, the Hermitian metric coincides with the standard Hermitian structure on $\mathbb{C}^n$, and its matrix is the identity. The Gram-Schmid orthonormalisation process is not holomorphic, so in general we cannot hope to find isometric holomorphic local frames.

In any local holomorphic frame, the Hermitian metric is represented by a hermitian matrix $H(z)$ with differentiable entries. However, if $P \in X$ is a point and $E_{|U} \simeq U \times \mathbb{C}^r$ is a local holomorphic frame in a neighbourhood of P, and $z_1, \cdots, z_n$ are local holomorphic coordinates centred at P, $H(0)$ is a positive definite Hermitian matrix, so that there is $B \in \mathrm{GL}(\mathbb{C}^r)$ with $B^t H(0)\overline{B} = \mathrm{Id}$. The matrix B determines a linear (hence holomorphic) change of coordinates for the local frame of E over U such that $H(0) = \mathrm{Id}$– that is such that the restriction of the Hermitian metric to the fibre over $P \in X$ is the standard Hermitian metric on $\mathbb{C}^r$.

Lemma 2.13. *Let $(E, h)\colon X$ be a holomorphic, Hermitian vector bundle over a complex manifold X, and $P \in X$ a point. There is a local holomorphic frame of E near P inducing local holomorphic coordinates $z_1, \cdots, z_n$ on $P \in U \subset X$ such that on U:*

1. $H(z) = \mathrm{Id} + O(|z|^2)$, *and*
2. $i\Theta_E(0) = -i\partial\overline{\partial H(0)}$.

Proof. The second assertion follows from the first by Corollary 2.11 and the relation $A = \overline{H}^{-1}\partial\overline{H}$. Take a local holomorphic frame for E such that in that frame $H(0) = \mathrm{Id}$. We now define a holomorphic change of coordinates on X given by a matrix $\mathrm{Id} + B$, where B is a matrix with holomorphic entries, such that the first assertion holds. After such a change of coordinates, the matrix of the Hermitian metric in the new frame is

$$H'(z) = (\mathrm{Id} + B(z)^t) \cdot H(z) \cdot (\mathrm{Id} + \overline{B(z)}.$$

Denote $B = (B_{j,k})1 \le j,k \le r$, and set

$$B_{j,k}(z) = \Sigma_{i=1}^n \frac{\partial H_{k,j}}{\partial z_i}(0) z_i,$$

so that $dH'(0) = (0)$, and we conclude by the Taylor formula. □

2.2 Kähler metrics

Let $V_\mathbb{C}$ be an n-dimensional $\mathbb{C}$-vector space, write $V_\mathbb{R}$ for its $\mathbb{R}$-vector space structure; recall that there is a direct sum decomposition $V_\mathbb{C} = V^{1,0} \oplus V^{0,1}$. Let $W_\mathbb{R} = \mathrm{Hom}_\mathbb{R}(V, \mathbb{R})$ be its dual and $W_\mathbb{C} = \mathrm{Hom}_\mathbb{R}(W, \mathbb{C})$; $W_\mathbb{C}$ inherits a direct sum decomposition $W_\mathbb{C} = W^{1,0} \oplus W^{0,1}$.

Lemma 2.14. *There is a* 1*-to-*1 *correspondence between Hermitian forms* $V \times V \to \mathbb{C}$ *and elements of* $W_\mathbb{R}^{1,1}$ *given by* $h \mapsto \omega = -\Im h$. *If* $\omega \in W_\mathbb{R}^{1,1}$, $h(\cdot,\cdot) = \omega(\cdot, I\cdot)$ *is Hermitian, where* I *is the complex structure endomorphism.*

Proof. Let $h = \Re h + i\Im h$ be a Hermitian form and $\omega = -\Im h$; for all $u, v \in V$,

$$h(u,v) = \Re h(u,v) + i\Im h(u,v) = \overline{h(u,v)} = \Re h(v,u) + i\Im h(v,u),$$

so that $\omega(u,v) = -\omega(v,u)$ is an alternating real form. We now show that $\omega \in W^{1,1}$, i.e. that $\omega(u,v) = 0$ for all $u, v \in V^{1,0}$ or $u, v \in V^{0,1}$. Note that since ω is a real form and for all $v \in V^{1,0}$, $\overline{v} \in V^{0,1}$, it is enough to check that $\omega(u,v) = 0$ for all $u, v \in V^{1,0}$. We have seen that elements of $V^{1,0}$ are of the form $v - iI(v)$ for $v \in V$; consider for $u, v \in V$:

$$\omega(u - iI(u), v - iI(v)) = \omega(u,v) - \omega(I(u), I(v)) - i(\omega(u, I(v)) + \omega(I(u), v)).$$

Since $h(I(u), I(v)) = ih(u, I(v)) = -i^2 h(u,v) = h(u,v)$ and $h(u, I(v)) = -h(I(u), v)$, we have $\omega(u,v) = \omega(I(u), I(v))$, and $\omega(u, I(v)) = -\omega(I(u), v)$. This finishes the proof. □

The isomorphism of Lemma 2.14 does not depend on the choice of a basis for V, but it is helpful to see its expression in a specific basis of V. Recall that if $v_1, \cdots, v_n$ is a basis for V, we write $v_1^*, \cdots, v_n^*$ for the associated dual basis. If $h = \Sigma h_{j,k} v_j^* \otimes \overline{v_k^*}$, then the associated $(1,1)$-form is $\omega = \frac{i}{2} \Sigma h_{j,k} v_j^* \wedge \overline{v_k^*}$.

The form ω is said to be *positive* if the associated hermitian form h is positive-definite.

Definition 2.15. Let X be a complex manifold. A *Hermitian structure* on X is a Hermitian metric h on the complex vector bundle $T_{X,\mathbb{C}}$. The Hermitian manifold (X, h) is *Kähler* if the real $(1,1)$-form ω associated to h is closed for the exterior differential, i.e. if $d\omega = 0$.

Example 2.16.

1. The real $(1,1)$-form $\omega = \frac{i}{2} \Sigma dz_i \wedge d\overline{z_i}$ associated to the standard Hermitian structure on $\mathbb{C}^n$ is Kähler.

2. Let X be a complex curve; then any Hermitian structure on X is Kähler.

3. Let $\Lambda \subset \mathbb{C}^n$ be a lattice of rank $2n$. The complex torus $X = \mathbb{C}^n/\Lambda$ is a Kähler manifold. Let $\omega' = \frac{i}{2} \Sigma h_{j,k} dz_j \wedge d\overline{z_k}$ be a $(1,1)$-form defined on $\mathbb{C}^n$, and assume that ω' has constant coefficients. Then ω' is invariant under the action of Λ, and hence $\omega' = \pi^*\omega$ for $\omega in \Omega_X^{1,1} \cap \Omega^2_{X,\mathbb{R}}$, where $\pi : \mathbb{C}^n \to X$. The form ω is Kähler.

Let h be a Hermitian form on a $\mathbb{C}$-vector space V; then $g = \Re h$ defines an Euclidian inner product on $T_{X,\mathbb{R}}$. In particular, any Hermitian manifold is also a Riemannian manifold. Assume that (X, h) is a Hermitian manifold; we have seen that X has a canonical orientation. Recall that if $v_1, \cdots, v_n$ is a basis of the holomorphic tangent bundle T_X, there is a canonical isomorphism $T_X = (T_X, \mathbb{C})^{1,0}$ and $v_1, I(v_1), \cdots, v_n, I(v_n)$ is an oriented basis of $T_X, \mathbb{R}$ over $\mathbb{R}$ (Exercise 1, Example Sheet 2).

Since (X, h) is in particular a Riemannian manifold, it has a canonical *volume form*, that is a nowhere vanishing section of $\Omega^{2n}_{X,\mathbb{R}}$ whose value at $P \in X$ is the unique form that is strictly positive on all oriented bases of $T_{X,P,\mathbb{R}}$ and has norm 1 for the induced metric on $\Omega^{2n}_{X,P,\mathbb{R}}$.

Lemma 2.17. *The volume form associated to the Hermitian metric h on X is $\frac{\omega^n}{n!}$.*

Proof. Let $z_1, \cdots z_n$ be local holomorphic coordinates near P such that $\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}$ is an isometric frame of $T_{X,\mathbb{C}}$ near P, i.e. $h_P(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}) = \delta_{i,j}$, then if $\Re(z_i) = x_i$ and $\Im(z_i) = y_i$, $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$ is an oriented basis of $T_{X,P,\mathbb{R}}$. The volume form of (X, h) is then the unique form with value 1 on $\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} \wedge \frac{\partial}{\partial y_n}$. It is enough to prove that $\frac{\omega^n}{n!}(\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} \wedge \frac{\partial}{\partial y_n}) = 1$.

By definition of the coordinates $z_1, \cdots, z_n$ near $P \in X$, $\omega_P = \Sigma_{j=1}^n dz_j \wedge d\overline{z_i}$; so that $\omega_P^n = (\frac{i}{2})^n \Pi_{j=1}^n dz_j \wedge d\overline{z_j}$ and the result follows from $dz_j \wedge d\overline{z_j} = (dx_j + idy_j) \wedge (dx_j - idy_j)$. □

When X is compact, this implies that $\int_X \frac{\omega^n}{n!} > 0$. We get as a corollary:

Lemma 2.18. *Let (X, ω) be a compact Kähler manifold, the form ω^k is not exact for $k = 1, \cdots, n$.*

Proof. Assume that $\omega^k = d\eta$, then $\omega^n = \omega^n \wedge d\eta = d(\omega^{n-k} \wedge \eta)$. By Stokes theorem,

$$\int_X \omega^n = \int_X d(\omega^{n-k} \wedge \eta) = 0:$$

this is a contradiction. □

Remark 2.19. For each $k = 1, \cdots, n$, $[\omega^k]$ is a nonzero class in $H^{2k}(X, \mathbb{R})$, and ω^k is a section of $\mathcal{A}_X^{k,k}$.

We now come to a useful characterisation of Kähler metrics. Let $P \in X$ and $z_1, \cdots, z_n$ holomorphic coordinates centred at P. Assume that in these coordinates, $\omega = \frac{i}{2} \Sigma h_{jk}(z) dz_j \wedge d\overline{z_k}$. The Kähler condition $\partial\omega = 0$ writes:

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j} \text{ for all } 1 \leq j, k, l \leq n \tag{5}$$

Theorem 2.20. *Let (X, ω) be a Hermitian manifold. Then (X, ω) is Kähler if and only if for all $P \in X$ there are local holomorphic coordinates $z_1, \cdots, z_n$ centred at $P \in X$ such that in these coordinates, $h_{jk}(z) = \delta_{jk} + O(|z|^2)$. Such a system of coordinates is called* normal *for the Kähler form ω.*

Proof. If the metric has the given expression, it is clear that it is Kähler. We prove that if ω is Kähler, there are such local holomorphic coordinates near each $P \in X$. As is noted in Remark 2.12, we may pick holomorphic coordinates $z_1, \cdots, z_n$ such that $h_{jk}(z) = \delta_{jk} + O(|z|)$. Consider the Taylor expansion of h_{jk} to the first order:

$$h_{jk}(z) = \delta_{jk} + \Sigma(a_{jkl} z_l + a'_{jkl} \overline{z_l}) + O(|z|^2),$$

where $a_{jkl} = \frac{\partial h_{jk}}{\partial z_l}$ and $a'_{jkl} = \frac{\partial h_{jk}}{\partial \bar{z}_l}$. We want to find a system of holomorphic coordinates $\zeta_1, \cdots, \zeta_n$ such that $\frac{\partial h_{jk}}{\partial \zeta_l} = \frac{\partial h_{jk}}{\partial \bar{\zeta}_l} = 0$. Since $h_{jk} = \overline{h_{kj}}$, $\overline{a_{kjl}} = a'_{jkl}$, and (5) implies that $a_{jkl} = a_{lkj}$. Define local holomorphic coordinates:

$$\zeta_k = z_k + \frac{1}{2}\Sigma a_{jkl} z_j z_j,$$

then

$$d\zeta_k = dz_k + \Sigma a_{jkl} z_l dz_j,$$

and $\omega = i\Sigma d\zeta_k \wedge d\overline{\zeta_k} + O(|\zeta|^2)$. □

This shows that in the neighbourhood of each point, a Kähler metric is isomorphic to a constant metric to the first order, ω *osculates the standard Hermitian metric on* $\mathbb{C}^n$ *to order* 2.

Remark 2.21. The holomorphic tangent bundle T_X, viewed as a complex vector bundle over the differentiable manifold X, is canonically isomorphic to $T_{X,\mathbb{R}}$. When X is Kähler, the matrix of the Levi-Civita connection of $(T_{X,\mathbb{R}}, g)$ coincides with the matrix of the Chern connection of (T_X, h), where $g = \Re h$. For more on this point of view, see [Voi02].

2.3 Examples of Kähler manifolds

Chern classes of Line bundles Let X be a compact complex manifold and $L \to X$ a complex line bundle over X. Recall that if D is a connection on E, the curvature $\Theta_D \in \mathcal{A}^2(X, \operatorname{End} E)$ is a global section of the sheaf $\mathcal{A}^2(\operatorname{End} E)$. Here, since $E = L$ is a line bundle,

$$\operatorname{End} L = L^* \otimes L \simeq X \times \mathbb{C},$$

and Θ_D is in fact a global 2-form. Fix a local frame $\{U, e_1\}$ for $L \to X$ over $U \subset X$ and write in this frame $D = d + A$, where A is a 1-form. Then $\Theta_D = dA$ is a closed 2-form. Consider $[\Theta_D] \in H^2(X, \mathbb{C})$ the corresponding De Rham cohomology class.

Claim 2.22. *The class* $[\Theta] = [\Theta_D] \in H^2(X, \mathbb{C})$ *is independent of the choice of connection* D *on* $L \to X$.

Proof. Let D' be another connection on $L \to X$ that is represented in the local frame $\{U, e_1\}$ by the 1-form A'. For any $s \in \mathcal{A}^0(L)$, $(D - D') \cdot s = (A - A') \cdot s$. Using the transition relations for A, A', we see that $A - A'$ glues to a global 1-form $B \in \mathcal{A}^1(X)$. Then, $\Theta_D - \Theta'_D = dB$, and $[\Theta_D] = [\Theta_{D'}] \in H^2(X, \mathbb{C})$. □

Proposition 2.23. *Let $L \to X$ be a Hermitian line bundle over a differentiable manifold. The* first Chern class *of L is*

$$c_1(L) = \left[\frac{i}{2\pi}\Theta_D\right] \in H^2(X, \mathbb{R}),$$

where D is an arbitrary connection on L.

Proof. We have seen that $c_1(L)$ is a well defined De Rham class in $H^2(X, \mathbb{C})$, we show that it is real. Being real or complex valued is a local property, so we can work in a local trivialisation $\{U, e_1\}$ of L. In that frame, the connection $D = d + A$, for $A \in \mathcal{A}^1(\operatorname{End} L)$, and $\Theta_D = dA$. Since we may assume that the frame is isometric, and the connection is Hermitian $-A^t = -A = \overline{A}$ and $\overline{i\Theta_D} = -id\overline{A} = i\Theta_D$. □

Remark 2.24. The constant $\frac{i}{2\pi}$ actually ensures that $c_1(L) \in H^2(X, \mathbb{Z})$.

Remark 2.25. The Lefschetz Theorem on $(1,1)$-classes (Example Sheet 4) states that if ω is a d-closed section of $\mathcal{A}^{1,1}(X)$ with $[\omega] \in H^2(X, \mathbb{Z})$, then $\omega = c_1(L)$ for some holomorphic Hermitian line bundle $L \to X$.

Remark 2.26. If $E \to X$ is a complex vector bundle, define $c_1(V) = c_1(\det V)$. If L is any line bundle, $c_1(E \otimes L) = c_1(E) + \operatorname{rk} E \cdot c_1(L)$.

Assume that $L \to X$ is a holomorphic line bundle, and let D_L be the Chern connection on L. The metric h defines a differentiable function $h \colon X \to \mathbb{R}_+^*$ with $h(z) = |e(z)|^2$, in a holomorphic local frame $\{U, e\}$. There is a well defined *weight function* $\varphi \colon U \to \mathbb{R}$ such that $\varphi(z) = 1/\log h(z)$. The Chern connection D_L is given in the local holomorphisc frame $\{U, e\}$ by $A = \overline{H}^{-1}\partial\overline{H}$, i.e. by $A = -\partial\varphi$, and

$$\Theta_L = \partial\overline{\partial}\varphi = -\overline{\partial}\partial\varphi.$$

It is clear that $\frac{i}{2\pi}\Theta_L$ is represented by a real valued, d-closed $(1,1)$-form ω_L. By Lemma 2.14, there is a Hermitian form h_L associated to ω_L.

Definition 2.27. The holomorphic line bundle $L \to X$ is *positive* if h_L defines a Hermitian metric on T_X, i.e. if in a holomorphic frame for $L \to X$, Θ_L is given by a weight function φ

$$\Theta_L = \Sigma \frac{\partial^2\varphi}{\partial z_j \partial \overline{z_k}} dz_j \wedge d\overline{z_k},$$

with $\frac{\partial^2\varphi}{\partial z_j \partial \overline{z_k}}$ positive-definite for every $z \in U$.

Exercise 2.28. Show that if $L \to X$ is globally generated, i.e. if for every $P \in X$, there is a section $\sigma \in \Gamma(X, L)$ with $\sigma(P) \neq 0$, then the Hermitian form h_L is defined by a weight function φ_L such that $\frac{\partial^2\varphi}{\partial z_j \partial \overline{z_k}}$ is positive semi-definite.

Exercise 2.29. Show that L is positive if and only if $L^{\otimes n}$ is positive for some $n \in \mathbb{N}^*$.

A positive holomorphic line bundle $L \to X$ defines a Kähler structure on X.

The Fubini-Study metric on $\mathbb{P}^n$ Recall that the tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathbb{P}^n$ is a subset of $\mathbb{P}^n \times \mathbb{C}^{n+1}$. The standard Hermitian product on $\mathbb{C}^{n+1}$ $< v, w >= v^t \cdot \overline{w}$ induces a Hermitian product h on the vector bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$, given by $h((x, v), (x, w)) =< v, w >= v^t \cdot \overline{w}$. The line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ inherits a Hermitian metric, which dualised gives a Hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(1)$. Explicitly, if $[z] = [z_0{:}\cdots{:}z_n] \in \mathbb{P}^n$ and $s_1, s_2 \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ are sections,

$$h_z(s_1(z), s_2(z)) = \frac{< s_1(z), s_2(z) >}{< z, z >}.$$

Recall that on the open set $U_i = \{z_i \neq 0\} \subset \mathbb{P}^n$, a local trivialisation of $\mathcal{O}_{\mathbb{P}^n}(1)$ is given by the section $s_i \in H^0(U_i, \mathcal{O}_{\mathbb{P}^n}(1))$ with $s_i([z]) = z_i$. Local holomorphic coordinates are given on U_i by $z_0, \cdots, \hat{z_i}, \cdots, z_n$ where $[z] = [z_0{:}\cdots{:}z_{i-1}{:}1{:}z_{i+1}{:}\cdots{:}z_n]$. Then

$$h_z(s_i([z]), s_i([z])) = \frac{1}{1 + \Sigma z_j^2},$$

and the associated hermitian form on U_i is:

$$\omega_{\mathcal{O}_{\mathbb{P}^n}(1)} = \frac{1}{2i\pi} \partial\overline{\partial} \log h_z(s_i([z]), s_i([z])) = -\frac{1}{2i\pi} \partial\overline{\partial} \log(1 + \Sigma z_j^2).$$

Lemma 2.30. *The form $\omega_{\mathcal{O}_{\mathbb{P}^n}(1)}$ is positive. This form is the* Fubini Study metric ω_{FS} *on* $\mathbb{P}^n$.

Proof. We check that ω_{FS} is positive definite at $0 \in U_i$ (that is at the point $[0{:}\cdots{:}1{:}\cdots{:}0] \in \mathbb{P}^n$);

$$\overline{\partial} \log(\frac{1}{1 + \Sigma z_i^2}) = \frac{\overline{\partial}(1 + \Sigma |z_i|^2)}{1 + \Sigma |z_i|^2} = \frac{\Sigma z_i d z_i}{1 + \Sigma |z_i|^2},$$

hence:

$$\partial\overline{\partial}\log(\frac{1}{1+\Sigma z_i^2}) = \frac{(1+\Sigma|z_i|^2)dz_i\wedge d\overline{z_i} - \Sigma\overline{z_i}dz_i\wedge\Sigma z_i d\overline{z_i}}{(1+\Sigma|z_i|^2)^2}.$$

At $0\in U_i$, $\omega = \frac{i}{2\pi}\Sigma dz_i\wedge d\overline{z_i}$ is positive. Since ω is invariant under the transitive action of $SU(n+1)$ on $\mathbb{P}^n$, it is positive everywhere on $\mathbb{P}^n$. □

Remark 2.31. Note that if $\pi\colon \mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{P}^n$ is the projection map, $\pi^*\omega_{FS} = \frac{i}{2\pi}\partial\overline{\partial}\log|z|^2$ on $\mathbb{C}^{n+1}\setminus\{0\}$.

Assume that $Y\subset X$ is a complex submanifold of X, if X is a Kähler manifold, $j^*\omega_X = \omega_Y$ is a Kähler form on Y. We then obtain:

Corollary 2.32. *Any projective manifold $X\subset\mathbb{P}^N$ is Kähler.*

The unexpected partial converse of this statement is the Kodaira embedding theorem, which we admit.

Theorem 2.33 (Kodaira embedding Theorem)**.** *Let X be a compact Kähler manifold and $L\to X$ a holomorphic line bundle. Then L is positive if and only if there is a holomorphic embedding $\phi\colon X\to\mathbb{P}^N$ of X into some projective space $\mathbb{P}^N$ such that $\phi^*\mathcal{O}_{\mathbb{P}^N}(1) = L^{\otimes m}$ for some $m\in\mathbb{N}$.*

Remark 2.34. Note that if there exists such an embedding, the pullback $\phi^*\mathcal{O}_{\mathbb{P}^N}(1) = L^{\otimes m}$ defines a form $\omega_{L^{\otimes m}} = \phi^*\omega_{FS}$, which is positive. Conversely, if X is known to be projective, then by the so called GAGA principle, L is an algebraic line bundle and if L is positive, $\int_V c_1(L)^{\dim V} > 0$ for any irreducible subvariety $V\subset X$, and hence, by Nakai's criterion, L is ample. The truly astonishing statement contained in the Kodaira Embedding Theorem is that the positivity of a line bundle on a compact Kähler manifold guarantees the existence of enough holomorphic sections to define an embedding.

Projective bundles Let $\pi\colon E\to X$ be a rank $r+1$ holomorphic vector bundle over a complex manifold X. Define the projective bundle $\mathbb{P}(E) = (E\setminus\{0\text{-section}\})/\mathbb{C}^*\to X$. If $E_{|U_i}\simeq U_i\times\mathbb{C}^{r+1}$ is a local trivialisation of E, $\mathbb{P}(E)_{|U_i}\simeq U_i\times\mathbb{P}^r$, and over $U_i\cap U_j$, the identification of the trivialisations $\mathbb{P}(E)_{|U_i}\simeq\mathbb{P}(E)_{|U_j}$ is given by $h_{i,j}\colon z\to P\,\mathrm{GL}(r,\mathbb{C})$, where $h_{i,j}\colon z\to\mathrm{GL}(r,\mathbb{C})$ is the cocycle of E.

Lemma 2.35. *If X is a compact Kähler manifold, $\mathbb{P}(E)$ is Kähler.*

Proof. Let $S \subset \pi^*E \to E$ be the tautological line bundle, i.e. the holomorphic line bundle with fibre over $(x, [V]) \in \mathbb{P}(E)$ the 1-dimensional subspace $V \subset E_x$. Similarly to the case of $\mathbb{P}^n$, the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is then defined as the dual S^*. On a fibre of π, by definition,

$$\mathcal{O}_{\mathbb{P}(E)}(1)_{|\pi^{-1}(x)} \simeq \mathcal{O}_{\mathbb{P}(E_x)}(1) \simeq \mathcal{O}_{\mathbb{P}^r}(1).$$

A Hermitian metric h on E induces a Hermitian metric on π^*E and hence on S and S^*. The associated Chern form ω_E is positive on each fibre $\pi^{-1}(x) = \mathbb{P}(E_x)$, because, there $\omega_{E\pi^{-1}(x)} \simeq \omega_{FS\mathbb{P}(E_x)}$. However, this form is not necessarily positive globally. Let ω_X be a Kähler form on X, then on $U \subset X$ there is a constant $\lambda_U \gg 0$ such that $\omega_{\lambda_U} = \omega_E + \lambda_U \omega_X$ is positive on $\mathbb{P}(E)_{|U}$. If X is compact, there is a constant $\lambda \gg 0$ such that ω_λ is positive everywhere, and $\mathbb{P}(E)$ is Kähler. □

Remark 2.36. Note that $\mathbb{P}(E)$ can be Kähler without X being Kähler itself.

Blowups Let $Y \subset X$ be a complex submanifold of codimension k. By Theorem 1.9, near each point $P \in Y$, there are local holomorphic coordinates $z_1, \cdots, z_n$ on X such that $Y = \{z_1 = \cdots = z_k = 0\} \subset X$. We will define the *blowup of X along Y*:

$$\sigma \colon \widetilde{X}_Y \to X.$$

The blowup σ is a proper holomorphic map σ from a complex manifold $\widetilde{X}_Y$ such that:

- $\sigma_{\widetilde{X}_Y} \colon \widetilde{X}_Y \smallsetminus \sigma^{-1}(Y) \simeq X \smallsetminus Y$,
- $E = \sigma^{-1}(Y) \subset \widetilde{X}_Y$ is a smooth hypersurface and $\sigma_{|E} \colon E \to Y$ is isomorphic to the natural projection $\mathbb{P}(\mathcal{N}_{Y/X}) \to Y$.

Remark 2.37. Blowups are very important in Algebraic Geometry; they are the simplest example of *birational maps*. In fact, if X is an algebraic variety and $Y \subset X$ is a subvariety, the blowup of X along Y has the following universality property: if $f \colon Z \to X$ is a morphism such that $f^{-1}(Y)$ is a Cartier divisor, then f factors through σ.

Let $U \subset X$ be an open set and let $z_1, \cdots, z_n$ be holomorphic coordinates on U such that $Y \cap U = \{z_1 = \cdots = z_k = 0\} \subset U$. Define $\sigma_{|U} \colon \widetilde{U}_Y \subset \mathbb{P}^{k-1} \times U \to U$, where $\widetilde{U}_Y = \{([Z], z) \in \mathbb{P}^{k-1} \times U | Z_i z_j = z_i Z_j\}$ and σ is the projection on the second factor. The map $\sigma_{|U}$ is clearly proper, and is an isomorphism over $U \smallsetminus (U \cap Y)$. For $y \in Y$, $\sigma^{-1}(y) \simeq \mathbb{P}^{k-1}$.

Recall that if $\{U_i, \phi_i\}$ is a complex atlas of X, the holomorphic tangent bundle T_X is defined by the cocycle $\{U_i, J(\phi_{i,j}) \circ \phi_j\}$, where J is the holomorphic Jacobian and $\phi_{i,j} = \phi_i \circ \phi_j^{-1}$ are the transition functions. If $Y \overset{i}{\subset} X$ is a complex submanifold, that corresponds to the embedding $i\colon (z_k, \cdots, z_n) \mapsto (0, \cdots, 0, z_k, \cdots, z_n)$, the holomorphic vector bundle $i^*T_X = T_{X|Y}$ is defined by the cocycle $\{U_i \cap Y, J(\phi_{i,j}) \circ \phi_j^{-1} \circ i\}$. There is an exact sequence of vector bundles

$$0 \to T_Y \to T_{X|Y} \to \mathcal{N}^*_{Y/X} \to 0$$

which defines the *normal bundle* $\mathcal{N}_{Y/X}$.

We identify the local constructions $\widetilde{U}_Y \to Y$ and $\widetilde{V}_Y \to V$ over $U \cap V$ by setting the biholomorphic map

$$\sigma_U^{-1}(U \cap V) \simeq \sigma_V^{-1}(U \cap V)$$

to be $([Z], z) \mapsto (M_{UV}[Z], z)$, where $M_{UV} \in P\,\mathrm{GL}(k-1, \mathbb{C})$ are the transition matrices of the cocycle $\mathcal{N}_{Y/X}$.

Note that if $Y \cap U = \{z_1 = \cdots z_k = 0\}$ and $V \cap Y = \{f_1 = \cdots = f_k = 0\}$ for holomorphic functions (with linearly independent differentials), then by construction of $\mathcal{N}_{Y/X}$, over $U \cap V \cap Y$:

$$(f_1, \cdots, f_k) = M_{UV}(z_1, \cdots, z_k) \text{ and } (df_1, \cdots, df_k) = M_{UV}(dz_1, \cdots, dz_k).$$

The biholomorphic identification over $U \cap V$ shows that $\widetilde{X}_Y \to X$ is well defined (the gluings satisfy the cocycle conditions..) and that over Y, σ coincides with $\mathbb{P}(\mathcal{N}_{Y/X}) \to Y$.

Remark 2.38. Note that when $Y = \{\sigma = 0\} \subset X$ is a smooth hypersurface ($d\sigma$ is injective), the blowup $\widetilde{X}_Y \to X$ is an isomorphism.

Example 2.39. Recall the construction of the tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$. The projection to the second factor $\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathbb{C}^{n+1}$ is the blowup of $\mathbb{C}^{n+1}$ at the origin $0 \in \mathbb{C}^{n+1}$. Note that $\mathcal{N}_{0/\mathbb{C}^{n+1}} \simeq \mathbb{C}^{n+1}$.

Claim 2.40. *$E = \sigma^{-1}(Y)$ is a smooth hypersurface in $\widetilde{X}_Y$.*

Denote $[Z] = [Z_0{:}\cdots{:}Z_k]$ and $y = (y_1, \cdots, y_n) = (0, \cdots, 0, z_{k+1}, \cdots, z_n)$, for $([Z], y) \in E$; there is an index i_0 such that $Z_{i_0} \neq 0$. A local holomorphic equation for E near $([Z], y)$ is given by $f_{i_0}\colon ([Z], y) \mapsto y_{i_0}$.

Claim 2.41. *There is a holomorphic line bundle $L \to \widetilde{X}_Y$ that is trivial outside E and such that $L_{|E} \simeq \mathcal{O}_{\mathbb{P}(\mathcal{N}_{Y/X})}(1)$.*

Ex. 5 in Example Sheet 3 constructs a line bundle $\mathcal{O}_{\widetilde{X}_Y}(E)$ associated to a smooth hypersurface $E \subset \widetilde{X}_Y$, Ex. 4(ii) in Example Sheet 2 shows that $\mathcal{N}_{E/\widetilde{X}_Y} \simeq \mathcal{O}_{\widetilde{X}_Y}(E) \otimes \mathcal{O}_E = \mathcal{O}_{\widetilde{X}_Y}(E)_{|E}$. It is enough to prove that $\mathcal{N}_{E/\widetilde{X}_Y} \simeq \mathcal{O}_{\mathbb{P}(\mathcal{N}_{Y/X})}(-1)$. This is clear on the explicit construction of $\widetilde{X}_Y$– this is the same idea as in Remark 2.38.

Proposition 2.42. *Let X be a Kähler manifold and $Y \subset X$ be a compact submanifold. The blowup $\widetilde{X}_Y$ is Kähler and compact if X is compact.*

Proof. The construction of σ shows that σ is proper, and hence $\widetilde{X}_Y$ is compact if X is. Let ω_X be a Kähler form on X; the pullback $\sigma^*\omega_X$ is a closed $(1,1)$-form on $\widetilde{X}_Y$ and it is positive at $P \in \widetilde{X}_Y$ that does not belong to E. The form $\sigma^*\omega_X$ is only semi-definite on E; indeed on E, the tangent vectors lying in the tangent spaces of fibres of σ are isotropic for $\sigma^*\omega_X$. Let $\omega' = \omega_L$ be the $(1,1)$-form associated to the holomorphic line bundle in Claim 2.41. Then $d\omega' = 0$, $\omega' = 0$ outside a compact neighbourhood of E and ω' positive definite on E and hence on the fibres of σ. Since Y is compact, there then exists a constant $C \gg 0$ such that $C\sigma^*\omega_X + \omega'$ is positive. □

3 Hodge Theory on Hermitian manifolds

In this section, we relate De Rham cohomology classes to harmonic forms on a Hermitian manifold (X, h). On the one hand, we have seen that De Rham cohomology coincides with Betti cohomology; this shows that De Rham cohomology classes really are topological objects. On the other hand, harmonic forms are solutions of PDEs that depend on the Riemannian/Hermitian structure. Hodge Theory shows that each cohomology class is represented uniquely by a harmonic form. This describes topological data on a manifold in terms of very explicit geometric objects.

3.1 Differential Operators

Let X be a compact oriented Riemannian manifold and (E, g_E) an Euclidian vector bundle. Recall that we may define a L^2-metric on $\mathcal{A}^0_c(E)$ the space of compactly supported sections of E by:

$$(s_1, s_2)_{L^2} = \int_X g_E(s_1, s_2) \operatorname{vol}_X .$$

If E, F are two Euclidian vector bundles on X and if $P\colon \mathcal{A}^0_c(E) \to \mathcal{A}^0_c(F)$ is a linear operator, then we may define the formal adjoint $P^*\colon \mathcal{A}^0_c(F) \to \mathcal{A}^0_c(E)$ by the requirement, for all $s_1 \in \mathcal{A}^0_c(E)$ and $s_2 \in \mathcal{A}^0_c(F)$:

$$(Ps_1, s_2)_{L^2} = (s_1, P^* s_2)_{L^2}.$$

Let X be a compact Riemannian manifold with $\dim_{\mathbb{R}} X = m$, the *Hodge operator* $\star\colon \mathcal{A}^k_{X,\mathbb{R}} \to \mathcal{A}^{m-k}_{X,\mathbb{R}}$ is defined by $\alpha \wedge \star\beta = <\alpha, \beta> \mathrm{vol}_X$, where $<,>$ is the metric induced on Ω^*_X by the Riemannian metric.

If $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}$ is an isometric (local) frame for the metric on $T_{X,\mathbb{R}}$, $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k}; i_1 < \cdots < i_k\}$ is an isometric frame for the induced metric on Ω^k_X, and the Hodge operator is then defined by linearity and permutations by imposing

$$\star(dx_1 \wedge \cdots \wedge dx_k) = dx_{k+1} \wedge \cdots \wedge dx_n.$$

In particular $\star 1 = \mathrm{vol}$, $\star\,\mathrm{vol} = 1$ and $\star\star = (-1)^{k(m-k)}$ on $\mathcal{A}^k_X$.

The adjoint $d^*\colon \mathcal{A}^{k+1}_{X,\mathbb{R}} \to \mathcal{A}^k_{X,\mathbb{R}}$ of $d\colon \mathcal{A}^k_{X,\mathbb{R}} \to \mathcal{A}^{k+1}_{X,\mathbb{R}}$ satisfies:

$$d^* = (-1)^{mk+1} \star d \star. \tag{6}$$

Assume now that (X, h) is a compact Hermitian manifold; recall that the underlying differentiable manifold X is orientable and endowed with a Riemannian structure $\Re h = g$. We extend the previous notions by $\mathbb{C}$-linearity. Let $n = \dim_{\mathbb{C}} X$, the Hodge operator $\star\colon \mathcal{A}^k_X \to \mathcal{A}^{2n-k}_X$ is such that for all $\alpha, \beta \in \mathcal{A}^k_X$, $\alpha \wedge \star\overline{\beta} = <\alpha, \beta> \mathrm{vol}$. In particular, it is clear from the definition that $\star\colon \mathcal{A}^{p,q}_X \to \mathcal{A}^{n-q,n-p}_X$ is a sheaf isomorphism, and as above, we check that $\star 1 = \mathrm{vol}$, $\star\,\mathrm{vol} = 1$ and $\star\star = (-1)^{k(2n-k)}$ on $\mathcal{A}^k_X$.

For any Hermitian vector bundle E, we define an associated L^2-metric on $\mathcal{A}^0_c(E)$ as above.

When X is compact, the space of differential forms (resp. the space of k-forms) admits a direct sum decomposition $\mathcal{A}^*_X = \bigoplus_{k=1}^{2n} \mathcal{A}^k_X$ (resp. $\mathcal{A}^k_X = \bigoplus_{p=1}^{k} \mathcal{A}^{p,k-p}_X$) that is orthogonal for the L^2-metric induced by the Hermitian structure:

$$(\alpha, \beta)_{L^2} = \int_X h(\alpha, \beta)\,\mathrm{vol} = \int_X \alpha \wedge \star\overline{\beta}.$$

Each $\mathcal{A}^{p,q}_X$ is an infinite-dimensional vector space endowed with a scalar product; $\mathcal{L}^{p,q}$ denotes its completion with respect to the associated L^2-norm $(\cdot,\cdot)_{L^2}$.

Exercise 3.1. Check that if h_k and $h_{p,q}$ are the restrictions of the Hermitian metric to $\Omega^k_{X,\mathbb{C}}$ and $\Omega^{p,q}_X$ respectively, $2^k h_k = \Sigma h^{p,q}$.

Since X is a complex manifold, the exterior differential splits as a sum $d = \partial + \overline{\partial}$ and from (5),

$$d^* = - \star d \star .$$

Definition-Lemma 3.2. Let X be a compact Hermitian manifold. The operators

$$\partial^* = - \star \overline{\partial} \star \colon \mathcal{A}_X^{p+1,q} \to \mathcal{A}_X^{p,q} \text{ and } \overline{\partial}^* = - \star \partial \star \colon \mathcal{A}_X^{p,q+1} \to \mathcal{A}_X^{p,q}$$

are formal adjoints for the operators ∂ and $\overline{\partial}$ respectively. They satisfy $d^* = \partial^* + \overline{\partial}^*$, $(\partial^*)^2 = (\overline{\partial}^*)^2 = 0$ and $\overline{\partial}^* \partial^* = -\partial^* \overline{\partial}^*$.

Proof. It is enough to prove that for any $\alpha \in \mathcal{A}_X^{p,q}$ and $\beta \in \mathcal{A}_X^{p,q+1}$, $(\overline{\partial}\alpha, \beta)_{L^2} = (\alpha, \overline{\partial}^* \beta)_{L^2}$. We have

$$(\overline{\partial}\alpha, \beta)_{L^2} = \int_X \overline{\partial}\alpha \wedge \star\overline{\beta},$$

and since $\alpha \wedge \star\overline{\beta}$ is of type $(n, n-1)$,

$$d(\alpha \wedge \star\overline{\beta}) = \overline{\partial}(\alpha \wedge \star\overline{\beta}) = \overline{\partial}\alpha \wedge \star\overline{\beta} + (-1)^{p+q}\alpha \wedge \overline{\partial}(\star\overline{\beta}).$$

Since $\star$ is a real operator, $\overline{\partial}(\star\overline{\beta}) = \overline{\partial \star \beta} = (-1)^{p+q} \star \overline{\star\partial \star \beta}$ and the result follows from the Stokes' formula. □

Definition 3.3. The *Laplace operators* associated to d, ∂ and $\overline{\partial}$ are:

$$\Delta_d = dd^* + d^*d \text{ , } \Delta_\partial = \partial\partial^* + \partial^*\partial \text{ , and } \Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}.$$

Remark 3.4. Note that while Δ_∂ and $\Delta_{\overline{\partial}}$ are bihomogeneous operators, i.e. they are operators $\mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q}$ that preserve both the degree of forms and their types, $\Delta_d \colon \mathcal{A}_X^k \to \mathcal{A}_X^{k+1}$ does not necessarily preserve the type of k-forms.

Since we assume that X is compact, if $\alpha \in \mathcal{A}_X^k$,

$$\begin{aligned}(\alpha, \Delta_d\alpha)_{L^2} = (d\alpha, d\alpha)_{L^2} + (d^*\alpha, d^*\alpha)_{L^2} &= ||d\alpha||_{L^2} + ||d^*\alpha||_{L^2} \\ (\alpha, \Delta_\partial\alpha)_{L^2} &= ||\partial\alpha||_{L^2} + ||\partial^*\alpha||_{L^2} \\ (\alpha, \Delta_{\overline{\partial}}\alpha)_{L^2} &= ||\overline{\partial}\alpha||_{L^2} + ||\overline{\partial}^*\alpha||_{L^2}\end{aligned}$$

Definition 3.5. A *d-harmonic* (resp. ∂-harmonic, $\overline{\partial}$-harmonic) is a form $\alpha \in \ker \Delta_d$ (resp. $\ker \Delta_\partial$, $\ker \Delta_{\overline{\partial}}$). When X is compact, α is d-(resp. ∂,$\overline{\partial}$-) harmonic when $\alpha \in \ker d \cap \ker d^*$ (resp. $\alpha \in \ker \partial \cap \ker \partial^*$, $\alpha \in \ker \overline{\partial} \cap \ker \overline{\partial}^*$). We denote $\mathcal{H}_d^k(X, h), \mathcal{H}_\partial^k(X, h)$ and $\mathcal{H}_{\overline{\partial}}^k(X, h)$ the spaces of d, ∂ and $\overline{\partial}$-harmonic forms on X.

From what we have seen, the spaces $\mathcal{H}^k_\partial(X,h)$ and $\mathcal{H}^k_{\overline{\partial}}(X,h)$ admit direct sum decompositions:

$$\mathcal{H}^k_\partial(X,h) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_\partial(X,h) \text{ and } \mathcal{H}^k_{\overline{\partial}}(X,h) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\overline{\partial}}(X,h),$$

but since the Laplacian Δ_d is not bihomogeneous, no such decomposition exists a priori on $\mathcal{H}^k_d(X,h)$.

Properties of harmonic spaces on Hermitian manifolds Recall that the Hodge operator $\star$ induces a $\mathbb{C}$-linear isomorphism $\star\colon \mathcal{A}^k_X \to \mathcal{A}^{2n-k}_X$; from the definition of d^*, this is an isomorphism $\star\colon \mathcal{H}^k_d(X,h) \to \mathcal{H}^{2n-k}_d(X,h)$, and more specifically $\star\colon \mathcal{H}^{p,q}_d(X,h) \to \mathcal{H}^{n-p,n-q}_d(X,h)$. From the definition of ∂^* and $\overline{\partial}^*$, we see that

$$\star\colon \mathcal{H}^{p,q}_{\overline{\partial}}(X,h) \to \mathcal{H}^{n-p,n-q}_{\partial}(X,h)$$

is an isomorphism. Complex conjugation interchanges $\mathcal{H}^{p,q}_{\overline{\partial}}(X,h)$ and $\mathcal{H}^{q,p}_{\partial}(X,h)$.

Assume that (X,h) is a compact, connected Hermitian manifold. The pairing $\mathcal{H}^{p,q}_{\overline{\partial}}(X,h) \times \mathcal{H}^{n-p,n-q}_{\overline{\partial}} \to \mathbb{C}$ defined by

$$(\alpha,\beta) \to \int_X \alpha \wedge \beta$$

is non degenerate because $(\alpha, \star\overline{\alpha}) > 0$ for $\alpha \neq 0$; this establishes *Serre Duality* for harmonic forms, that is:

$$\mathcal{H}^{p,q}_{\overline{\partial}}(X,h) \simeq \mathcal{H}^{n-p,n-q}_{\overline{\partial}}(X,h)^*, \tag{7}$$

where $\mathcal{H}^{n-p,n-q}_{\overline{\partial}}(X,h)^*$ denotes the dual vector space.

Elliptic Operators

Definition 3.6. Let E,F be two complex vector bundles and $P\colon \mathcal{A}^0(E) \to \mathcal{A}^0(F)$ be a $\mathbb{C}$-linear morphism of sheaves. P is a *differential operator of order k* if for any simultaneous trivialisation $E_{|U} \simeq U \times \mathbb{C}^p$, and $F_{|U} \simeq U \times \mathbb{C}^q$ over $U \subset X$,

$$P(s_1,\cdots,s_p) = (t_1,\cdots,t_q), \text{ where } t_j = \Sigma P_{I,i,j}\frac{\partial s_i}{\partial x_I},$$

and $P_{I,i,j} = 0$ for any multi-index $|I| > k$ and $P_{I,i,j} \neq 0$ for some I with $|I| = k$.

The operator P decomposes as a sum $P = P_1 + \cdots + P_k$, where each P_i is a section of $\mathrm{Hom}(E, F) \otimes \mathrm{Sym}^i T_X$.

The *symbol* of P is $\sigma_P = P_k$. For $x \in X$, $\sigma_P(x) \in \mathrm{Hom}(E_x, F_x) \otimes \mathrm{Sym}^k T_X$, i.e. is a degree k homogeneous map $\Omega_{X,x} \to \mathrm{Hom}(E_x, F_x)$. The operator P is *elliptic* if for all $x \in X$, $x \neq 0$, $\sigma_P(x)$ is injective.

We admit the following.

Lemma 3.7. *The Laplacian operators $\Delta_d, \Delta_\partial$ and $\Delta_{\overline{\partial}}$ are elliptic and self-adjoint. Their symbols are*

$$\sigma_d(\alpha) = -||\alpha||^2_{L^2} \cdot \mathrm{Id} \ \ , \ \textit{and} \ \sigma_\partial(\alpha) = \sigma_{\overline{\partial}}(\alpha) = -\frac{1}{2}||\alpha||^2_{L^2} \cdot \mathrm{Id}\,.$$

Proof. This is an easy computation in a local holomorphic coordinate system for X. □

We admit the following Fundamental Theorem on elliptic operators.

Theorem 3.8. *Let X be a compact complex manifold and $E, F \to X$ be two Hermitian vector bundles with $\mathrm{rk}\, E = \mathrm{rk}\, F$. Let $P\colon E \to F$ be an elliptic differential operator, then $\ker P \subset \mathcal{A}^0(E)$ has finite dimension and $P(\mathcal{A}^0(E)) \subset \mathcal{A}^0(F)$ is closed and has finite codimension.There is a decomposition as a direct sum:*

$$\mathcal{A}^0(E) = \ker P \oplus P^*(\mathcal{A}^0(F))$$

that is orthogonal for the L^2 metric (induced by the metric on E).

Corollary 3.9. *Let (X, h) be a compact Hermitian manifold. Then there is a orthogonal decomposition:*

$$\mathcal{A}^k(X) = \mathcal{H}^k_d(X, h) \oplus d\mathcal{A}^{k-1}(X) \oplus d^*\mathcal{A}^{k+1}(X),$$

and $\mathcal{H}^k_d(X, h)$ has finite dimension. Similarly,

$$\mathcal{A}^{p,q}(X) = \mathcal{H}^{p,q}_\partial(X, h) \oplus \partial\mathcal{A}^{p-1,q}(X) \oplus \partial^*\mathcal{A}^{p+1,q}(X),$$
$$\mathcal{A}^{p,q}(X) = \mathcal{H}^{p,q}_{\overline{\partial}}(X, h) \oplus \overline{\partial}\mathcal{A}^{p,q-1}(X) \oplus \overline{\partial}^*\mathcal{A}^{p,q+1}(X)$$

and $\mathcal{H}^{p,q}_\partial(X, h)$ and $\mathcal{H}^{p,q}_{\overline{\partial}}(X, h)$ have finite dimension.

Remark 3.10. The crucial point is the existence of such a decomposition (orthogonality is then easy). For a proof of this theorem and more on Harmonic Theory, see [Dem96].

Corollary 3.11. *Let (X, h) be a compact Hermitian manifold. Then, the projections*

$$\mathcal{H}^k_d(X, h) \to H^k_{DR}(X, \mathbb{C}) \text{ and } \mathcal{H}^{p,q}_{\overline{\partial}}(X, h) \to H^{p,q}(X)$$

are isomorphisms.

Proof. We prove the statement for the operator $\overline{\partial}$ (the proof for d is identical). Let $\alpha \in \mathcal{H}^{p,q}_{\overline{\partial}}(X, h)$ be a $\overline{\partial}$- harmonic form; since α is $\overline{\partial}$-closed, α defines a Dolbeault cohomology class $[\alpha] \in H^{p,q}(X)$. Assume that $\beta \in \mathcal{A}^{p,q}(X)$ is a $\overline{\partial}$-closed form. By Corollary 3.9,

$$\beta = \alpha + \Delta_{\overline{\partial}}\gamma,$$

where $\alpha \in \mathcal{H}^{p,q}_{\overline{\partial}}(X, h)$ and $\gamma \in \mathcal{A}^{p,q}(X)$. since $\overline{\partial}\beta = \overline{\partial}\alpha = 0$, $\overline{\partial}(\overline{\partial}^*\overline{\partial}\gamma) = 0$, so that $\overline{\partial}^*(\overline{\partial}\gamma) \in \ker\overline{\partial} \cap \operatorname{im}\overline{\partial}^*$. By Corollary 3.9, $\ker\overline{\partial} \cap \operatorname{im}\overline{\partial}^* = \{0\}$, so that $\Delta_{\overline{\partial}}\gamma = \overline{\partial}\overline{\partial}^*\gamma$ is $\overline{\partial}$-exact. This shows that $[\beta] = [\alpha]$, and the projection $\mathcal{H}^{p,q}_{\overline{\partial}}(X, h) \to H^{p,q}(X)$ is an isomorphism. □

Remark 3.12. Corollary 3.11 shows that every De Rham cohomology class (resp. Dolbeault cohomology class) has a unique d-harmonic (resp. $\overline{\partial}$-harmonic) representative.

3.2 The case of Kähler manifolds

We will show that if (X, h) is a Hermitian manifold and if the metric is Kähler–we have seen that this is a local condition–then the various Laplace operators are proportional, so that d-harmonic forms are ∂ and $\overline{\partial}$-harmonic. By definition, a Kähler metric coincides with the standard Hermitian metric on $\mathbb{C}^n$ up to order 2; this will imply relations between linear and differential operators on X.

In this section, we assume that (X, h) is a Kähler manifold, and we denote ω its Kähler form.

Definition-Lemma 3.13. The *Lefschetz operator* is the linear operator $\mathcal{A}^k_X \to \mathcal{A}^{k+2}_X$ defined by

$$\alpha \mapsto \omega \wedge \alpha.$$

Its formal adjoint $\Lambda\colon \mathcal{A}^{k-2}_X \to \mathcal{A}^k_X$ satisfies $\Lambda = (-1)^k \star L\star$.

Proof. The only thing there is to prove is that $(L\alpha, \beta)_{L^2} = (\alpha, (-1)^k\star L\star\beta)_{L^2}$ for $\alpha \in \mathcal{A}^k(X)$ and $\beta \in \mathcal{A}^{k+2}(X)$. For any $x \in X$,

$$L\alpha \wedge \star\overline{\beta} = \omega \wedge \alpha \wedge \star\overline{\beta} = (-1)^k\alpha \wedge \star\star\omega \wedge \star\overline{\beta} = (-1)^k\alpha \wedge \star(\overline{\star\omega \wedge \star\beta})$$

because ω is a real form. □

Recall that for any differential operators P, Q of degrees p, q, the *Lie bracket*

$$[P, Q] = P \circ Q - Q \circ P$$

is a differential operator of degree $p + q$.

Proposition 3.14 (The Kähler identities). *Let (X, ω) be a Kähler manifold, then:*

1. $[\Lambda, \overline{\partial}] = -i\partial^*$,
2. $[\Lambda, \partial] = i\overline{\partial}^*$,
3. $[\overline{\partial}^*, L] = i\partial$,
4. $[\partial^*, L] = -i\overline{\partial}$.

Proof. Note that the first two equalities imply the third and fourth by adjunction. Also the first implies the second by complex conjugation because L and Λ are real operators and hence for any $\alpha \in \mathcal{A}_X^k$,

$$[\Lambda, \partial](\alpha) = \overline{[\Lambda, \overline{\partial}](\overline{\alpha})}.$$

We therefore just need to prove the first equality. Note that since Λ is an operator of degree 0, $[\Lambda, \overline{\partial}]$ has degree 1 and hence it is enough to prove it on $\mathbb{C}^n$ equipped with the standard Hermitian metric by Theorem 2.20. This is the content of Lemma 3.15. □

Lemma 3.15. *Let $U \subset \mathbb{C}^n$ be an open set endowed with the standard Hermitian metric, i.e. $\omega = i\Sigma dz_j \wedge d\overline{z}_j$. Then $[\overline{\partial}^*, L] = i\partial$.*

Recall that if $\theta \in \mathcal{A}^0(T_{X,\mathbb{C}})$ is a vector field on X, the *interior product* with θ is the map

$$i(\theta)\colon \mathcal{A}_X^k \to \mathcal{A}_X^{k-1}, \tag{8}$$

such that for any vector fields $\eta_1, \cdots, \eta_{k-1} \in \mathcal{A}^0(T_{X,\mathbb{C}})$ and for any k-form $\alpha \in \mathcal{A}_X^k$,

$$i(\theta)(\alpha)(\eta_1, \cdots, \eta_{k-1}) = \alpha(\theta, \eta_1, \cdots, \eta_{k-1}).$$

It follows immediately that if $u, v \in \mathcal{A}_X^\bullet$ are two differential forms:

$$i(\theta)(u \wedge v) = i(\theta)(u) \wedge v + (-1)^{\deg u} u \wedge i(\theta)(v).$$

If $\theta = \theta^{1,0} + \theta^{0,1}$ is the decomposition of the vector field θ into its components of type $(1,0)$ and $(0,1)$, the interior products with $\theta^{1,0}$ and $\theta^{0,1}$ define maps:

$$i(\theta^{1,0})\colon \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p-1,q} \text{ and } i(\theta^{0,1})\colon \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q-1}.$$

Recall that if $z_1, \cdots, z_n$ are local holomorphic coordinates on X, $\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\}$ and $\{\frac{\partial}{\partial \overline{z_1}}, \cdots, \frac{\partial}{\partial \overline{z_n}}\}$ are local frames for $T_X^{1,0}$ and $T_X^{0,1}$ respectively. If I, J are multi-indices with $|I| = p$ and $|J| = q$, we have:

$$\begin{aligned} i(\frac{\partial}{\partial z_i})(dz_I \wedge d\overline{z_J}) &= 0 \text{ if } i \notin I \\ &= (-1)^{l-1} dz_{I-\{i\}} \wedge d\overline{z_J} \text{ for } I = \{i_1 < \cdots < i_l = i < \cdots < i_p\} \end{aligned}$$

and:

$$\begin{aligned} i(\frac{\partial}{\partial \overline{z_j}})(dz_I \wedge d\overline{z_J}) &= 0 \text{ if } j \notin J \\ &= (-1)^{p+l-1} dz_I \wedge d\overline{z_{J-\{j\}}} \text{ for } J = \{j_1 < \cdots < j_l = j < \cdots < j_q\} \end{aligned}$$

For a k-form $\alpha \in \mathcal{A}_X^k$ (on a differentiable manifold X),

$$d^*\alpha = -\Sigma_{l=1}^n \Sigma_{|J|=k} \frac{\partial \alpha_J}{\partial x_l} i(\frac{\partial}{\partial x_l})(dx_J). \tag{9}$$

For a complex manifold X, (9) implies that if $\alpha = \Sigma_{I,J}\alpha_{I,J} dz_I \wedge d\overline{z_J}$, then:

$$\partial^*\alpha = -\Sigma_{l=1}^n \Sigma_{I,J} \frac{\partial \alpha_{I,J}}{\partial z_l} i(\frac{\partial}{\partial \overline{z_l}})(dz_I \wedge d\overline{z_J})$$

and

$$\overline{\partial}^*\alpha = -\Sigma_{l=1}^n \Sigma_{I,J} \frac{\partial \alpha_{I,J}}{\partial \overline{z_l}} i(\frac{\partial}{\partial z_l})(dz_I \wedge d\overline{z_J}).$$

Proof of Lemma 3.15. Fix $\alpha \in \mathcal{A}_X^{p,q}$, then:

$$[\overline{\partial}^*, L](\alpha) = -\Sigma_{l=1}^n i(\frac{\partial}{\partial \overline{z_l}})\frac{\partial(}{\partial \omega} \wedge \alpha)z_l + \omega \wedge \Sigma_{l=1}^n i(\frac{\partial}{\partial \overline{z_l}})(\frac{\partial \alpha}{\partial z_l}).$$

Since $X = \mathbb{C}^n$ with the standard hermitian structure, $\frac{\partial(\omega\wedge\alpha)}{\partial z_l} = \omega \wedge \frac{\partial \alpha}{\partial z_l}$, and

$$i(\frac{\partial}{\partial \overline{z_l}})(\omega \wedge \frac{\partial \alpha}{\partial z_l}) = i(\frac{\partial}{\partial \overline{z_l}}))(\omega) \wedge \frac{\partial \alpha}{\partial z_l} + \omega \wedge i(\frac{\partial}{\partial \overline{z_l}})(\frac{\partial \alpha}{\partial z_l}).$$

Since $i(\frac{\partial}{\partial \overline{z_l}})(i\Sigma dz_j \wedge d\overline{z_j}) = -i dz_l$,

$$[\overline{\partial}^*, L](\alpha) = i\Sigma_{l=1}^n dz_l \wedge \frac{\partial \alpha}{\partial z_l} = i\partial\alpha.$$

□

Theorem 3.16. *Let (X, ω) be a Kähler manifold. The Laplace operators $\Delta_d, \Delta_\partial$ and $\Delta_{\overline{\partial}}$ associated to d, ∂ and $\overline{\partial}$ satisfy:*

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}.$$

Proof. We use the Kähler identities of Proposition 3.14 to express the d-Laplacian in terms of the ∂-Laplacian. Write:

$$\begin{aligned} \Delta_d = dd^* + d^*d &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\ &= (\partial + \overline{\partial})(\partial^* - i[\Lambda, \partial]) + (\partial^* - i[\Lambda, \partial]) \\ &= \Delta_\partial - i\partial[\Lambda, \partial] - i\overline{\partial}[\Lambda, \partial] + \overline{\partial}^*\partial - i[\Lambda, \partial] - i[\Lambda, \partial]\overline{\partial} \end{aligned}$$

Using that $\overline{\partial}\partial^* = i\overline{\partial}[\Lambda, \overline{\partial}] = i\overline{\partial}\Lambda\overline{\partial}$ and $\partial^*\overline{\partial} = -i\overline{\partial}\Lambda\overline{\partial}$, and $\partial\overline{\partial} = -\overline{\partial}\partial$, $\partial^*\overline{\partial}^* = -\overline{\partial}^*\partial^*$, the expression above reduces to:

$$\Delta_d = \Delta_\partial + i\partial[\Lambda, \overline{\partial}] + i[\Lambda, \overline{\partial}]\partial = 2\Delta_\partial.$$

The proof of $\Delta_d = 2\Delta_{\overline{\partial}}$ is similar. □

Corollary 3.17. *If X is Kähler, the d-Laplacian is bihomogeneous–i.e. if $\omega \in \mathcal{A}_X^{p,q}$, $\Delta_d(\omega) \in \mathcal{A}_X^{p,q}$. If α is a d-harmonic form and if $\alpha = \Sigma\alpha^{p,q}$ is the decomposition of α into forms of type (p, q), then for each (p, q), $\alpha^{p,q}$ is a d-harmonic form.*

Proof. Since $\ker \Delta_d = \ker \Delta_\partial = \ker \Delta_{\overline{\partial}}$, d-harmonic forms are ∂ and $\overline{\partial}$-harmonic, the result then follows (see Remark 3.4). □

The properties of $\overline{\partial}$-harmonic forms on X, and the identification between d, ∂ and $\overline{\partial}$-harmonic forms then imply the Hodge Decomposition for compact Kähler manifolds:

Theorem 3.18 (Hodge Decomposition)**.** *Let X be a compact Kähler manifold, there is a decomposition:*

$$H^k(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X)$$

such that $H^{p,q}(X) \simeq \overline{H^{q,p}(X)}$.

Remark 3.19. The isomorphism between the Betti cohomology group and the direct sum of the Dolbeault cohomology groups in the above statement reflects the fact that this follows from the isomorphisms between the cohomology groups and harmonic forms; these depend on the choice of metric *a priori*.

Proposition 3.20. *The Hodge decomposition* $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$ *does not depend on the choice of Kähler metrics on* X*. The isomorphism* $H^{p,q}(X) \simeq \overline{H^{q,p}(X)}$ *does not depend on the choice of Kähler metrics.*

Proof. Let $K^{p,q} \subset H^k(X,\mathbb{C})$ be the set of De Rham classes that can be represented by a form of type (p,q) on X. Since the Dolbeault classes are identified with $\overline{\partial}$-harmonic forms of type (p,q), $H^{p,q}(X) \subset K^{p,q}$. We want to show the reverse inclusion. Let ω be a closed (p,q)-form corresponding to a class $[\omega] \in K^{p,q}$. By Corollary 3.9,

$$\omega = \alpha^{p,q} + \Delta_d \eta,$$

for $\alpha^{p,q}$ a harmonic form of type (p,q) and $\eta \in \mathcal{A}_X^{p,q}$. Since $d\omega = 0$, $dd^*d\eta = 0$, and $d^*d\eta \in \ker d \cap \operatorname{im} d^*$. By Corollary 3.9 again, we obtain that $d^*d\eta = 0$, so that $[\omega] = [\alpha^{p,q}]$ which is canonically a class in $H^{p,q}(X)$. The identification $K^{p,q} = H^{p,q}(X)$ does not depend on the choice of Kähler metric.

Similarly, since $K^{p,q} = \overline{K^{q,p}}$, the second assertion holds. □

The existence of a unique harmonic representative in each (Dolbeault or De Rham) cohomology class also implies:

Corollary 3.21. *If a (Dolbeault or De Rham) cohomology class on* X *is representable by a form of type* (p,q) *and by a form of type* (p',q') *for* $(p,q) \neq (p',q')$*, it is zero.*

The properties of harmonic forms explained in Section 3 imply the following duality statements on cohomology groups of a Kähler manifold. The map

$$\alpha \in \mathcal{H}_{\overline{\partial}}^{p,q}(X) \mapsto \star\overline{\alpha} \in \mathcal{H}_{\overline{\partial}}^{n-p,n-q}$$

defines a duality $\mathcal{H}_{\overline{\partial}}^{p,q}(X) \simeq (\mathcal{H}_{\overline{\partial}}^{n-p,n-q})^*$. This implies that the Dolbeault cohomology groups of X satisfy *Serre Duality*:

$$H^{p,q}(X) \simeq H^{n-p,n-q}(X)^*.$$

The Hodge Decomposition Theorem then implies *Poincaré Duality*:

$$H^k(X,\mathbb{C}) = \bigoplus H^{p,q}(X) \simeq \bigoplus H^{n-p,n-q}(X)^* = H^{2n-k}(X,\mathbb{C})^*.$$

The Hodge Decomposition Theorem imposes conditions on the cohomology of compact Kähler manifolds.

Corollary 3.22. *Let* X *be a compact Kähler manifold.*

- *For all* $0 \leq p, q \leq n$, $h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-q,n-p}$,
- $b_k = \Sigma_{p+q=k} h^{p,q}$, *and in particular,* b_k *is even when* k *is odd,*
- $H^{k,k}(X) \neq (0)$ *for* $0 \leq k \leq n$.

Proof. The first statement follows from the action of complex conjugation and Serre Duality, the second from the Hodge decomposition theorem. The last statement is a consequence of Remark 4.44. □

Example 3.23. The Hopf surface (see Example sheet 1) is not Kähler because $b_1(S) = 1$. In fact, for a complex compact surface S, it can be shown that S is Kähler precisely when $b_1(S)$ is even.

Exercise 3.24. Show that $H^{p,q}(\mathbb{P}^n) \simeq \mathbb{C}$ when $0 \leq p = q \leq n$, and $H^{p,q}(\mathbb{P}^n) = (0)$ otherwise.

The Kähler condition implies that the spaces of harmonic forms for the three Laplace operators on X coincide. A reformulation of this is the following lemma, which allows one to identify the Dolbeault cohomology groups with the Bott-Chern cohomology groups (see ex.2 on Example Sheet 5.), whose definition is independent of the Kähler metric.

Lemma 3.25 (The $\partial\overline{\partial}$ lemma)**.** *Let X be a compact Kähler manifold and α a d-closed form of type (p,q). The following are equivalent:*

1. *α is d-exact, i.e. $\alpha = d\beta$ for $\beta \in \mathcal{A}^{k-1}(X)$,*
2. *α is ∂-exact, i.e. $\alpha = \partial\beta$ for $\beta \in \mathcal{A}^{p-1,q}(X)$,*
3. *α is $\overline{\partial}$-exact, i.e. $\alpha = \overline{\partial}\beta$ for $\beta \in \mathcal{A}^{p,q-1}(X)$,*
4. *α is $\partial\overline{\partial}$-exact, i.e. $\alpha = \partial\overline{\partial}\beta$ for $\beta \in \mathcal{A}^{p-1,q-1}(X)$,*

Proof. It is enough to prove the equivalence of (4) and (2) for instance, because on a Kähler manifold, $\overline{\partial}\partial = -\partial\overline{\partial}$ and (4) implies (1–3). Assume that $\alpha \in \mathcal{A}^k(X)$ is a closed form; since $d\alpha = 0$, $\partial\alpha = \overline{\partial}\alpha = 0$. Assume that $\alpha = \partial\beta$ for some form $\beta \in \mathcal{A}^{p-1,q}(X)$. By Corollary 3.9, $\beta = \gamma + \Delta_{\overline{\partial}}\epsilon$ for some $\overline{\partial}$-harmonic form γ and some $\epsilon \in \mathcal{A}^{p,q}(X)$. The term $\Delta_{\overline{\partial}}\epsilon$ can be written $\overline{\partial}\epsilon' + \overline{\partial}^*\epsilon$" for forms $\epsilon' \in \mathcal{A}^{p,q-1}(X)$ and ϵ" $\in \mathcal{A}^{p,q+1}(X)$. It is enough to prove that $\partial\overline{\partial}^*\epsilon$" $= 0$. Since $\overline{\partial}\alpha = 0$, we have $\overline{\partial}\partial\beta = 0$. It follows that $\overline{\partial}\partial\overline{\partial}^*\epsilon$" $= 0$. We may write $\overline{\partial}\partial\overline{\partial}^*\epsilon$" $= -\overline{\partial}\overline{\partial}^*\partial\epsilon$" because X is a complex manifold, so that $\partial\overline{\partial} = -\overline{\partial}\partial$ and $\overline{\partial}^*\partial = -\partial\overline{\partial}^*$. But then the L^2-product

$$(\partial\epsilon", \overline{\partial}\overline{\partial}^*\partial\epsilon") = ||\overline{\partial}^*\partial\epsilon"||^2 = 0$$

and $\overline{\partial}^* \partial \epsilon" = 0$ and $\partial \overline{\partial}^* \epsilon" = 0$, so that $\alpha = \partial \beta$ is of the form $\partial(\gamma + \overline{\partial} \epsilon')$ and the result follows. □

Remark 3.26. The $\partial\overline{\partial}$-lemma is a direct consequence of the identification of the spaces of harmonic forms for the three Laplace operators and of the fact that there is a unique harmonic representative for each cohomology class for De Rham, Dolbeault or Bott-Chern cohomology.

Exercise 3.27. Let X be a Kähler manifold. Show that any holomorphic p-form $\alpha \in \Gamma(X, \Omega_X^p) = H^{p,0}(X)$ is harmonic.

3.3 The Lefschetz Decomposition

The De Rham cohomology groups $H^k(X, \mathbb{R})$ of a compact Kähler manifold X admit another decomposition, which is of a topological nature. The Lefschetz operator L and its adjoint Λ (see 3.13) define linear operators on differential forms.

Exercise 3.28. L and Λ commute with the Laplace operator Δ_d.

Lemma 3.29. *The commutator* $[L, \Lambda]$ *restricts to* $(k - n)\,\mathrm{Id}$ *on* $\mathcal{A}_X^k$.

Proof. Since L and Λ are operators of degree 0, we may assume that (X, ω) is $\mathbb{C}^n$ equipped with the standard Hermitian structure in order to compute their commutator; we therefore consider:

$$\omega = \frac{i}{2} \Sigma_{j=1}^n dz_j \wedge d\overline{z_j}.$$

Recall from Definition-Lemma 3.13 that $\Lambda = \star^{-1} L \star$. On $\mathcal{A}_X^k$, we have (check it!)

$$\star^{-1} \circ (d\overline{z_j} \wedge) \circ \star = (-1)^{k+1} 2i(\frac{\partial}{\partial z_j}), \text{ and } \star^{-1} \circ (dz_j \wedge) \circ \star = (-1)^{k+1} 2i(\frac{\partial}{\partial \overline{z_j}}).$$

where $i(u)$ for a vector field u denotes the interior product with u. This shows that for every j,

$$\star^{-1} \circ (\frac{i}{2}(d\overline{z_j} \wedge d\overline{z_j})\wedge) \circ \star = -2i \cdot i(\frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial \overline{z_j}}),$$

and

$$[L, \Lambda] = \Sigma_{j,k} [(d\overline{z_j} \wedge d\overline{z_j})\wedge, i(\frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial \overline{z_k}})].$$

It is clear that each operator $[(d\overline{z_j}\wedge d\overline{z_j})\wedge, i(\frac{\partial}{\partial z_k}\wedge\frac{\partial}{\partial \overline{z_k}})]$ is zero unless $j=k$, so that

$$[L,\Lambda]=\Sigma_j[(d\overline{z_j}\wedge d\overline{z_j})\wedge, i(\frac{\partial}{\partial z_j}\wedge\frac{\partial}{\partial \overline{z_j}})].$$

Any form $\alpha\in\mathcal{A}_X^k$ can be written $\alpha=\Sigma_{I,J,M}\alpha_{I,J,M}dz_I\wedge d\overline{z_J}\wedge\omega_M$, where $I,J,M\subset\{1,\cdots,n\}$ are disjoint sets of indices, and $\omega_M=dz_{i_1}\wedge d\overline{z_{i_1}}\wedge\cdots\wedge dz_{i_m}\wedge d\overline{z_{i_m}}$, for $M=\{i_1<\cdots<i_m\}$. The sets I,J,K satisfy $|I|+|J|+2|M|=k$.

Write $K=\{1,\cdots,n\}\setminus(I\cup J\cup M)$, then

$$dz_l\wedge d\overline{z}_l\wedge(\alpha_{I,J,M}dz_I\wedge d\overline{z_J}\wedge\omega_M)=0 \text{ unless } l\in K \text{ , and}$$

$$i(\frac{\partial}{\partial z_l}\wedge\frac{\partial}{\partial \overline{z_l}})(\alpha_{I,J,M}dz_I\wedge d\overline{z_J}\wedge\omega_M))=0 \text{ unless } l\in M$$

. It follows that:

$$[L,\Lambda](\alpha_{I,J,M}dz_I\wedge d\overline{z_J}\wedge\omega_M)=(|M|-|K|)\cdot\alpha_{I,J,M}dz_I\wedge d\overline{z_J}\wedge\omega_M,$$

and since $|M|-|K|=k-n$, the result follows. □

Lemma 3.30. *The operator $L^{n-k}\colon\Omega^k_{X,\mathbb{R}}\to\Omega^{2n-k}_{X,\mathbb{R}}$ is an isomorphism (and hence $L^{n-k}\colon\mathcal{A}^k_X\to\mathcal{A}^{n-k}_X$ is an isomorphism of sheaves).*

Proof. Since $\operatorname{rk}\Omega^k_{X,\mathbb{R}}=\operatorname{rk}\Omega^{2n-k}_{X,\mathbb{R}}$, it is enough to prove that L^{n-k} is injective. By induction, one shows that $[L^r,\Lambda]=\{r(k-n)+r(r-1)\}L^{r-1}$ for all $r\geq 1$. If $L^r\alpha=0$ for some $\alpha\in\Omega^k_{X,\mathbb{R}}$, assume that L^{r-1} is injective, then since

$$L^{r-1}(L\Lambda-(r(k-n)+r(r-1))\operatorname{Id})(\alpha)=0,$$

α is of the form $L\beta$ for some $\beta\in\Omega^{k-2}_X$ with $L^{r+1}\beta=0$. Conclude by induction of the degree of α. □

Denote $\Pi^k\colon\mathcal{A}^*_X\to\mathcal{A}^k_X$ be the projection of differential forms on their degree k component, and define the operator:

$$h=\Sigma_{k=0}^{2n}(n-k)\Pi^k.$$

Then, the operators h,L and Λ satisfy the commutator relations:

$$[h,\Lambda]=2\Lambda \text{ ; } [h,L]=-2L \text{ , and } [\Lambda,L]=h. \tag{10}$$

The operators L,Λ and h all commute with Δ_d, they act on the space of harmonic forms $\mathcal{H}^*_d(X)$, and hence on the cohomology ring $H^*(X,\mathbb{R})$. This proves:

Lemma 3.31. *Let X be a compact Kähler manifold. The operators L, Λ and h induce a representation of $\mathfrak{sl}_2(\mathbb{C})$ on the cohomology of X.*

Remark 3.32. These operators also define a representation of $\mathfrak{sl}_2$ on the space of differential forms $\mathcal{A}_X^*$; we concentrate on the representation on cohomology because $H^*(X, \mathbb{R})$ is finite dimensional, and we will apply some results from the theory of finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

Representations of $\mathfrak{sl}_2(\mathbb{C})$ Recall that $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra of $\mathrm{SL}(2, \mathbb{C})$, it is the space of 2×2 matrices of trace 0. A basis of $\mathfrak{sl}_2(C)$ is given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

that satisfy the commutator relations $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$.

Definition 3.33. Let V be a finite dimensional $\mathbb{C}$-vector space, and $\mathfrak{gl}(V)$ the Lie algebra of endomorphisms of V (endowed with the commutator as Lie bracket). A Lie algebra representation of $\mathfrak{sl}_2(\mathbb{C})$ on V is a Lie algebra homomorphism

$$\rho\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V),$$

i.e. such that for any $A, B \in \mathfrak{sl}_2(\mathbb{C})$, $\rho([A, B]) = \rho(A)\rho(B) - \rho(B)\rho(A)$. The vector space V is a $\mathfrak{sl}_2$-*module.*

Any subspace $W \subset V$such that $\rho(W) \subset W$ is an $\mathfrak{sl}_2$-subrepresentation; for any such subrepresentation, there is a well defined complement $W^\perp$ such that $V = W \oplus W^\perp$.

A representation V is *irreducible* if there is no non-trivial subrepresentation.

Any finite dimensional representation V of $\mathfrak{sl}_2$ is a direct sum of irreducible representations. When the representation ρ is fixed, we write H, X, Y for $\rho(H), \rho(X)$ and $\rho(Y)$.

Let λ be an eigenvalue for H; λ is called a *weight* for ρ, and the associated eigenspace V_λ is a *weight space.* If λ is aneigenvalue for $\rho(H)$ and $v \in V_\lambda$,

$$Hv = \lambda \cdot v \ , \ HXv = (\lambda + 2)v \text{ and } HYv = (\lambda - 2)v.$$

Since V is assumed to be finite dimensional, there is a finite number of weights, and X and Y are nilpotent. A *primitive vector* $v \in V$ is an eigenvalue for H such that $Xv = 0$. The vector subspace of primitive elements is denoted PV.

Proposition 3.34. *Let V be an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. If $v \in V$ is a primitive element, V is generated as a vector space by the elements $v, Yv, Y^2v, \cdots$.*

Proof. It is easy to check that the elements $Y^n v$ are linearly independent, and that they generate a subrepresentation of $\mathfrak{sl}_2(\mathbb{C})$. □

The weights of ρ are easily seen to be integers, the irreducible representation of $\mathfrak{sl}_2$ are of the form $V(n)$, where $V(n) \simeq \operatorname{Sym}^n \mathbb{C}^2$, where n is the order of nilpotence of a primitive element $v \in V$ for the operator Y. Each such irreducible representation has dimension $n+1$, and

$$V(n) = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n,$$

where each V_{n-2i} is generated by the element $Y^i v$.

Any representation V can therefore be written:

$$V = V_n \oplus V_{n-2} \oplus \cdots \oplus V_{-n+2} \oplus V_{-n} \tag{11}$$

The *Lefschetz decomposition theorem* states that there is a direct sum decomposition

$$V = PV \oplus YPV \oplus Y^2PV \oplus \cdots \oplus Y^n PV, \tag{12}$$

that is compatible with the decomposition (11). In particular, the weight spaces V_m and V_{-m} are isomorphic, with $X^m \colon V_{-m} \to V_m$ and $Y^m \colon V_m \to V_{-m}$. The primitive elements in V_m are

$$PV \cap V_m = \ker(Y^{m+1} \colon V_m \to V_{-m-2}).$$

The cohomology $H^*(X, \mathbb{R})$ is an $\mathfrak{sl}_2$-representation, defined by

$$X \mapsto \Lambda \ , \ Y \mapsto L \ , \text{ and } H \mapsto H.$$

The weight space for the eigenvalue $n-k$ of H is $H^k(X, \mathbb{R})$. The results of the theory of representations of $\mathfrak{sl}_2(\mathbb{C})$ imply:

Theorem 3.35 (Hard Lefschetz theorem)**.** *Let X be a compact Kähler manifold of dimension n. The operator L^k defines an isomorphism*

$$L^k \colon H^{n-k}(X, \mathbb{R}) \to H^{n+k}(X, \mathbb{R}).$$

Define

$$P^{n-k} = \ker(\Lambda) \cap H^{n-k}(X, \mathbb{R}) = \ker(L^{k+1} \colon H^{n-k}(X, \mathbb{R}) \to H^{n+k+2}(X, \mathbb{R})),$$

there is a Lefschetz decomposition *on the cohomology of X:*

$$H^m(X, \mathbb{R}) = \bigoplus_{2k \leq m} L^k P^{m-2k}(X).$$

Remark 3.36. Note that since the operator L is of type $(1,1)$ on forms and since L and Λ commute with Δ_d, the Lefschetz decomposition is compatible with the Hodge Decomposition.

Remark 3.37. By definition of the operator L on $H^*(X,\mathbb{R})$, the Lefschetz decomposition depends on the choice of Kähler class $[\omega] \in H^2(X,\mathbb{R})$; it does not depend on the choice of Kähler metric itself (that is, it does not depend on the representative in a fixed class).

Since the class ω is $\overline{\partial}$-closed, for all $p+q \leq n$, L^{n-p-q} defines an isomorphism

$$L^{n-p-q}\colon H^{p,q}(X) \to H^{n-p,n-q}(X). \tag{13}$$

Exercise 3.38. Show that if X is a compact Kähler manifold, then for all p,q such that $k = p+q \leq n$ (resp. $k = p+q \geq n$), $h^{p-1,q-1} \leq h^{p,q}$ (resp. $h^{p-1,q-1} \geq h^{p,q}$) and $b_k \leq b_{k+2}$ (resp. $b_k \geq b_{k+2}$).

Define an *intersection form* on the cohomology of X:

$$Q\colon H^k(X,\mathbb{R}) \times H^{2n-k}(X,\mathbb{R})t\ \mathbb{R}$$
$$Q(\alpha,\beta) = \int_X \omega^{n-k} \wedge \alpha \wedge \beta =< L^{n-k}\alpha, \beta > .$$

When k is even, Q is symmetric; when k is odd, Q is alternating. Define a Hermitian form on $H^k(X,\mathbb{C})$ by:

$$H_k(\alpha,\beta) = i^k Q(\alpha,\overline{\beta}). \tag{14}$$

Lemma 3.39. *The Lefschetz Decomposition*

$$H^k(X,\mathbb{C}) = \bigoplus_{2r\leq k} L^r P^{k-2r}(X,\mathbb{C})$$

is orthogonal for H_k. On each summand $L^rP^{k-2r}(X,\mathbb{C})$, the form H_k induces $(-1)^r H_{k-2r}$.

Proof. Assume that $\alpha = L^r\alpha'$ and $\beta = L^s\beta'$ for some primitive classes $\alpha' \in P^{k-2r}(X,\mathbb{C})$ and $\beta' \in P^{k-2s}(X,\mathbb{C})$, where $s \geq r$. Then:

$$H_k(\alpha,\beta) = \int_X L^{n-k}L^r\alpha' \wedge L^s\overline{\beta'} = (-1)^r \int_X L^{n-k+r+s}\alpha' \wedge \overline{\beta'}.$$

This vanishes if $s \neq r$ because $\alpha' \in \ker L^{n-k+2r+1}$, and coincides with $(-1)^r H_{k-2r}(\alpha',\overline{\beta'})$ otherwise. □

Lemma 3.40. *The Hodge Decomposition* $H^k(X,\mathbb{C}) = \bigoplus H^{p,q}(X)$ *is an orthogonal direct sum for* H_k.

If $H^{p,q}(X)_{prim} = H^{p,q}(X) \cap P^k(X,\mathbb{C})$*, the form* $(-1)^{k(k-1)/2} i^{p-q-k} H_k$ *is positive definite on* $H^{p,q}(X)_{prim}$.

Proof. If $[\alpha^{p,q}]$ and $[\beta^{p',q'}]$ are classes in $H^k(X,\mathbb{C})$, $L^{n-k}\alpha^{p,q} \wedge \overline{\beta^{p',q'}}$ is of type $(n-k+p+q', n-k+p'+q)$; this vanishes if $(p,q) \neq (p',q')$ because $H^{2n}(X,\mathbb{C}) = H^{n,n}(X)$. The Hodge Decomposition on H^k is therefore orthogonal for H_k. The second part follows from noting that if a cohomology class is represented by a harmonic form α, with α primitive, then α is primitive at every point $x \in X$, and $\overline{\alpha}$ is primitive because L and Λ are real operators. □

The Hodge Index Theorem (This was not covered in Lectures)

Exercise 3.41. If $\omega \in \Omega^{p,q}_{X,x}$ is primitive, then

$$\star\omega = (-1)^{k(k+1)/2} i^{p-q} \frac{L^{n-k}\omega}{(n-k)!}.$$

Theorem 3.42 (Hodge Index Theorem)**.** *Let X be a compact Kähler manifold of even dimension* $\dim_{\mathbb{C}} X = n$*. The signature of the intersection form*

$$Q(\alpha,\beta) = \int_X \alpha \wedge \beta$$

on $H^n(X,\mathbb{R})$ *is equal to* $\Sigma_{p,q}(-1)^p h^{p,q}(X)$.

Proof. The signature of Q is equal to the signature of the Hermitian form $H(\alpha,\beta) = \int_X \alpha \wedge \overline{\beta}$. Since

$$H^n(X,\mathbb{C}) = \bigoplus_{2r\leq n} L^r P^{n-2r}(X,\mathbb{C}), \quad \bigoplus_{p+q=n-2r\geq 0} L^r H^{p,q}(X)_{prim}$$

and the sign of H on $L^r H^{p,q}(X)_{prim}$ is equal to $(-1)^p$ because n is even, so that:

$$\operatorname{sgn} Q = \Sigma_{p+q=n-2r}(-1)^p h^{p,q}(X)_{prim}.$$

By Exercise 3.38, this is equal to

$$\operatorname{sgn} Q = \Sigma_{p+q=n-2r, 2r\leq n}(-1)^p (h^{p,q}(X) - h^{p-1,q-1}(X)),$$

By Poincaré Duality, this is equal to:

$$\operatorname{sgn} Q = \Sigma_{p+q\equiv n \mod 2}(-1)^p h^{p,q}(X),$$

and since, by complex conjugation, $\Sigma_{p+q\equiv 1 \mod 2}(-1)^p h^{p,q}(X) = 0$, this is:

$$\operatorname{sgn} Q = \Sigma_{p,q}(-1)^p h^{p,q}(X).$$

□

Exercise 3.43. Let X be a compact Kähler surface. Show that the signature of the intersection form

$$Q(\alpha, \beta) = \int_X \alpha \wedge \beta$$

on $H^2(X, \mathbb{R}) \cap H^{1,1}(X)$ is $(1, h^{1,1} - 1)$.

4 Hodge Structures

We have seen that the cohomology groups of compact Kähler manifolds are endowed with a very rich structure. If X is a compact Kähler manifold, the decomposition of differential forms into components of different types descends to its Betti or De Rham cohomology groups $H^k(X, \mathbb{C}) \simeq H^k_{DR}(X, \mathbb{C})$ and yield a direct sum decomposition:

$$H^k(X, \mathbb{C}) = \bigoplus_{p=0}^{k} H^p(X, \Omega_X^{k-p}) = \bigoplus_{p=0}^{k} H^{p,k-p}(X),$$

where the summands are related by $H^{p,k-p}(X) = \overline{H^{k-p,p}(X)}$. This decomposition is obtained at level of harmonic forms, that is solutions of partial differential equations on X. The Betti cohomology groups also contain a lattice, the integral cohomolohy modulo torsion which is the image of $H^k(X, \mathbb{Z}) \otimes \mathbb{C} \to H^k(X, \mathbb{C})$. Further, cup product with the class of the Kähler form induces a Lefschetz decomposition which is compatible with the Hodge Decomposition, and reflects the existence of a polarisation on the cohomology of X.

These structures are an example of the more general notion of *Hodge structures*, which will be well suited to the study of deformations of Kähler or projective manifolds.

4.1 Hodge Structures

Definition 4.1. An *integral Hodge structure of weight k* is a free abelian group of finite type $V_{\mathbb{Z}}$ and a decomposition on the complexification:

$$V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$

such that $V^{p,q} = \overline{V^{q,p}}$.

Remark 4.2. One can define analogously rational Hodge structures, where the lattice $V_{\mathbb{Z}}$ is replaced with a finite dimensional $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$.

Remark 4.3. It is sometimes useful to identify Hodge Structures with certain representations of $\mathbb{C}^*$ on $V_{\mathbb{R}}$. For more on this, see Exercise 5, Example Sheet 6.

Example 4.4. Let X be a compact Kähler manifold, then $(H^k_{\mathbb{Z}}, H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X))$ is a Hodge structure of weight k, where $H^k_{\mathbb{Z}}$ is the torsion free part of the integral cohomology $H^k(X,\mathbb{Z})$ (i.e. the image of $H^k(X,\mathbb{Z})$ in $H^k(X,\mathbb{Q}), H^k(X,\mathbb{R})$ or $H^k(X,\mathbb{C})$).

Example 4.5. The *Tate twist* $\mathbb{Q}(k)$ is the 1-dimensional weight $-2k$ rational Hodge structure with decomposition:

$$(\mathbb{Q}(k)\otimes\mathbb{C})^{p,q} = (0) \text{ if } (p,q) \neq (-k,-k) \text{ and } \mathbb{Q}(k)\otimes\mathbb{C})^{-k,-k} = \mathbb{C}.$$

Note that $\mathbb{Q}(-k)$ is the Hodge structure on the degree $2k$ cohomology of $\mathbb{P}^n$ for $n \geq k$. (This is the Hodge structure associated to the representation $z \mapsto z^{-k}\overline{z}^{-k}$.)

Example 4.6. Let V be a rational vector space such that $V_{\mathbb{R}}$ is endowed with an almost complex structure J. Then the exterior product $\bigwedge^k V_{\mathbb{C}}$ always admit a bidegree decomposition $V^{p,q}$, which makes it into a Hodge structure of weight k. For example, when (X,h) is a Hermitian manifold, $\Omega^k_{X,x}$ has a Hodge structure of weight k for all $x \in X$, but this Hodge structure does not descends to cohomology unless (X,h) is Kähler.

Example 4.7. The Dolbeault cohomology group $H^q(X,\Omega^p_X)$ of a Kähler manifold is endowed with a weight $p+q$ Hodge structure– where the (p,q)-summand is $H^q(X,\Omega^p_X)$, and the others are zero.(This is the Hodge structure associated to the representation $z \mapsto z^p\overline{z}^{-q}$.)

Example 4.8. Let (X,ω) be a Kähler manifold, such that $[\omega] \in H^k(X,\mathbb{Z})$ (resp. $[\omega] \in H^k(X,\mathbb{Q})$). The primitive cohomology $P^k(X,\mathbb{C})$ is a weight k integral (resp. rational) Hodge structure.

Definition 4.9. If $(V_{\mathbb{Z}}, V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q})$ is a weight k Hodge structure, the associated *Hodge Filtration* is the decreasing filtration

$$F^\bullet V = \{F^p V_{\mathbb{C}} = \bigoplus_{r\geq p} V^{r,k-r}\}.$$

Remark 4.10. A Hodge structure can alternatively be defined by its integral lattice and the Hodge filtration: the decomposition of $V_{\mathbb{C}} = \bigoplus V^{p,q}$ and the filtration $F^{\bullet}V$ are equivalent data. By definition of a Hodge Structure,

$$V_{\mathbb{C}} = F^p V_{\mathbb{C}} \oplus \overline{F^{k-p+1} V_{\mathbb{C}}} \text{ and } V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}.$$

In light of Example 4.1, it is natural to ask when Hodge structures on spaces of differential forms descend to Hodge structures on cohomology, or more generally:

Question 4.11. How do the decompositions on differential forms of degree k and the degree k cohomology classes interact?

Define a filtration on the sheaf of differential forms as follows:

$$F^p \mathcal{A}^k_X = \{\alpha \in \mathcal{A}^k(X) | \text{ if } U \subset X, \alpha_{|U} = \Sigma_{r \geq p} \alpha^{r,k-r}, \text{ with } \alpha^{r,k-r} \in \mathcal{A}^{r,k-r}_X(U)\},$$

and denote $\Pi^{p,k-p} \colon \mathcal{A}^k_X \to \mathcal{A}^{p,k-p}_X$ the projection maps.

Remark 4.12. The diagram

$$\begin{array}{ccc} \mathcal{A}^k(X) & \xrightarrow{d} & \mathcal{A}^{k+1}(X) \\ \downarrow{\scriptstyle \Pi^{p,k-p}} & & \downarrow{\scriptstyle \Pi^{p,k+1-p}} \\ \mathcal{A}^{p,k-p}(X) & \xrightarrow{\overline{\partial}} & \mathcal{A}^{p,k-p+1}(X) \end{array}$$

is not in general commutative when $p \neq 0$.

Exercise 4.13. Show that if X is compact and Kähler, the diagram

$$\begin{array}{ccccccc} \mathbb{C} & \xrightarrow{d} & \mathcal{A}^0(X) & \xrightarrow{d} & \mathcal{A}^1(X) & \xrightarrow{d} & \cdots \\ \downarrow & & \| & & \downarrow{\scriptstyle \Pi^{0,1}} & & \\ \mathcal{O}_X & \longrightarrow & \mathcal{A}^0(X) & \xrightarrow{\overline{\partial}} & \mathcal{A}^{0,1}(X) & \xrightarrow{\overline{\partial}} & \cdots \end{array}$$

where the first vertical map is induced by the inclusion of sheaves $0 \to \mathbb{C} \to \mathcal{O}_X$, and the other vertical maps are the projections to the $(0,q)$ component, is commutative. This shows that the map on cohomology $H^k(X,\mathbb{C}) \to H^k(X,\mathcal{O}_X)$ induced by the inclusion of sheaves $\mathbb{C} \subset \mathcal{O}_X$ coincides with that induced by the bidegree decomposition.

Proposition 4.14. *If X is a compact Kähler manifold, then:*

$$F^p H^k(X,\mathbb{C}) = \frac{\ker(d \colon F^p\mathcal{A}^k(X) \to F^p\mathcal{A}^{k+1}(X))}{\operatorname{im}(d \colon F^p\mathcal{A}^{k-1}(X) \to F^p\mathcal{A}^k(X))}.$$

Proof. There is a natural map f from $\ker(d\colon F^p\mathcal{A}^k(X) \to F^p\mathcal{A}^{k+1}(X))$ to $H^k(X,\mathbb{C})$ which sends a closed form α to its De Rham cohomology class. The image of f is contained in $F^pH^k(X,\mathbb{C})$.

By Corollary 3.11, every class in $F^pH^k(X,\mathbb{C})$ has a unique harmonic representative. Let $[\gamma] \in F^pH^k(X,\mathbb{C})$ be a cohomology class with γ harmonic. Since $\gamma \in \ker(d\colon F^p\mathcal{A}^k(X) \to F^p\mathcal{A}^{k+1}(X))$, the map f is surjective. We want to show that if $\alpha \in \ker f$, $\alpha \in \operatorname{im}(d\colon F^p\mathcal{A}^{k-1}(X) \to F^p\mathcal{A}^k(X))$. By induction, first assume that $\alpha \in F^p\mathcal{A}^p_X$ is such that $d\alpha = 0$ and $[\alpha] = 0$. The form $\alpha \in \mathcal{A}^{p,0}_X$, and by Lemma 3.25, $\alpha = \partial\overline{\partial}\beta$ with $\beta \in \mathcal{A}^{p-1,-1}_X$. This shows that $\alpha = 0$ because $\mathcal{A}^{p-1,-1}_X = 0$.

Assume that $\alpha \in F^p\mathcal{A}^k_X$ is a closed form such that $[\alpha] = 0$. Write $\alpha = \alpha^{p,q} + \alpha'$ for $\alpha' \in F^{p+1}\mathcal{A}^k_X$. Write $\alpha^{p,q} = \beta^{p,q} + \Delta_{\overline{\partial}}\epsilon^{p,q}$, where $\beta^{p,q}$ is $\overline{\partial}$ (and d)-harmonic. Since $\alpha^{p,q}$ is d-exact, by Lemma 3.25, the Dolbeault cohomology class of $\alpha^{p,q}$ is zero and $\beta^{p,q} = 0$, so that $\alpha^{p,q} = \Delta_{\overline{\partial}}\epsilon^{p,q}$. Since $\overline{\partial}\alpha^{p,q} = 0$, $\Delta_{\overline{\partial}}\epsilon^{p,q} = \overline{\partial}\overline{\partial}^*\epsilon^{p,q}$. The form $\overline{\partial}^*\epsilon^{p,q}$ is a section of $\mathcal{A}^{p,q-1}_X$, so that $\delta\overline{\partial}^*\epsilon^{p,q} \in \mathcal{A}^{p+1,q-1}_X$, so that the form $\alpha - d(\overline{\partial}^*\epsilon^{p,q}) \in F^{p+1}\mathcal{A}^k_X$, and this form is d-closed and d-exact. By induction, $\alpha - d(\overline{\partial}^*\epsilon^{p,q}) \in \operatorname{im}(d\colon F^{p+1}\mathcal{A}^{k-1}_X \to F^{p+1}\mathcal{A}^k_X)$, and hence $\alpha \in \operatorname{im}(d\colon F^p\mathcal{A}^{k-1}(X) \to F^p\mathcal{A}^k(X))$. □

An immediate consequence of this proposition is:

Corollary 4.15. *For all* $p \in \mathbb{N}$, $H^{p,0}(X) \simeq \Gamma(X,\Omega^p_X)$.

Polarisations

Definition 4.16. An integral *Polarised Hodge Structure* of weight k is an integral Hodge Structure of weight k $(V_\mathbb{Z}, V_\mathbb{C} = \bigoplus V^{p,q})$ endowed with an intersection form

$$Q\colon V_\mathbb{Z} \times V_\mathbb{Z} \to \mathbb{Z}$$

such that

1. Q is symmetric if k is even and alternating otherwise,
2. For all non-zero $\alpha \in V^{p,q}$, $i^{p-q-k}(-1)^{k(k-1)/2}H(\alpha) > 0$, where $H(\alpha,\beta) = i^kQ(\alpha,\overline{\beta})$.

Remark 4.17. In terms of representations of $\mathbb{C}^*$ on $V_\mathbb{R}$ (see Ex.5, ES 6), these conditions can be reformulated as follows: if

$$Q\colon V_\mathbb{Z} \times V_\mathbb{Z} \to \mathbb{Z},$$

setting $(\rho(z)\alpha, \rho(z)\beta) = z^k\overline{z}^k(\alpha,\beta)$, the form $(\cdot, \rho(i)\cdot)$ is symmetric and definite positive.

Example 4.18. Let (X, ω) be a Kähler manifold with $[\omega] \in H^2(X, \mathbb{Z})$. Then the primitive cohomology $P^k(X, \mathbb{C})$ is a polarised Hodge structure of weight k.

Remark 4.19. As above, taking $Q\colon V_{\mathbb{Q}} \times V_{\mathbb{Q}} \to \mathbb{Q}$, one defines a rational polarised Hodge structure. If the Kähler class is rational, the primitive cohomology $P^k(X, \mathbb{C})$ is a rational polarised Hodge structure of weight k.

Definition 4.20. A *polarised manifold* (X, ω) is a compact Kähler manifold whose Kähler class $[\omega]$ is integral, i.e. $[\omega] \in H^2(X, \mathbb{Z})$.

The significance of this notion is illustrated in the following theorem– which we have already seen (Theorem 2.33)– but recall here.

Theorem 4.21 (Kodaira Embedding Theorem)**.** *Let X be a compact complex manifold. The following are equivalent:*

1. *X is projective, i.e. there exists a holomorphic embedding $X \to \mathbb{P}^N$ for some $N \in \mathbb{N}$.*
2. *There exists an integral Kähler form ω on X; $[\omega] \in H^2(X, \mathbb{R}) \cap H^{1,1}(X)$.*
3. *There exists a positive holomorphic line bundle $L \to X$.*

Corollary 4.22. *Let X be a compact Kähler manifold. If $H^2(X, \mathcal{O}_X) = (0)$, X is projective.*

Proof. The compact manifold X is Kähler, hence $H^2(X, \mathcal{O}_X) = H^{0,2}(X) = \overline{H^{2,0}(X)}$, and $H^2(X, \mathbb{C}) = H^2(X, \mathbb{Q}) \otimes \mathbb{C} = H^{1,1}(X)$. Since $H^2(X, \mathbb{C}) \simeq \mathcal{H}^2(X)$, there exists a basis $\alpha_1, \cdots, \alpha_r$ of $H^2(X, \mathbb{Q})$ consisting of harmonic forms. The Kähler form ω itself is harmonic (see Ex.7, ES 5), and real, so it is a linear combination

$$\omega = \Sigma_{j=1}^r \lambda_j \alpha_j,$$

where $\lambda_j \in \mathbb{R}$ for all j. By Ex. 6, ES 5, $\mathcal{K}_X \subset H^{1,1}(X)$, the set of Kähler classes on X, is an open convex cone, so that if $q_j \in \mathbb{Q}$ are such that $|q_j - \lambda_j| \leq \varepsilon$ for some $0 < \varepsilon \ll 1$, and the class of

$$\omega' = \Sigma_{j=1}^r q_j \alpha_j$$

is Kähler and lies in $H^2(X, \mathbb{Q})$. An appropriate multiple of ω' is then an integral Kähler form, and $(X, [n\omega'])$ is a polarised manifold. The result follows by the Kodaira Embedding Theorem. ☐

4.2 Weight 1 Hodge Structures and complex tori

We first establish some results on the cohomology of complex tori.

Let $V_{\mathbb{R}}$ be a $2n$-dimensional $\mathbb{R}$-vector space, and let $\mathbb{Z}^{2n} \simeq \Lambda \subset V_{\mathbb{R}}$ be a full lattice. The complexified vector space $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$ has a natural decomposition $V \oplus \overline{V}$. The image of Λ by the inclusion map $V_{\mathbb{R}} \to V_{\mathbb{C}}$ is a full lattice $\mathbb{Z}^{2n}$ in $V \simeq \mathbb{C}^n$, and $T = V/\Lambda$ is a complex torus.

Exercise 4.23. Every complex torus $T = V/\Lambda$ is of this form (set $V_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$).

The complex torus T inherits a group structure from Λ as follows. Let:

$$\pi \colon V \to T$$

be the natural projection map. If $a \in T$, let v_a be a point in V with $\pi(v_a) = a$. The *translation* automorphism $\tau_a \colon T \to T$ is defined by $\tau_a(x) = \pi(v + v_a)$, where $v \in V$ is such that $\pi(v) = x$.

Exercise 4.24. Check that the automorphism τ_a is well defined.

Near a point $P \in T$, a complex basis of V yields local holomorphic coordinates of T, and there is a natural identification

$$T^{1,0}_{T,p} \simeq V.$$

A Hermitian inner product on V yields a Kähler metric on T, which is invariant under translations τ_a for $a \in T$. We assume that T is endowed with such a metric.

Denote $\mathcal{I}^*(T) = \{\alpha \in \mathcal{A}^*_T | \tau_a^*(\alpha) = \alpha \text{ for all } a \in T\}$

Lemma 4.25. *The space of harmonic forms $\mathcal{H}^*(T)$ coincides with $\mathcal{I}^*(T)$. If $z_1, \cdots, z_n$ are local holomorphic coordinates on T induced by a basis of V, $\mathcal{H}^*(T) = \mathbb{C} \cdot \{dz_I \wedge d\overline{z}_J\}_{I,J}$, for $I, J \subset \{1, \cdots n\}$.*

Proof. Let $a \in T$, since $\tau_a^* \colon \mathcal{A}^*_T \to \mathcal{A}^*_T$ preserves the metric, $\tau_a^* \colon \mathcal{H}^*(T) \to \mathcal{H}^*(T)$. The space $\mathcal{H}^*(T)$ is isomorphic to $H^*(T, \mathbb{C})$ because T is Kähler and τ_a^* is homotopic to the identity for all $a \in T$, it follows that $\mathcal{H}^*(T) \subset \mathcal{I}^*(T)$.

Conversely, if $\alpha \in \mathcal{I}^*(T)$, α is entirely determined by its value at any point $P \in T$, and since $T_{T,P,\mathbb{C}} = V \oplus \overline{V}$, $\mathcal{I}^*(T) \simeq \bigwedge^*(V^*) \otimes \bigwedge^*(\overline{V}^*)$, where V^* is the dual of V.

Since $T \simeq (S^1)^{2n}$, $b_k(T, \mathbb{C}) = \dim \mathcal{H}^k(T) = \dim \mathcal{I}^k(T)$, and this concludes the proof. ☐

The homology of T is well known–any loop γ with base point $0 \in T$ lifts to a path $\tilde{\gamma}$ in $V \simeq \mathbb{C}^n$, the universal covering space of T, with end point $\lambda \in \Lambda$, so that $H_1(T, \mathbb{Z}) \simeq \Lambda$. We have shown the following:

Lemma 4.26. *If T is a complex torus,*

$$T \simeq H^{1,0}(T)^*/H_1(T,\mathbb{Z}).$$

A complex torus T is entirely determined by the weight 1 Hodge Structure $(H^1(T,\mathbb{Z}), H^1(T,\mathbb{C}))$.

There are two distinguished bases of $H^k(T,\mathbb{Z})$:

- A $\mathbb{Z}$-basis $\{\lambda_1, \cdots, \lambda_{2n}\}$ of $H_1(T,\mathbb{Z})$ yields a dual basis $\{x_1, \cdots, x_{2n}\}$ for V as a $\mathbb{R}$-vector space. Denote $dx_1, \cdots, dx_n$ the associated 1-forms, then, by definition,
$$\int_{\lambda_i} dx_j = \delta_{i,j},$$
and $H^k(T,\mathbb{Z})$ is generated by forms dx_I, for $|I| = k$;
- Lemma 4.25 shows that $H^k(T,\mathbb{Z})$ is generated by forms $dz_I \wedge d\overline{z_J}$, with $|I| + |J| = k$.

The first basis reflects the integral structure, while the second reflects the complex structure. Conditions can be expressed for T to be polarised– or equivalently, projective–in terms of transition matrices between these bases (see [GH78]).

Let now X be a compact Kähler manifold, by Theorem 3.18, there is a decomposition:

$$H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \text{ and } H^{1,0}(X) \simeq \overline{H^{0,1}(X)}.$$

Assume that $H^i(X,\mathbb{Z})$ is torsion free for all $i > 0$–if this is not the case, we replace $H^i(X,\mathbb{Z})$ by its image in $H^i(X,\mathbb{R})$). The long exact sequence associated to the exponential exact sequence

$$0 \to \mathbb{Z} \overset{2i\pi\cdot}{\to} \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}_X^* \to 0 \tag{15}$$

shows that $H^1(X,\mathbb{Z}) \hookrightarrow H^1(X,\mathcal{O}_X)$. From the computations on singular cohomology, we have seen that:

$$H^1(X,\mathbb{Z}) \overset{i_1}{\hookrightarrow} H^1(X,\mathbb{R}) \overset{i_2}{\hookrightarrow} H^1(X,\mathbb{R}) \otimes \mathbb{C} = H^1(X,\mathbb{C}) \overset{pr}{\to} H^{0,1}(X),$$

and we have seen that $pr \circ i_2 \colon H^1(X,\mathbb{R}) \to H^{0,1}(X)$ is an isomorphism of $\mathbb{R}$-vector spaces. The image of $H^1(X,\mathbb{Z})$ in $H^{0,1}(X)$ is a lattice of rank $b_1(X)$.

We have proved:

Definition-Lemma 4.27. Let X be a compact Kähler manifold, the *Picard variety* of X is $\mathrm{Pic}^0(X) = H^{0,1}(X)/H^1(X,\mathbb{Z})$; $\mathrm{Pic}^0(X)$ is a complex torus.

The Picard variety $\mathrm{Pic}^0(X)$ is Kähler, the long exact sequence in cohomology associated to (15) shows that $\mathrm{Pic}^0(X)$ fits in an exact sequence:

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{c_1} \mathrm{im}\, c_1 \to 0. \qquad (16)$$

By Theorem 3.18, the subgroup $\mathrm{im}\, c_1 \subset H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ is called the Néron-Severi group of X and denoted $\mathrm{NS}(X)$.

By (16), $\mathrm{Pic}^0(X) \simeq \{L \in \mathrm{Pic}\, X : c_1(L) = 0\}$; $\mathrm{Pic}^0(X)$ parametrises *numerically trivial line bundles*.

Remark 4.28. The Lefschetz theorem on $(1,1)$-classes (see Exercise 2, ES 4) shows that $\mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$.

The Picard group $\mathrm{Pic}(X)$ is therefore made of a discrete part, $\mathrm{NS}(X)$ and a continuous part, $\mathrm{Pic}^0(X)$, that is a complex torus.

Let $\alpha \in \mathrm{NS}(X)$ and $L_\alpha \in \mathrm{Pic}(X)$ be a line bundle such that $c_1(L_\alpha) = \alpha$. Denote

$$\mathrm{Pic}^\alpha(X) = \{L \in \mathrm{Pic}(X) : c_1(L) = \alpha\},$$

then, by Remark 1.36, the map $L \mapsto L \otimes L_\alpha^{-1}$ defines an isomorphism $\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^\alpha(X)$. This isomorphism depends on the choice of a distinguished line bundle in $\mathrm{Pic}^\alpha(X)$, it is not canonical.

If X is a compact Kähler manifold and $H^2(X,\mathbb{Z})$ is torsion free, $\mathrm{Pic}(X)$ is fibered by complex tori of dimension $b_1(X)$ over a (discrete) subgroup of $H^2(X,\mathbb{Z})$.

There is another complex torus canonically associated to a compact Kähler manifold X, the Albanese variety. Indeed, by Serre duality, $H^1(X,\mathbb{C})$ is dual to $H^{2n-1}(X,\mathbb{C})$, with

$$H^{1,0}(X) \simeq H^{n-1,n}(X)^* \text{ and } H^{0,1}(X) \simeq H^{n,n-1}(X)^*;$$

while Poincaré duality states $H^{2n-1}(X,\mathbb{Z}) \simeq H_1(X,\mathbb{Z})^*$. We identify $H_1(X,\mathbb{Z})^*$ with its image in $H^{2n-1}(X,\mathbb{Z})$.

As in the case of the Picard variety, the image of $H^{2n-1}(X,\mathbb{Z})$ by the composition of maps

$$H^{2n-1}(X,\mathbb{Z}) \overset{i_1}{\hookrightarrow} H^{2n-1}(X,\mathbb{R}) \overset{i_2}{\hookrightarrow} H^{2n-1}(X,\mathbb{R}) \otimes \mathbb{C} = H^{2n-1}(X,\mathbb{C}) \xrightarrow{pr} H^{n-1,n}(X)$$

is a full lattice.

Definition 4.29. The *Albanese variety* of X is the complex torus

$$\mathrm{Alb}(X) = H^{n-1,n}(X)/H^{2n-1}(X,\mathbb{Z}).$$

Remark 4.30. We may identify $H_1(X,\mathbb{Z})$ with its image by the map $H_1(X,\mathbb{Z}) \to H^0(X,\Omega_X)^*$ defined by:

$$[\gamma] \mapsto I_\gamma \text{ where } I_\gamma(\alpha) = \int_\gamma \alpha.$$

Check that when X is Kähler, this map is well defined (i.e. that I_γ depends only on the class of the loop $[\gamma]$). The Albanese and Picard varieties are *dual tori*:

$$\mathrm{Alb}(X) \simeq H^0(X,\Omega_X)^*/(H_1(X,\mathbb{Z})).$$

We now show that the Albanese variety is *universal* from the point of view of maps from X to complex tori. Fix a base point $x_0 \in X$ and define the *Albanese map*:

$$\begin{aligned} \mathrm{alb}\colon X &\to \mathrm{Alb}(X) \\ x &\mapsto (\alpha \to \int_{x_0}^{x} \alpha). \end{aligned}$$

Exercise 4.31. Check that the Albanese map is well defined and determine the effect of a change of base point in its definition.

Proposition 4.32. *The Albanese map is holomorphic and the pullback of forms induces an isomorphism* $H^0(X,\Omega_X) \simeq H^0(\mathrm{Alb}(X),\Omega_{\mathrm{Alb}(X)})$. *If* $f\colon X \to Y$ *is a holomorphic map between Kähler manifolds,* f *induces a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{alb}}\downarrow & & \downarrow{\scriptstyle \mathrm{alb}} \\ \mathrm{Alb}(X) & \longrightarrow & \mathrm{Alb}(Y) \end{array}$$

where the basepoint of $\mathrm{Alb}(Y)$ *is* $f(x_0)$*, and the map* $\mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ *is induced by the pullback of forms.*

Sketch proof. Holomorphicity is a local question: for any 1-form α, in the neighbourhood of $x_1 \in X$, the holomorphicity of $\mathrm{alb}(x) = \int_{x_0}^{x_1} \alpha + \int_{x_1}^{x} \alpha$ is equivalent to the holomorphicity of $x \mapsto \int_0^x \alpha$ on a polydisc $D \subset \mathbb{C}^n$, and this is clear.

Since f is holomorphic, the pullback of forms preserves the type of forms, and hence defines a map $H^{1,0}(Y) \to H^{1,0}(X)$. The existence of the diagram then follows from the definition of pullbacks because for any path γ from x_0 to x in X, $\int_{x_0}^{x} f^*\alpha = \int_{f(x_0)}^{f(x)} \alpha$. □

Exercise 4.33. Show that $\mathrm{Alb}(\mathrm{Alb}(X)) = \mathrm{Alb}(X)$.

Exercise 4.34. Show that when $\dim X = 1$, $\mathrm{Alb}(X) \simeq \mathrm{Pic}^0(X)$ (this torus is then called the *Jacobian of* X).

Exercise 4.35. Show that $\mathrm{Pic}^0(\mathrm{Alb}(X)) \simeq \mathrm{Pic}^0(X)$, and that $\mathrm{Alb}(\mathrm{Pic}^0(X)) \simeq \mathrm{Alb}(X)$.

Exercise 4.36. Show that if X is a compact Kähler manifold, with $b_1(X) = 0$, then every holomorphic map $X \to T$, where T is a complex torus is constant, and that $\mathrm{Pic}\, X \simeq \mathrm{NS}(X)$.

Exercise 4.36 can be generalised as follows:

Corollary 4.37. *Let $f\colon X \to T$ be a holomorphic map from X to a complex torus, then f decomposes as $X \to \mathrm{Alb}(X) \to T$.*

Proof. This is a direct consequence of the diagram of Proposition 4.32, because $T = \mathrm{Alb}(T)$. □

It is natural to ask under which conditions these complex tori naturally associated to X are projective.

Definition 4.38. A complex torus is an *abelian variety* if it is projective.

By the Kodaira Embedding Theorem (2.33), since a complex torus T is Kähler, this is equivalent to asking whether there exists an integral Kähler form on T.

Remark 4.39. It is possible to write precise conditions for a complex torus to be projective; these (the *Riemann conditions*) are formulated in terms of the transition matrices between the bases of $H^k(T, \mathbb{Z})$ associated to the integral and complex structures mentioned above.

Lemma 4.40. *If X is projective,* $\mathrm{Pic}^0(X)$ *is an abelian variety.*

Proof. Since X is projective, $H^1(X, \mathbb{C}) = P^1(X, \mathbb{C})$ is an integral Polarised Hodge Structure, i.e. there exists an alternating form

$$Q\colon H^1(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}) \to \mathbb{Z},$$

with $Q(\alpha, \beta) = (L^{n-1}\alpha, \beta)_{L^2}$, where L is the Lefschetz operator and $n = \dim X$.

The form Q takes integral values on $H^1(X, \mathbb{Z})$, $H^{1,0}(X)$ and $H^{0,1}(X)$ are isotropic subspaces for Q, and the Hermitian form $H(\alpha, \beta) = iQ(\alpha, \overline{\beta})$ is positive definite on $H^{1,0}(X)$. By the description of differential forms on a torus given above, for any $P \in \mathrm{Pic}^0(X)$, we have:

$$T_{\mathrm{Pic}^0(X),P} \simeq H^1(\mathrm{Pic}^0(X), \mathbb{Z}) \otimes \mathbb{R} \simeq H^1(X, \mathbb{Z})^* \otimes \mathbb{R}, \text{ and } H^{1,0}(\mathrm{Pic}^0(X)) \simeq H^{0,1}(X)$$

so that $Q \in \bigwedge^2 (H^1(X, \mathbb{Z}))^*$ defines an element of $\bigwedge^2 (H^1(\mathrm{Pic}^0(X), \mathbb{Z})^*)^*$ and hence of $H^2(\mathrm{Pic}^0(X), \mathbb{Z})^*$.

We denote $[\Omega]$ the class of the element of $H^2(\mathrm{Pic}^0(X), \mathbb{R})$ obtained by extending this element by $\mathbb{R}$-linearity. Note that by construction, $[\Omega] \in H^2(\mathrm{Pic}^0(X), \mathbb{Z})$ is an integral class. The form Ω is of type $(1,1)$ because Q vanishes on $\bigwedge^2 T^{1,0}_{\mathrm{Pic}^0(X)} \simeq \Lambda^2 H^{0,1}(X)$.

Last, Ω is positive if $u \in T_{\mathrm{Pic}^0(X),\mathbb{R}}$, and $u = 2\Re(u_1)$ for $u_1 \in H^{0,1}(X) \simeq T^{1,0}_{\mathrm{Pic}^0(X),\mathbb{C}}$, we have:

$$\Omega(u, Iu) = \Omega(u_1 + \overline{u_1}, -iu_1 + i\overline{u_1}) = 2iQ(u_1, \overline{u_1}) > 0$$

because H is definite positive on $H^{1,0}(X)$. The class $[\Omega] \in H^2(\mathrm{Pic}^0(X), \mathbb{R})$ is an integral Kähler class and the result follows by Theorem 2.33. $\square$

4.3 Functoriality

Let $(V_{\mathbb{Z}}, \{F^p V_{\mathbb{C}}\}_p)$ and $(W_{\mathbb{Z}}, \{F^q W_{\mathbb{C}}\}_q)$ be integral Hodge structures of weights n and $m = n + 2r$ respectively, for $r \in \mathbb{Z}$.

Definition 4.41. A *morphism of Hodge structures* is a group homomorphism $\varphi \colon V_{\mathbb{Z}} \to W_{\mathbb{Z}}$ whose extension by $\mathbb{C}$-linearity $\varphi_{\mathbb{C}} \colon V_{\mathbb{C}} \to W_{\mathbb{C}}$ satisfies $\varphi(F^p V_{\mathbb{C}}) \subset F^{p+r}(W_{\mathbb{C}})$ for all $p \in \mathbb{N}$. The morphism φ is said to be of type (r, r).

Remark 4.42. Note that the condition $\varphi(F^p V_{\mathbb{C}}) \subset F^{p+r} W_{\mathbb{C}}$ is equivalent to requiring that $\varphi(V^{p,q}) \subset W^{p+r,q+r}$ for all $p, q \geq 0$.

Assume that $\varphi \colon (V_{\mathbb{Z}}, \{F^p V_{\mathbb{C}}\}_p) \to (W_{\mathbb{Z}}, \{F^q W_{\mathbb{C}}\}_q)$ is a morphism of type (r, r). It is easy to check that φ is *strict*, that is: $\operatorname{im} \varphi \cap W^{p+r,q+r} = \varphi(V^{p,q})$, and hence induces a Hodge structure of weight m on $\operatorname{im} \varphi$. Similarly, φ induces a natural Hodge structure of weight n on $\ker \varphi$, and a natural Hodge structure of weight m on $\operatorname{Coker} \varphi$.

There are several "natural" morphisms of Hodge structures from the point of view of geometry.

Pullbacks Let $\varphi\colon X \to Y$ be a holomorphic map.

First, φ is a continuous map of topological spaces, so that there is a pullback map

$$\varphi^*\colon H^k_{\text{sing}}(Y, \mathbb{Z}) \to H^k_{\text{sing}}(X, \mathbb{Z}) \tag{17}$$

induced by the morphism of complexes given by $\varphi^*(\alpha)(\psi) = \alpha(\varphi \circ \psi)$ for any singular cochain α on Y, and singular chain ψ on X.

Second, since φ is a differentiable map, there is a pullback map of differential forms $\varphi^*\colon \mathcal{A}^\bullet_Y \to \mathcal{A}^\bullet_X$ such that $\varphi^* \circ d_Y = d_X \circ \varphi^*$, so that φ^* lifts to the cohomology groups. The map φ^* between the De Rham complexes $\varphi^*\colon \mathcal{A}^\bullet_Y \to \varphi_*\mathcal{A}^\bullet_X$ naturally extends the map $\varphi^*\colon \mathbb{Z}_Y \to \varphi_*\mathbb{Z}_X$ inducing (17), and defines a map

$$\varphi^*\colon H^k(Y, \mathbb{C}) \to H^k(X, \mathbb{C}).$$

Since φ is holomorphic, φ^* preserves the type of forms, and φ^* is a morphism of Hodge structures of type $(0, 0)$. Note that by definition, φ^* is compatible with cup products–that is, $\varphi^*(\alpha \cup \beta) = \varphi^*\alpha \cup \varphi^*\beta$.

Lemma 4.43. *Let $\varphi\colon X \to Y$ be a holomorphic and surjective map of compact Kähler manifolds. Then, $\varphi^*\colon H^k(Y, Z) \to H^k(X, \mathbb{Z})$ is injective.*

Remark 4.44. Here we assume implicitly that the cohomology groups $H^k(X, \mathbb{Z})$ and $H^k(Y, \mathbb{Z})$ are torsion free; if this is not the case, the statement of the Lemma concerns the torsion free part of the integral cohomology.

Proof. By Remark 4.44, it is enough to prove the result for the cohomology with real or complex coefficients. Let $n = \dim Y$ and $n + r = \dim X$. We first prove that $H^{2n}(Y, \mathbb{R}) \to H^{2n}(X, \mathbb{R})$ is injective. To this end, consider a nonzero class $[\alpha] \in H^{2n}(Y, \mathbb{R}) \simeq \mathbb{R}$, we prove that $\varphi^*\alpha \neq 0$. We may assume that $[\alpha] \in H^{2n}(Y, \mathbb{C})$ is represented by a positively oriented form α. Let ω be a Kähler form on X, then $\varphi^*\alpha \wedge \omega^r$ is positive or 0 at every point of X. At every point where $d\varphi$ is a submersion, $\varphi^*\alpha \wedge \omega^r \neq 0$; this is the case on an open set $U \subset X$ and hence $\int_X \varphi^*\alpha \wedge \omega^r > 0$ in $H^{2n+2r}(X, \mathbb{R})$. It follows that $\varphi^*[\alpha] \neq 0$ in $H^{2n}(X, \mathbb{R})$. Consider an arbitrary nonzero class $[\alpha] \in H^k(Y, \mathbb{R})$. By Poincaré Duality, there is a class $[\beta] \in H^{2n-k}(Y, \mathbb{R})$ such that $[\alpha] \cup [\beta] = [\eta]$ is a positive class in $H^{2n}(Y, \mathbb{R})$. We have noted that φ^* is compatible with cup products, and the result follows. □

Remark 4.45. Note that the assumption that X and Y are Kähler is only used when the dimensions of X and Y are different.

Pushforwards Let $\varphi\colon X \to Y$ be a holomorphic map between complex manifolds of dimensions $\dim X = n$ and $\dim Y = n + r = m$. Consider the map on singular homology dual to (17):

$$H_{2n-k}(X,\mathbb{Z}) \to H_{2n-k}(Y,\mathbb{Z}) = H_{2m-2r-k}(Y,\mathbb{Z}). \tag{18}$$

The *pushforward map*

$$\varphi_*\colon H^k(X,\mathbb{Z}) \to H^{k+2r}(Y,\mathbb{Z})$$

is the map that is Poincaré dual to (18). If $\psi\colon \Delta \to X$ is a singular chain, $\varphi_*(\psi)$ is the singular chain $\varphi \circ \psi\colon \Delta \to Y$.

The induced morphism of Hodge structures φ_* is of type (r,r). We need to check that if a differential form α is of type (p,q) on X, $\varphi_*\alpha$ is of type $(p+r, q+r)$. This follows immediately from the fact

$$H^{k,l}(Y) = \Big(\bigoplus_{p'+q'=2m-k-l,(p',q')\neq(p,q)} H^{p',q'}(Y)\Big)^{\perp},$$

where the "orthogonality" is with respect to Poincaré duality.

Hodge structure of a blowup Let $\sigma\colon \widetilde{X}_Y \to X$ be the blowup of a Kähler manifold X along a submanifold Y; denote E the exceptional divisor of σ and recall that $E \simeq \mathbb{P}(\mathcal{N}_{Y/X})$. If Y has codimension r, E is a $\mathbb{P}^{r-1}$-bundle over Y. We want to determine the relations between the cohomology groups $H^k(\widetilde{X}_Y,\mathbb{Z}), H^k(X,\mathbb{Z})$ and $H^k(Y,\mathbb{Z})$.

Intuitively, since $E \to Y$ is a projective bundle, we expect that $H^k(E,\mathbb{Z})$ and $H^k(Y,\mathbb{Z})$. The following theorem, which we admit, formalizes this intuition.

Theorem 4.46 (Leray-Hirsch Theorem)**.** *Let $\pi\colon F \to Z$ be a fibration over a locally contractible base Z. Suppose that $H^\bullet(F_z,\mathbb{Z})$ is torsion free for all $z \in \mathbb{Z}$. Assume there are classes $\alpha_1, \cdots, \alpha_N \in H^\bullet(F,\mathbb{Z})$ such that the subgroup $A =< \alpha_1, \cdots, \alpha_N >$ of $H^\bullet(F,\mathbb{Z})$ is isomorphic by restriction to $H^k(F_z,\mathbb{Z})$ for all $z \in Z$, then*

$$H^k(F,\mathbb{Z}) \simeq A \otimes_{\mathbb{Z}} H^\bullet(Z,\mathbb{Z}),$$

where the isomorphism is obtained by restriction and cup product. More precisely, this means that any cohomology class in $H^k(F,\mathbb{Z})$ is of the form $h \cup \alpha$, where h is a class of degree j in A, and $\alpha \in H^{k-j}(Z,\mathbb{Z})$.

Remark 4.47. You may compare this result with the construction of the Hodge bundles $\mathcal{H}$ in the next section.

Lemma 4.48. *The cohomology $H^\bullet(E,\mathbb{Z})$ is a free module over $H^\bullet(Y,\mathbb{Z})$ with basis $1, h, \cdots, h^{r-1}$, where $h = c_1(\mathcal{O}_E(1))$.*

Proof. This is a direct consequence of the Leray-Hirsch theorem. Indeed, $H^k(E_z,\mathbb{Z})$ is torsion free for all $z \in Z$ and we have seen that $c_1(\mathcal{O}_E(1))_{|E_x} = c_1(\mathcal{O}_{\mathbb{P}(E_x)}(1))$ for all $x \in Y$, so that $H^\bullet(E,\mathbb{Z}) =< 1, h, \cdots, h^{r-1} >$, with $h \in H^2(E,\mathbb{Z})$. Leray-Hirsch then implies that any class in $H^k(E,\mathbb{Z})$ is of the form $h^i \cup \alpha$ for $\alpha \in H^{k-2i}(Z,\mathbb{Z})$. $\square$

Theorem 4.49. *Let X be a Kähler manifold, $Y \subset X$ a submanifold of codimension r, and let $\widetilde{X}_Y \overset{\sigma}{\to} X$ be the blowup of X along Y; then*

$$H^k(X,\mathbb{Z}) \oplus (\bigoplus_{i=0}^{r-2} H^{k-2i-2}(Y,\mathbb{Z})) \overset{\sigma^* + \Sigma j_* \circ h^i \circ \sigma^*_{|E}}{\longrightarrow} H^k(\widetilde{X}_Y,\mathbb{Z}) \qquad (19)$$

is an isomorphism of Hodge Structures. In (19), h^i denotes the cup product with the class $h^i = (c_1(\mathcal{O}_E(1)))^i$, and is a morphism of Hodge structures of type $(i+1, i+1)$, $j \colon Y \to X$ is the inclusion and $\sigma_{|E}$ is the restriction of σ to $E = \operatorname{Exc}\sigma$.

Proof. By definition of the blowup map, $X \smallsetminus Y \simeq \widetilde{X}_Y \smallsetminus E \simeq U$. The blowup thus defines an isomorphism of pairs $\sigma \colon (\widetilde{X}_Y, U) \overset{\simeq}{\to} (X,U)$ which induces a diagram:

$$\begin{array}{ccccccc} H^{k-1}(U) & \longrightarrow & H^{k-1}(X,U) & \xrightarrow{j_{Y*}} & H^k(X) & \longrightarrow & H^k(U) \\ \downarrow & & \downarrow{\scriptstyle \sigma^*_{X,U}} & & \downarrow{\scriptstyle \sigma^*_X} & & \downarrow \\ H^{k-1}(U) & \longrightarrow & H^{k-1}(\widetilde{X}_Y,U) & \xrightarrow{j_*} & H^k(\widetilde{X}_Y) & \longrightarrow & H^k(U) \end{array} \qquad (20)$$

where the first and last vertical arrows are isomorphisms. Here, we omit the coefficient ring $\mathbb{Z}$ from the notation of the cohomology groups. The Excision and Thom isomorphism Theorems in Algebraic Topology imply that:

$$H^{k-1}(X,U) \simeq H^{k-2r}(Z) \text{ and } H^{k-1}(\widetilde{X}_Y,U) \simeq H^{k-2}(E).$$

Lemma 4.43 shows that σ^* and $\sigma^*_{X,U}$ are injective and Lemma 4.48 shows that $\sigma^*_{X,U}$ coincides with the map:

$$\alpha \colon H^{k-2r}(Z) \to \bigoplus_{i=0}^{r-1} h^i(\sigma^*_{|E} H^{k-2i-2}(Z,\mathbb{Z})) \simeq H^{k-2}(E).$$

The commutativity of the diagram (20) shows that

$$(\sigma^*, j_*)\colon H^k(X) \oplus H^{k-2}(E) \to H^k(\widetilde{X}_Y)$$

is surjective. Also, injectivity of $\sigma^*_{X,U}\colon H^{k-1}(X,U) \to H^{k-1}(\widetilde{X}_Y, U)$ implies that

$$\ker(\sigma^*, j^*) = \operatorname{im}((j_{Z*}, -\sigma^*_{X,U})\colon H^k(Z) \to H^k(X) \oplus H^{k-2}(E).$$

The theorem then follows from the description of the cohomology of E in Lemma 4.48. □

5 Deformations, families and the Kodaira-Spencer map

We have seen in the previous sections that the cohomology of a compact Kähler manifold X is endowed with a Hodge Structure $(H^k(X,\mathbb{Z}), H^k(X,\mathbb{C}) = \oplus H^{p,q}(X))$, which encodes much geometric information. This Hodge Structure does not depend on the Kähler metric itself, but on the complex structure on X. In a sense, the Hodge Decomposition on the cohomology of X means that, under the Kähler hypothesis, the decomposition induced by the complex structure on the exterior algebra of differential forms descends to cohomology. It is natural to try and understand the dependency of the Hodge Structure on the complex structure. There are several ways to formulate this problem.

First, we could ask how the Hodge structure on a Kähler manifold X changes when the underlying differentiable manifold X is fixed while the complex structure varies. Recall that the complex structure on X can be defined in the following two equivalent ways.

1. X is endowed with a complex atlas, i.e. an open cover $X = \cup U_i$ and a collection of holomorphic charts $\varphi_i\colon U_i \to \mathbb{C}^n$. Two such structures $\{U_i, \varphi_i\}$ and $\{V_j, \psi_j\}$ are equivalent if there is a map $f\colon X \to X$ such that $\varphi_i \circ f \circ \psi_j^{-1}$ is holomorphic for all i, j.

2. X is endowed with a complex structure, i.e. an endomorphism $I\colon T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$, with $I^2 = \mathrm{Id}$ that is integrable. Two such structures (X, I) and (X, I') are equivalent if there is a diffeomorphism $F\colon X \to X$ such that $dF \circ I = I' \circ dF$.

Alternatively, we could ask how the Hodge structure varies in a family of complex manifolds. In other words, if $\mathcal{X} \to B$ is a proper submersive map between complex manifolds, such that $\mathcal{X}_t$ is Kähler for all $t \in B$, we ask whether the Hodge structures on the cohomology of $\mathcal{X}_t$ and on $\mathcal{X}_{t'}$ for $t \neq t'$ can be related.

We will see that these two formulations are in fact equivalent; a *family* of compact complex manifolds is locally a family of complex structures on a fixed differentiable manifold.

5.1 Small deformations of complex manifolds

Definition 5.1. A *family of complex manifolds* is a proper holomorphic and submersive map $\pi\colon \mathcal{X} \to B$ between connected complex manifolds.

Remark 5.2. The assumption that π is proper and holomorphic implies that $\mathcal{X}_t$ is a compact complex submanifold of $\mathcal{X}$ for all $t \in B$. Since π is submersive, π has maximal rank everywhere, $\mathcal{X}$ is locally diffeomorphic to a product, i.e. for all $t \in B$, there is a neighbourhood V of t and a diffeomorphism $T\colon \mathcal{X}_{|\pi^{-1}(V)} \simeq U \times V$, where $U \subset \mathbb{C}^n$ is an open set and $pr_2 \circ T = \varphi$.

Theorem 5.3 (Ehresmann Theorem)**.** *Let $\pi\colon \mathcal{X} \to B$ be a proper submersive map between differentiable manifolds. If B is contractible, there is a diffeomorphism $\mathcal{X} \overset{T}{\simeq} X_0 \times B$, where $0 \in B$ is a basepoint and $X_0 = \pi^{-1}(0)$.*

Proof. We use the following property of real differentiable manifolds (Tubular Neighbourhood Theorem): there is a neighbourhood $\mathcal{U}$ of X_0 in $\mathcal{X}$ that is diffeomorphic to a neighbourhood $\mathcal{V}$ of X_0 in the normal bundle $\mathcal{N}_{X_0/\mathcal{X}}$, and a differentiable retraction $T\colon \mathcal{U} \to X_0$. The differentiable map $F = (T, \pi)\colon \mathcal{U} \to X_0 \times B$ has invertible differential dF along X_0, and hence defines an embedding in a neighbourhood of X_0. There is an open set $V \subset B$ such that $\mathcal{U} \subset \pi^{-1}(V)$, and $F = (T, \pi)\colon \pi^{-1}(V) \to X_0 \times V$ is a diffeomorpshim.

By assumption, B is contractible, hence there is a vector field $v \in H^0(B, T_B)$ with flow Φ that exists for all t such that $\operatorname{im} \Phi_t \subset V$ for $t \gg 0$. Let $k = \dim_{\mathbb{R}} B$. There is a differentiable chart of $\{\cup \mathcal{U}_i, \varphi\colon U_i \to \mathbb{R}^{m+k}\}$ that trivialises π, i.e. such that $\pi \circ \varphi_i^{-1}$ is the projection to the last k coordinates. The vector field v lifts to each U_i, and by using a partition of unity, there is a lift $\tilde{v}$ of v to $\widetilde{X}$. The associated flow Ψ exists for all t because π is proper, and by definition of $\tilde{v}$, $\pi \circ \Phi = \Psi \circ \pi$; Ψ_t defines a diffeomorphism between $\mathcal{X}$ and $\pi^{-1}(V)$ that is compatible with π. □

Remark 5.4. Note that the map T is a diffeomorphism and *not* biholomorphic. The fibres of T are not in general biholomorphic, but the diffeomorphisms $T_t\colon \mathcal{X}_t \to \mathcal{X}_0$ allow us to compare the complex structures on fibres of π.

Proposition 5.5. *Let $\pi\colon \mathcal{X} \to B$ be a family of complex manifolds and $0 \in B$. Possibly shrinking B to a neighbourhood of $0 \in B$, there is a differentiable trivialisation $T = (T_0, \pi)\colon \mathcal{X} \to X_0 \times B$ such that for all $x \in X_0$, $T_0^{-1}(x)$ is a complex manifold. Note that the submanifold $T_0^{-1}(x)$ is diffeomorphic to B for every $x \in X_0$.*

Sketch Proof. As in the proof of Theorem 5.3, there is a diffeomorphism $\psi\colon \mathcal{V} \to \mathcal{U}$ between a neighbourhood of X_0 in $\mathcal{N}_{X_0/\mathcal{X}}$ and a neighbourhood of X_0 in $\mathcal{X}$. We may assume that the restriction of ψ_{X_0} to the 0-section $X_0 \subset \mathcal{N}_{X_0/\mathcal{X}}$ is the inclusion $X_0 \to \mathcal{U}$, that $d\psi_{X_0}$ induces a canonical isomorpshim $\mathcal{N}_{X_0/\mathcal{X}} \simeq \mathcal{N}_{X_0/\mathcal{U}}$, and that ψ is real analytic in the neighbourhood of 0 in each fibre of $\mathcal{V} \to X_0$ (this is part of the statement of the tubular neighbourhood theorem).

The map ψ associates to every $(x, u) \in \mathcal{V} \subset \mathcal{N}_{X_0/\mathcal{X}}$ the end point of the geodesic γ defined by $\gamma(0) = x, \dot{\gamma}(0) = u$. For all $x \in X_0$, $\psi_x\colon \pi^{-1}(x) = \mathcal{U}_x \to \mathcal{X}$ is real analytic –we admit this fact, it is a consequence of the tubular neighbourhood theorem–and gives expansions of holomorphic coordinates $z_1, \cdots, z_n$ on $\mathcal{X}$ in terms of real analytic coordinates $x_1, \cdots, x_m$ on $\mathcal{U}$. Define ψ_x^h as the holomorphic part of ψ_x. The definition of ψ_x^h is independent of the choice of coordinates on $\mathcal{X}$.

The differentiable map $\psi\colon \mathcal{U} \to \mathcal{X}$ is holomorphic on the fibres of π. The restriction to the 0-section of $d(\pi \circ \psi)_{|X_0} = d(\pi \circ \psi^h)_{|X_0}$ is $\mathbb{C}$-linear and surjective. Denote $\pi'\colon \mathcal{U} \to X_0$ the map induced by ψ, then $T = (\pi' \circ \psi^h, \pi)\colon \mathcal{X} \to X_0 \times B$ is a diffeomorpshim such that for all $x \in X_0$, $T_0^{-1}(x)$ is a complex manifold. ☐

Remark 5.6. By Theorem 5.3, in a neighbourhood of $0 \in B$, the fibres of π are diffeomorphic to X_0. Proposition 5.5 shows that the family of complex structures on X_0 parametrised by B varies holomorphically with $t \in B$.

We now try and *classify deformations* of complex manifolds. Let $\pi\colon \mathcal{X} \to B$ be a family of complex manifolds and $X_0 = \pi^{-1}(0)$ its central fibre. The differential $d\pi\colon T_{\mathcal{X}} \to \pi^* T_B$ is a surjective morphism of holomorphic vector bundles and we define the *relative tangent bundle* $T_{\mathcal{X}/B}$ as the kernel of $d\pi$. Restricting $d\pi$ to X_0 yields an exact sequence:

$$0 \to T_{X_0} \to (T_{\mathcal{X}})_{|X_0} \to (\pi^* T_B)_{|X_0} \to 0, \tag{21}$$

where $(\pi^*T_B)_{|X_0} \simeq \pi^*T_{B,0} \times X_0$. Hence, (21) defines an extension of T_{X_0} by $(\pi^*T_B)_{|X_0}$; this extension is characterised by the map (see ES3, ex 7):

$$\rho\colon H^0(X_0, (\pi^*T_B)_{|X_0}) \to H^1(X_0, T_{X_0}),$$

the connecting map in the long exact sequence associated to (21). Note that since $(\pi^*T_B)_{|X_0}$ is trivial,

$$\rho\colon T_{B,0} \to H^1(X_0, T_{X_0}).$$

Definition 5.7. The map $\rho\colon T_{B,0} \to H^1(X_0, T_{X_0})$ is the *Kodaira-Spencer map* of the family $\mathcal{X} \to B$ at $0 \in B$.

There are several ways to think about the Kodaira-Spencer map. Let $T = (T_0, \pi)$ be the map of Proposition 5.5. For $t \in B$, $\mathcal{X}_t$ and X_0 are diffeomorphic, denote T_t the induced diffeomorphism.

Using the diffeomorphism T_t, the complex structure $I_t \in \mathrm{End}(T_{\mathcal{X}_t,\mathbb{R}})$ on $\mathcal{X}_t$ defines a complex structure on X_0. We consider the map

$$t \in B \to I_t \in \mathrm{End}(T_{X_0,\mathbb{R}}).$$

Giving a complex structure I on X_0 is equivalent to giving a direct sum decomposition of $T_{X_0,\mathbb{R}}$ into eigenspaces for I. In particular, giving I_t is equivalent to specifying a subspace $(T^{1,0}_{X_0,x})_t \subset T_{X_0,x,\mathbb{C}}$ that varies differentiably with $x \in X_0$. We show that the map above corresponds to a map:

$$t \mapsto \alpha_t \in \mathcal{A}^{0,1}(T^{1,0}_{X_0})$$

where $\alpha_t(x) \in \Omega^{0,1}_{X_0,x} \otimes T^{1,0}_{X_0,x}$, is the map

$$T^{0,1}_{X_0,x} \overset{(T_t^{-1})^*}{\simeq} (T^{0,1}_{X_0,x})_t \overset{pr_2}{\to} T^{1,0}_{X_0,x}.$$

Here pr_2 denotes the projection onto the second factor of the decomposition $T_{X_0,x,\mathbb{C}} = T^{0,1}_{X_0,x} \oplus T^{1,0}_{X_0,x}$.

Conversely, if $\alpha_t \in \Omega^{0,1}_{X_0,x} \otimes T^{1,0}_{X_0,x}$, α_t defines a complex structure on X_0 (and hence on $\mathcal{X}_t$, using the diffeomorphism T_t), for which the vectors of type $(0,1)$ are of the form $u - \alpha_t(u)$ for all $u \in T^{0,1}_{X_0,x}$.

Proposition 5.5 implies that $t \mapsto \alpha_t$ is holomorphic; also, by definition, $\alpha_0 = 0$.

Proposition 5.8. *The map*

$$\begin{aligned} T_{B,0} &\to \mathcal{A}^{0,1}(T_X) \\ u &\mapsto d_u(\alpha_t) \end{aligned}$$

takes values in $\{\sigma \in \mathcal{A}^{0,1}(T_X) : \overline{\partial}\sigma = 0\}$*. If* $u \in T_{B,0}$*, the Dolbeault cohomology class of* $d_u(\alpha_t)$ *in* $H^1(T_X)$ *is* $\rho(u)$*, the image of* u *by the Kodaira-Spencer map.*

Proof. Let $T^{-1}\colon X_0 \times B \stackrel{\simeq}{\to} \mathcal{X}$ be the inverse of the diffeomorphism of Proposition 5.5. We have seen that for all $x \in X_0$, $T^{-1}(x \times B)$ is a complex submanifold, which defines a complex sub-vector bundle $T_*^{-1}(T_B) = \pi^*(T_B) \subset T_{\mathcal{X}}^{1,0}$,. To this subbundle, is associated a differentiable map $\sigma\colon \pi^* T_B \to T_{\mathcal{X}}$, and hence a differentiable splitting of the exact sequence

$$0 \to T_{\mathcal{X}/B} \to T_{\mathcal{X}} \to \pi^* T_B \to 0.$$

When restricted to X_0, this defines a differentiable splitting of

$$0 \to T_{X_0} \to (T_{\mathcal{X}})_{|X_0} \to \pi^* T_{B,0} \to 0.$$

By definition, the Dolbeault cohomology class $\rho(u)$ is the class of $\overline{\partial}(\sigma(u))$ for $u \in T_{B,0}$.

We have to prove that $\overline{\partial}\sigma(u) = d_u(\alpha_t) \in A^{0,1}(T_X)$. Since this is a local statement, we will introduce coordinate systems. Let $t_1, \cdots, t_k$ be local holomorphic coordinates for B and $z_1, \cdots, z_n, t_1, \cdots, t_k$ be local holomorphic coordinates for $\mathcal{X}$ (there exist such coordinates for $\mathcal{X}$ because π is holomorphic and submersive). The diffeomorphism $\mathcal{X} \to X_0$ is

$$T_0\colon (z_1, \cdots, z_n, t_1, \cdots, t_k) \mapsto (f_1, \cdots, f_n),$$

where $\{f_i\}_{1 \le i \le n}$ are differentiable functions that are holomorphic in the variables t_i. Vector fields of type $(0,1)$ for I_t are of the form:

$$(T_0)_*(\frac{\partial}{\partial \overline{z_i}}) = \Sigma_j \frac{\partial f_j}{\partial \overline{z_i}}(\frac{\partial}{\partial z_j}) + \Sigma_j \frac{\partial \overline{f_j}}{\partial \overline{z_i}}(\frac{\partial}{\partial \overline{z_j}}). \tag{22}$$

We have seen that vectors of type $(0,1)$ for I_t are of the form $u - \alpha_t(u)$ for u a vector field of type $(0,1)$ for I_0, i.e for u a C^∞ combination of the vector fields $\frac{\partial}{\partial \overline{z_k}}$. From (22), we have:

$$\alpha_t(\Sigma_j \frac{\partial \overline{f_j}}{\partial \overline{z_i}}(\frac{\partial}{\partial \overline{z_j}})) = -\Sigma_j \frac{\partial f_j}{\partial \overline{z_i}}(\frac{\partial}{\partial z_j})$$

Since $\alpha_0 = 0$ and $f(z, 0) = z$, at the first order in t, this shows that

$$\alpha_t(\frac{\partial}{\partial \overline{z_i}}) = -\Sigma_j \frac{\partial f_j}{\partial \overline{z_i}}(\frac{\partial}{\partial z_j})$$

Differentiate this with respect to t_k to obtain:

$$\frac{\partial \alpha_t}{\partial t_k}|_{t=0}(\frac{\partial}{\partial \overline{z_i}}) = -\frac{\partial}{\partial \overline{z_i}} \Sigma \frac{\partial f_j}{\partial t_k}(\frac{\partial}{\partial z_j}).$$

Now consider $\sigma(\frac{\partial}{\partial t_k})$. By definition of σ, the vector field $\sigma(\frac{\partial}{\partial t_k})$ is of type $(1,0)$ and $\pi_*(\sigma(\frac{\partial}{\partial t_k})) = \frac{\partial}{\partial t_k}$ on B and $T_{0*}(\sigma(\frac{\partial}{\partial t_k})) = 0$. Since $T_{0*}(\frac{\partial}{\partial t_k}) = \Sigma_j \frac{\partial f_j}{\partial t_k} \frac{\partial}{\partial z_j}$, we obtain:

$$\sigma(\frac{\partial}{\partial t_k}) = \frac{\partial}{\partial t_k} - \Sigma_j \frac{\partial f_j}{\partial t_k} \frac{\partial}{\partial z_j},$$

and the result follows. □

There is another way to think about the action of the Kodaira-Spencer map, which is slightly more algebraic.

Define the *thick point at* 0 as the complex manifold B_ϵ whose underlying topological space is the point $\{0\} \in B$, endowed with the sheaf of holomorphic functions $\mathcal{O}_{B_\epsilon} = \mathcal{O}_B/(\mathfrak{m}_0)^2$, where $\mathfrak{m}_0$ is the maximal ideal of $0 \in B$.

The *first order neighbourhood* X_ϵ of X is defined by the commutative diagram:

$$\begin{array}{ccc} X_\epsilon & \longrightarrow & X \\ \downarrow & & \downarrow \\ B_\epsilon & \longrightarrow & B, \end{array}$$

where the bottom map is the scheme theoretic inclusion. In other words, X_ϵ has the same underlying topological space as X, and its sheaf of holomorphic functions is $\mathcal{O}_{X_\epsilon} = \{\pi^* z_i \mod \mathfrak{m}_0^2; z_i \text{ a normal direction to } X_0 \text{ in } \mathcal{X}\}$.

Consider a trivialisation $\{V_i, \varphi_i\}$ of $\mathcal{X}$ which is of the form $V_i = U_i \times B_i$, where $\{U_i = V_i \cap X_0, \varphi_{i|X_0}\}$ and $\{B_i, \varphi_{i|B}$ are complex atlases of X_0 and B.

Assume that $\dim B = 1$, so that $\mathcal{O}_{B_\epsilon} = \mathbb{C}[\epsilon](\epsilon^2)$. For all i, $\Theta_i \colon \mathcal{O}_{X_{\epsilon|V_i}} \simeq \mathcal{O}_{U_i}[\epsilon]/(\epsilon^2)$ is an isomorphism. On $V_i \cap V_j$, these identifications determine an automorphism $\theta_{i,j} \colon : \mathcal{O}_{X_{\epsilon|V_i \cap V_j}} \simeq \mathcal{O}_{U_i \cap U_j}[\epsilon]/(\epsilon^2)$ which is compatible with π, so that we have an exact sequence:

$$0 \to \mathcal{O}_{U_{ij}} \cdot \epsilon \to \mathcal{O}_{U_{ij}}[\epsilon]/(\epsilon^2) \to \mathcal{O}_{U_{ij}} \to 0.$$

By the definition of derivations (cf. Remark 1.36), the morphism of rings θ_{ij} corresponds to a derivation $\chi_{ij} \in \Gamma(U_{ij}, T_{U_{ij}})$ of $\mathcal{O}_{U_{ij}}$ by

$$\phi \in \mathcal{O}_{U_{ij}} \mapsto \phi + \theta_{ij}(\phi)\epsilon,$$

and by definition the χ_{ij} satisfy the cocycle condition. A change of trivialisation of $\mathcal{X}$ modifies χ_{ij} by a coboundary, so that $\{\chi_{ij}\}$ defines a (Čech) cohomology class in $H^1(X_0, T_{X_0})$.

Claim 5.9. *The class of the cocycle* $\{\chi_{ij}\}$ *is* $\rho(\frac{\partial}{\partial\epsilon})$.

Indeed, $T_{B_{\epsilon},0}$ is the sheaf generated by the vector field $\frac{\partial}{\partial\epsilon}$, and $(T_{X_\epsilon})_{|X}$ is the sheaf of free $\mathcal{O}_X$-modules generated by T_{X_0} and $\frac{\partial}{\partial\epsilon}$. On U_i, $\chi_i = \theta_i^*(\frac{\partial}{\partial\epsilon})$; χ_i defines a splitting of the exact sequence:

$$0 \to T_{X_0} \to (T_{X_\epsilon})_{X_0} \to T_{B_\epsilon,0} \otimes \mathcal{O}_{X_0} \to 0.$$

On U_i, $\chi_i = \frac{\partial}{\partial\epsilon} \circ \theta_i = \theta_i^*(\frac{\partial}{\partial\epsilon})$ and $\chi_{ij} = \chi_i - \chi_j$, and one checks–as in the proof of Proposition 5.8– that $\rho(\frac{\partial}{\partial\epsilon}) = [\chi_{ij}]$.

Remark 5.10. Conversely, an interesting question is to ask under what conditions does a class $\theta \in H^1(X, T_X)$ define a deformation of the complex structure of X. This is the object of the theory of *obstructions*, and as in the case of integrability conditions, this leads to a non-linear condition.

In fact, if there exists a *universal family of deformations of* X over the germ of an analytic set $(B, 0)$, $(B, 0) \simeq H^1(X, T_X)$ and the existence of obstructed deformations corresponds to the presence of a singularity at $0 \in B$.

After this digression, let us go back to the study of a family of complex manifolds $\pi\colon \mathcal{X} \to B$, where B is locally contractible and $X_0 = \pi^{-1}(0)$ is the central fibre. By Proposition 5.5–or even by Theorem 5.3–for all $t \in B$ in a neighbourhood of $0 \in B$, $H^k(\mathcal{X}_t, \mathbb{C}) \simeq H^k(X_0, \mathbb{C})$.

In an appropriate trivialisation of $\mathcal{X}, B$ and π (with open sets of the form $V_i = U_i \times B_i$ as above),

$$V \mapsto H^k(\pi^{-1}V, G)$$

is determined by the Künneth formula; where $V \subset B$ is an open set and $G = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

Definition 5.11. Let G be an abelian group. A *local system of stalk* G is a sheaf of abelian groups/ vector spaces that is locally isomorphic to the constant sheaf of stalk G.

Definition 5.12. The *Hodge bundles* $\mathcal{H}^k = R^k\pi_*G$ are the sheaves of free $\mathcal{O}_B$-modules (resp. free $\mathcal{A}_B$-modules) for $G = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ assocciated to the presheaves $V \mapsto H^k(\pi^{-1}V, G)$; $R^k\pi_*G$ is the local system of stalk $H^k(X_0, G)$.

Definition 5.13. The *Gauss-Manin connection* $\nabla\colon \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega_B$ is the natural connection which coincides with the exterior differential d in a local trivialisation of $\mathcal{X}, B$ and π as above.

We will need the following:

Lemma 5.14 (The Lie-Cartan formula)**.** *Let $u \in \Gamma(T_{B,0})$ and $v \in \Gamma((T_{\mathcal{X}})_{X_0})$ be a vector field such that $\pi_*(v) = u$. Let $\Omega \in \mathcal{A}^k_{\mathcal{X}}$ be such that $\Omega_t = \Omega_{|\mathcal{X}_t}$ is closed for all $t \in B$ and denote $\omega\colon B \to H^k(\mathcal{X}_t, \mathbb{C})$ be the map $t \mapsto [\Omega_t]$. Then:*

$$\nabla(\omega)_{|0}(u) = [i(v)(d\Omega)_{X_0}],$$

where $i(v)$ is the interior product with v as in (23)*, and $[\alpha]$ denotes the class of α.*

Proof. Admitted. □

5.2 The case of Kähler manifolds

When the central fibre of a complex family is Kähler, we obtain more precise results, and the Hodge Decomposition on the cohomology "deforms" with the family.

Theorem 5.15. *Let $\pi\colon \mathcal{X} \to B$ be a family of complex manifolds. Assume that the central fibre $X = \pi^{-1}(0)$ is Kähler and let $\mathcal{F}$ be a holomorphic vector bundle over $\mathcal{X}$. Then, the function $b \mapsto \dim H^q(\mathcal{X}_b, \mathcal{F}_{|\mathcal{X}_b})$ is upper semi-continuous, i.e. in a neighbourhood of* 0*, $\dim H^q(\mathcal{X}_b, \mathcal{F}_{|\mathcal{X}_b}) \leq \dim H^q(X, \mathcal{F}_{|X})$.*

Corollary 5.16. *For b in a neighbourhood of* 0 *in B, the function $b \mapsto h^{p,q}(\mathcal{X}_b)$ is upper semi-continuous and*

$$H^k(\mathcal{X}_b, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(\mathcal{X}_b),$$

where $H^{p,q}(\mathcal{X}_b) = \overline{H^{q,p}(\mathcal{X}_b)}$ and $H^{p,q}(\mathcal{X}_b) = H^q(\mathcal{X}_b, \Omega^p_{\mathcal{X}_b})$.

Remark 5.17. It is important to note that the statement of the corollary is that a Hodge Decomposition holds on the cohomology of fibres near 0, but we do not know a priori whether these fibres are Kähler or not.

The result will follow from the following proposition, which we admit.

Proposition 5.18 (Kodaira)**.** *Let $\pi\colon \mathcal{X} \to B$ be a family of manifolds, and $G \to \mathcal{X}$ a vector bundle. Let Δ be a* relative differential operator *acting on G, that is $\Delta_b = \Delta_{\mathcal{X}_b}\colon G_{\mathcal{X}_b} \to G_{\mathcal{X}_b}$ is a differential operator. If each Δ_b is elliptic of fixed order, then $b \mapsto \dim\ker\Delta_b$ is upper semi-continuous, and $\ker\Delta_b \subset C^\infty(G_b)$ varies in a C^∞ way, and forms a complex subbundle of G.*

Remark 5.19. Note that since in a suitable trivialisation $\mathcal{X}_{\pi^{-1}(V)} \simeq U \times V$, for U, V open sets of $\mathcal{X}$ and B respectively, $b \mapsto \Delta_b$ can be seen as a family of differential operators acting on $G_{|X} \to X$.

Proof of Theorem 5.15. Endow $\mathcal{F}$ and $\mathcal{X}$ with hermitian metrics; these induce Hermitian metrics on $\mathcal{F}_b$ and $\mathcal{X}_b$ for all $b \in B$. Apply Proposition 5.18 to the $\overline{\partial}$-Laplacian, that acts on the sections $\mathcal{A}^{0,q}_{\mathcal{X}_b}(\mathcal{F}_{\mathcal{X}_b})$. Here, $\ker\Delta_b = \mathcal{H}^{0,q}_{\overline{\partial}}(\mathcal{F}_{\mathcal{X}_b})$, by Corollary 3.11, $\mathcal{H}^{0,q}(\mathcal{F}_{\mathcal{X}_b}) \simeq H^{p,q}(\mathcal{X}_b)$ (note that we only need $\mathcal{X}_b$ to be a compact complex manifold, which is the case because π is proper and holomorphic) , and the result follows. □

We may define a filtration on the cohomology of the fibres $\mathcal{X}_b$ without assuming that they are Kähler (however this filtration will not necessarily have the properties of a Hodge filtration) as follows. Recall that

$$F^pA^k(\mathcal{X}_b) = \{\alpha = \Sigma_i\alpha^{i,k-i} | \alpha^{i,k-i} = 0 \text{ for all } i < p\},$$

we define:

$$F^pH^k(\mathcal{X}_b, \mathbb{C}) = \ker(d\colon F^pA^k \to F^{p+1}A^k)/\operatorname{im}(d\colon F^{p-1}A^k \to F^pA^k), \quad (23)$$

so that $F^pH^k(\mathcal{X}_b, \mathbb{C})$ is the set of classes representable by closed forms that are sums of forms of type $(i, k-i)$ with $i \geq p$.

When $\mathcal{X}_b$ is Kähler, we have seen that $F^pH^k(\mathcal{X}_b, \mathbb{C})/F^{p+1}H^k(\mathcal{X}_b, \mathbb{C}) \simeq H^k(\mathcal{X}_b, \Omega_{\mathcal{X}_b})$. In general, $F^pH^k(\mathcal{X}_b, \mathbb{C})/F^{p+1}H^k(\mathcal{X}_b, \mathbb{C})$ is isomorphic to a quotient of $H^q(\mathcal{X}_b, \Omega^p_{\mathcal{X}_b})$.

Proof of Corollary 5.16. Apply Theorem 5.15 to the holomorphic vector bundle $\Omega^p_{\mathcal{X}/b} = \Lambda^p\Omega_{\mathcal{X}/B}$, where $\Omega_{\mathcal{X}/B}$ is defined by the exact sequence:

$$0 \to \pi^*\Omega_B \to \Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}/B} \to 0.$$

By Theorem 5.3, $b_k = \dim H^k(X, \mathbb{C}) = \dim H^k(\mathcal{X}_b, \mathbb{C})$ for all b near 0. Since for all p, $F^p H^k(\mathcal{X}_b, \mathbb{C})/F^{p+1}H^k(\mathcal{X}_b, \mathbb{C})$ is isomorphic to a quotient of $H^q(\mathcal{X}_b, \Omega^p_{\mathcal{X}_b})$, we may write

$$b_k = \dim H^k(\mathcal{X}_b, \mathbb{C}) = \Sigma \dim(F^p H^k(\mathcal{X}_b, \mathbb{C})/F^{p+1}H^k(\mathcal{X}_b, \mathbb{C})) \leq \Sigma h^{p,k-p}(\mathcal{X}_b). \tag{24}$$

Theorem 5.15 shows that for all p, $h^{p,k-p}(\mathcal{X}_b) \leq h^{p,k-p}(X)$, and hence $\Sigma h^{p,k-p}(\mathcal{X}_\lfloor) \leq \Sigma h^{p,k-p}(X) = b_k$, so that all inequalities in (24) are equalities and this forces

$$F^p H^k(\mathcal{X}_b, \mathbb{C})/F^{p+1}H^k(\mathcal{X}_b, \mathbb{C}) \simeq H^q(\mathcal{X}_b, \Omega^p_{\mathcal{X}_b})$$

for all p. □

Proposition 5.20. *For b in a neighbourhood of $0 \in B$, there is a decomposition on the Betti cohomology of the fibre $\mathcal{X}_b$:*

$$H^k(\mathcal{X}_b, \mathbb{C}) = \bigoplus H^{p,k-p}(\mathcal{X}_b, \mathbb{C}),$$

where $H^{p,q}(\mathcal{X}_b) = \overline{H^{q,p}(\mathcal{X}_b)}$, and $H^{p,q}(\mathcal{X}_b) \simeq H^q(\mathcal{X}_b, \Omega^p_{\mathcal{X}_b})$ for all p, q.

Sketch Proof. By Proposition 5.18, $F^p H^k(\mathcal{X}_b, \mathbb{C}) \subset H^k(\mathcal{X}_b, \mathbb{C}) \simeq H^k(X, \mathbb{C})$ varies in a C^∞ way, and the Hodge decomposition on $H^k(X, \mathbb{C})$ yields that

$$F^p H^k(\mathcal{X}_b, \mathbb{C}) \cap \overline{F^{k-p+1}H^k(\mathcal{X}_b, \mathbb{C})} = H^{p,k-p}(\mathcal{X}_b)$$

varies in a C^∞ way, has constant dimension for b near 0, and hence is isomorphic to $H^{k-p}(\mathcal{X}_b, \Omega^p_{\mathcal{X}_b})$; the direct sum decomposition on $H^k(\mathcal{X}_b, \mathbb{C})$ follows.

The statement on complex conjugation and direct sum decomposition on the cohomology of $\mathcal{X}_b$ follow from the Hodge Decomposition on $H^k(X, \mathbb{C})$ by continuity (the dimensions of summands are constant in a neighbourhood of $0 \in B$). □

Remark 5.21. In fact, more is true: Kodaira proves that if the central fibre X is Kähler, $\mathcal{X}_b$ is Kähler for all b in a neighbourhood of $0 \in B$.

5.3 Period Domains and the Period map

We fix the setup for this section. Let $\pi \colon \mathcal{X} \to S$ be a family of complex manifolds, whose central fibre X is assumed to be Kähler, and where B is a contractible neighbourhood of $0 \in B$. By Theorem 5.3, the fibres of

π are diffeomorphic, and hence $H^k(\mathcal{X}_b, \mathbb{C}) \simeq H^k(X, \mathbb{C})$ for all $b \in B$, and we have defined the local system $\mathcal{H}^k$ for all k (definition 5.12). Further, Theorem 5.15 shows that the function $b \mapsto \dim F^p H^k(\mathcal{X}_b, \mathbb{C})$ is constant on B (possibly shrinking B). Denote $b^{p,k}$ the value of $\dim F^p H^k(\mathcal{X}_b, \mathbb{C})$ on B.

Definition 5.22. The *period map* of the family $\pi\colon \mathcal{X} \to B$ is:

$$\begin{aligned} \mathcal{P}^{p,k}\colon B \to & \quad \operatorname{Gr}(b^{p,k}, H^k(X, \mathbb{C})) \\ b \mapsto & \quad (F^p H^k(\mathcal{X}_b, \mathbb{C}) \subset H^k(X, \mathbb{C})) \end{aligned}$$

By Proposition 5.18, we may in fact define complex subbundles $F^p\mathcal{H}^k$ over B, whose fibre over $b \in B$ is $F^p H^k(\mathcal{X}_b, \mathbb{C})$. The period map is then the map:

$$\mathcal{P}^{p,k}\colon b \mapsto [(F^p\mathcal{H}^k)_b] \in \operatorname{Gr}(b^{p,k}, H^k(X, \mathbb{C})).$$

Recall (Example Sheet 1) that the Grassmannian $G = Gr(k, n)$ of k-planes in $\mathbb{C}^n$ is a projective manifold of dimension $k(n-k)$, and that there is a tautological vector bundle $\mathcal{S} \to G$ over G, whose fibre over a point $[K] \in G$ is the subspace $K \subset \mathbb{C}^n$.

We recall briefly a construction of charts for the Grassmannian G. Let $[K] \in G$ and let V be a supplementary subspace of K, i. e. such that $K \oplus V = \mathbb{C}^n$. Denote G_V the open set $\{Z \subset \mathbb{C}^n; \dim Z = k \text{ and } Z \cap V = \{0\}\}$. Denote π_V and π_K the projections of $\mathbb{C}^n$ on V and K; a complex chart on G_V is defined by identifying $Z \in G_V$ to the element $h_Z = \pi_V \circ (\pi_K^{-1})_{|Z}\colon K \to V \in \operatorname{Hom}_{\mathbb{C}}(K, V)$. This yields a canonical isomorphism:

$$T_{G,[K]} \simeq T_{\operatorname{Hom}_{\mathbb{C}}(K,V),0} \simeq \operatorname{Hom}(K, \mathbb{C}^n/K)$$

Note that if $s_1, \cdots, s_k$ is a basis of K, and $\widetilde{s_i}\colon G \to \mathcal{S}$ are the holomorphic sections defined by $\widetilde{s_i}(K) = s_i$, then if $u \in T_{G,[K]}$, $h_u \in \operatorname{Hom}(K, \mathbb{C}^n/K)$ is the map

$$h_u\colon K \to \mathbb{C}^n/K$$

such that $h_u(s_i) = u(\widetilde{s_i}) \mod K$, where $u(\widetilde{s_i})$ is the derivative of $\widetilde{s_i}$ along u.

We now come to one of the main results of this course.

Theorem 5.23. *The period map $\mathcal{P}^{p,k}\colon B \to \operatorname{Gr}(b^{p,k}, H^k(X, \mathbb{C}))$ is holomorphic for all $p, k \geq 0$.*

Proof. It is clear from Definition 5.22 that the period map $\mathcal{P}^{p,k}$ is C^∞ (this follows from Proposition 5.18). It is enough to prove that $d\mathcal{P}^{p,k}$ is $\mathbb{C}$-linear, or equivalently that $d\mathcal{P}^{p,k}(u) = 0$ for all $u \in T_{B,b,\mathbb{C}}$ of type $(0, 1)$.

Step 1. From what we have just seen on tangent spaces of Grassmannians, the differential of $\mathcal{P}^{p,k}$ is a linear map

$$d\mathcal{P}^{p,k}\colon T_{B,b} \to \mathrm{Hom}(F^pH^k(\mathcal{X}_b,\mathbb{C}), H^k(\mathcal{X}_b,\mathbb{C})/F^pH^k(\mathcal{X}_b,\mathbb{C})).$$

Let $s_1, \cdots, s_k$ be a basis of $F^pH^k(\mathcal{X}_b,\mathbb{C})$ and denote $\widetilde{s_i}$ the associated holomorphic sections of the tautological line bundle $\mathcal{S} \to \mathrm{Gr}$. We may see $\widetilde{s_i}$ as C^∞-sections of $\mathcal{H}^k$ over B with $\widetilde{s_i}(b) = s_i$, and $s_i(b') \in F^pH^k(\mathcal{X}_{b'},\mathbb{C})$ for all b' in a neighbourhood of $b \in B$ (in other words: $\widetilde{s_i}$ are C^∞-sections of $F^p\mathcal{H}^k$). Then, $d\mathcal{P}^{p,k}(u)(s_i) = u(\widetilde{s_i}) \mod F^pH^k(\widetilde{X}_b,\mathbb{C})$, or equivalently, $d\mathcal{P}^{p,k}(u)(s_i) = \nabla_u(\widetilde{s_i})$, where ∇ is the Gauss-Manin connection.

Step 2. We want to compute the directional derivative $\nabla_u(\widetilde{s_i})$. For this, denote $T\colon \mathcal{X} \simeq X \times B$ a diffeomorphism as in Proposition 5.5, so that $T\colon \{x\} \times B \to \mathcal{X}$ is holomorphic for all $x \in X$. By Proposition 5.18, possibly after shrinking B, the subspace of $H^k(\mathcal{X}_{b'},\mathbb{C}) \simeq H^k(X,\mathbb{C})$ consisting of cohomology classes represented by elements of $F^p\mathcal{A}^k_{\mathcal{X}_{b'}}$ is of constant rank, there is a differential form $\Omega \in F^p\mathcal{A}^k_{\mathcal{X}/B}$– or after lifting $\Omega \in F^p\mathcal{A}^k_{\mathcal{X}}$ such that $\Omega_{|\mathcal{X}_{b'}}$ is closed for all $b' \in B$ and $[\Omega_{|\mathcal{X}_{b'}}] = \widetilde{s_i}(b')$. We view Ω as a section $b' \mapsto \widetilde{s_i}(b')$ of $\mathcal{H}^k$.

As we have seen, T defines a C^∞ splitting of

$$0 \to (T_{\mathcal{X}/B})_{|\mathcal{X}_b} \to (T_{\mathcal{X}})_{|\mathcal{X}_b} \to (\pi^*T_B)_{|\mathcal{X}_b} \to 0$$

that defines a direct sum decomposition $(T_{\mathcal{X}})_{|\mathcal{X}_b} = T_{\mathcal{X}_b} \oplus M$ with $M \simeq \pi^*(T_{B,b})$. This decomposition remains valid on the real tangent bundle because the map π is holomorphic.

Pick v a section of M such that $\pi_*v = u$. The Lie-Cartan formula (Lemma 5.14) implies

$$\nabla(\widetilde{s_i})_{|b}(u) = [i(v)(d\Omega)_{\widetilde{X}_b},$$

so that

$$d\mathcal{P}^{p,k}(u)(s_i) = [i(v)(d\Omega)_{|\mathcal{X}_b}] \mod F^pH^k(\mathcal{X}_b,\mathbb{C}).$$

The form $d\Omega \in F^p\mathcal{A}^{k+1}_{\mathcal{X}}$, so that the class of $i(v)(d\Omega)_{|\mathcal{X}_b}$ is in $F^pH^k(\mathcal{X}_b,\mathbb{C})$. If we assume that $u \in T^{0,1}_{B,b,\mathbb{C}}$, then the restriction of v to $\mathcal{X}_b$ is of type $(0,1)$, and $d\mathcal{P}^{p,k}(u)(s_i) = 0$ for $i = 1, \cdots, k$. This shows that $\mathcal{P}^{p,k}$ is holomorphic. □

Definition 5.24. For all $p, k \geq 0$, there is a holomorphic vector bundle

$$F^p\mathcal{H}^k \subset \mathcal{H}^k \to B$$

with fibre $F^p\mathcal{H}^k(\mathcal{X}_b,\mathbb{C})$, defined as $(\mathcal{P}^{p,k})^*\mathcal{S}$, where $\mathcal{S}\to \mathrm{Gr}(b^{p,k}, F^pH^k(X,\mathbb{C}))$ is the tautological vector bundle.

For all $k\geq 0$, there is a filtration

$$\cdots \subset F^p\mathcal{H}^k \subset F^{p-1}\mathcal{H}^k \subset \cdots \subset F^0\mathcal{H}^k = \mathcal{H}^k$$

of the Hodge bundle by holomorphic sub-bundles.

Define $\mathcal{H}^{p,k-p} = F^p\mathcal{H}^k/F^{p+1}\mathcal{H}^k$ as the successive quotients of these holomorphic vector bundles. When X is Kähler, Corollary 5.16 shows that the fibre over $b\in B$ is $\mathcal{H}^{p,k-p}_b = H^{k-p}(\mathcal{X}_b,\Omega^p_{\mathcal{X}_b})$.

Remark 5.25. We have already mentioned the vector bundles $F^p\mathcal{H}^k$; it is a consequence of Theorem 5.23 that they are in fact holomorphic. Compare this definition with E.S. 2, Ex.6.

Theorem 5.26 (Griffiths' transversality)**.** *The differential of the period map*

$$d\mathcal{P}^{p,k}\colon T_{B,b}\to \mathrm{Hom}(F^pH^k(\mathcal{X}_b,\mathbb{C}), H^k(\mathcal{X}_b,\mathbb{C})/F^p\mathcal{H}^k(\mathcal{X}_b,\mathbb{C}))$$

takes values in $\mathrm{Hom}(F^pH^k(\mathcal{X}_b,\mathbb{C}), F^{p-1}H^k(\mathcal{X}_b,\mathbb{C})/F^p\mathcal{H}^k(\mathcal{X}_b,\mathbb{C}))$.

Proof. We use the notation of the proof of Theorem 5.23. As in Step 2. of the proof of Theorem 5.23, for all $u\in T_{B,b}$, $i(v)(d\Omega)_{|\mathcal{X}_b}$ is a differential form in $F^{p-1}\mathcal{A}^k_{\mathcal{X}}$, and $d\mathcal{P}^{p,k}(u)(s_i)\in F^{p-1}H^k(\mathcal{X}_b,\mathbb{C})$ for all i. The result follows. □

Corollary 5.27. *The holomorphic subbundles $F^p\mathcal{H}^k$ satisfy the* transversality property*:*

$$\nabla F^p\mathcal{H}^k \subset F^{p-1}\mathcal{H}^k.$$

Proof. This is a direct consequence of Theorem 5.26. Indeed, a holomorphic section s of $F^p\mathcal{H}^k$ is in particular a differentiable section of $\mathcal{H}^k$ such that $s(b)\in F^pH^k(\mathcal{X}_b,\mathbb{C})$ for all $b\in B$, and for all $u\in T_{B,b}$,

$$d\mathcal{P}^{p,k}(u)(s(b)) = \nabla_u(s) \mod F^pH^k(\mathcal{X}_b,\mathbb{C}).$$

□

Definition 5.28. The *flag space* $F_{b^\bullet,W}$ associated to a $(k+1)$-tuple of integers $b^\bullet = (b^i)_{0\leq i\leq k}$ with $b^i\leq b^{i-1}$ and to a $\mathbb{C}$-vector space $W\simeq\mathbb{C}^n$ is:

$$F_{b^\bullet,W} = \{(W_0,\cdots,W_k)\in \Pi\,\mathrm{Gr}(b^i,W)|W^i\subset W^{i-1}\}.$$

Exercise 5.29. Check that $F_{b^\bullet,W}$ is a complex manifold and that its tangent space at $[W] = (W^0, \cdots, W^k)$ is:

$$T_{F_{b^\bullet,W},[W]} = \{(h_0, \cdots, h_k) \in \oplus \operatorname{Hom}(W^i, W/W^i) | (h_i)_{|W^{i+1}} = h_{i+1} \mod W^i\}.$$

Definition 5.30. For all $k \geq 0$, the map $\mathcal{P} = (\mathcal{P}^1, \cdots, \mathcal{P}^k)$

$$\mathcal{P} \colon b \in B \mapsto [(F^0 H^k(\mathcal{X}_b, \mathbb{C}), \cdots, F^k H^k(\mathcal{X}_b, \mathbb{C}))] \in F_{b^{p,k}, H^k(X,\mathbb{C})}$$

is holomorphic. The open set $\mathcal{D} = \operatorname{im} \mathcal{P}$ in $F_{b^{p,k}, H^k(X,\mathbb{C})}$ is the *non-polarised period domain.*

Exercise 5.31. Determine the flag spaces associated to weight 1 and 2 Hodge structures on a Kähler manifold.

Assume further that there is a class $[\omega] \in H^2(\mathcal{X}, \mathbb{Z})$ such that $\omega_{|\mathcal{X}_b}$ is a Kähler form and its class is locally constant for $b \in B$ (recall that B is assumed to be a contractible neighbourhood of 0). Cup product with the class $[\omega]$ defines a global Lefschetz operator, that is a morphism of local systems (this is a morphism of Hodge Structures of type $(1, 1)$):

$$L \colon R^k \pi_* \mathbb{C} \to R^{k+2} \pi_* \mathbb{C} \text{ or, equivalently: } L \colon \mathcal{H}^k \to \mathcal{H}^{k+2}.$$

We generalize the Lefschetz decomposition (see Lemma 3.39) to a relative setting as follows.

Let $n = \dim \mathcal{X}_b$ and define:

$$(R^i \pi_* \mathbb{C})_{\text{prim}} = \ker L^{n-i+1},$$

then there is a Lefschetz Decomposition:

$$R^k \pi_* \mathbb{C} = \bigoplus_{2k-2n \leq 2r \leq k} L^r (R^{k-2r} \pi_* \mathbb{C})_{\text{prim}}.$$

The restriction to each fibre of this decomposition is compatible with the Hodge Decomposition.

The intersection form Q associated to the polarisation $\omega_{\mathcal{X}_b}$ is globalised in the obvious way (i. e. $Q(\alpha, \beta) =< L^{n-k}\alpha, \beta >$ for forms of the appropriate degree on $\mathcal{X}$).

There is a *polarised period map* $\mathcal{P}_{\text{pol}}$ associated to the polarised Hodge structure $H^k(\mathcal{X}_b, \mathbb{C})_{\text{prim}}$ for all $k \geq 0$ that keeps track of the Hodge decomposition and of the intersection form on the cohomology of the fibres. The image of $\mathcal{P}_{\text{pol}}$ is the *polarised period domain* $\mathcal{D}_{\text{pol}}$.

Example 5.32 (ES 6, ex.3)**.** Let X be a compact Kähler manifold and $H = H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$. If X is polarised, there is an alternating intersection form

$$Q\colon H \times H \to \mathbb{Z},$$

such that for all $x \in F^1H$, $Q(x,x) = 0$ and $Q(x,\overline{x}) > 0$. These two conditions imply that the polarised period domain $\mathcal{D}_{\mathrm{pol}}$ is an open set in a quadric hypersurface.

We now turn to the infinitesimal study of the period map; i.e. we will study *variations of Hodge structures.* We now draw some consequences of Theorem 5.26 and Corollary 5.27. Consider the following diagram, which defines the operator $\overline{\nabla}^{p,k-p}$:

$$\begin{array}{ccc}
F^{p+1}\mathcal{H}^k & \xrightarrow{\nabla} & F^p\mathcal{H}^k \otimes \Omega_B \\
\downarrow & & \downarrow \\
F^p\mathcal{H}^k & \xrightarrow{\nabla} & F^{p-1}\mathcal{H}^k \otimes \Omega_B \\
\downarrow & & \downarrow \\
\mathcal{H}^{p,k-p} & \xrightarrow{\overline{\nabla}^{p,k-p}} & \mathcal{H}^{p-1,k-p+1} \otimes \Omega_B \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Definition 5.33. The operator $\overline{\nabla}_b^{p,k-p}\colon \mathcal{H}_b^{p,k-p} \to \mathcal{H}_b^{p-1,k-p+1} \otimes \Omega_{B,b}$ is the *Infinitesimal Variation of Hodge Structure* at $b \in B$.

Lemma 5.34. *The differential $d\mathcal{P}_b^{p,k}\colon T_{B,b} \to T_{\mathrm{Gr},F^pH^k(\mathcal{X}_b)}$ is the map constructed by adjunction from*

$$\overline{\nabla}_b^{p,k-p}\colon \mathcal{H}_b^{p,q} \to \mathcal{H}_b^{p-1,q+1} \otimes \Omega_{B,b}.$$

Proof. This is a manipulation of the definitions and a consequence of Corollary 5.27: check that

$$\begin{aligned}
d\mathcal{P}^{p,k}\colon \quad & T_{B,b} \to \mathrm{Hom}(F^pH^k(\mathcal{X}_b)/F^{p+1}H^k(\mathcal{X}_b), F^{p-1}H^k(\mathcal{X}_b)/F^pH^k(\mathcal{X}_b)) \\
& T_{B,b} \to \mathrm{Hom}(H^q(\mathcal{X}_b, \Omega^p_{\mathcal{X}_b}), H^{q+1}(\mathcal{X}_b, \Omega^{p-1}_{\mathcal{X}_b}))
\end{aligned}$$

is the map obtained from $\overline{\nabla}_b^{p,k-p}$ by adjunction. ☐

Theorem 5.35 (Griffiths). *Let u be a tangent vector in $T_{B,b}$, the map*

$$d\mathcal{P}^{p,k}(u)\colon H^q(\mathcal{X}_b, \Omega^p_{\mathcal{X}_b}) \to H^{q+1}(\mathcal{X}_b, \Omega^{p-1}_{\mathcal{X}_b}))$$

is the map obtained as the composition of the cup product with the Kodaira-Spencer class with the map induced on cohomology by the interior product

$$T_{\mathcal{X}_b} \otimes \Omega^p_{\mathcal{X}_b} \to \Omega^{p-1}_{\mathcal{X}_b}.$$

Remark 5.36. Let X be a complex manifold and E, F holomorphic vector bundles on X. Denote $\mathcal{E}, \mathcal{F}$ the associated sheaves of free $\mathcal{O}_X$-modules. The *cup-product*

$$\cup\colon H^r(X, \mathcal{E}) \otimes H^s(X, \mathcal{F}) \to H^{r+s}(X, \mathcal{E} \otimes \mathcal{F})$$

is the map on cohomology induced by the exterior product of forms on Dolbeault cohomology representatives. Recall from Example 1.83 the Dolbeault resolution of the sheaves $\mathcal{E}$, $\mathcal{F}$, the cup product is the map on cohomology induced by the natural map:

$$\mathcal{A}^{0,r}(E) \otimes \mathcal{A}^{0,s}(F) \to \mathcal{A}^{0,r+s}(E \otimes F).$$

Proof. We have seen in the proof of Proposition 5.8 that if $u \in T_{B,b}$, the class $\rho(u)$ is represented by $\overline{\partial} v_{|\mathcal{X}_b}$, where v is a C^∞ vector field on $T_{\mathcal{X}}$ such that $\pi_* v = u$ and $v_{|\mathcal{X}_b}$ is of type $(1,0)$. Here, we assume that $u \in T_B$ is a holomorphic tangent vector, i.e. when viewed as an element of the complexified tangent bundle, u is of type $(1,0)$.

If $\sigma \in H^{k-p}(\mathcal{X}_b, \Omega_{\mathcal{X}_b})$, by Proposition 5.18, there is a form $\Omega \in F^p\mathcal{A}_{\mathcal{X}}$ such that the restriction $\Omega_{|\mathcal{X}_{b'}} = \Omega_{b'}$ is closed for all $b' \in B$ and whose Dolbeault cohomology class $[\Omega_b] = \sigma$. Then, as in the proof of Theorem 5.23, Lemma 5.14 implies that:

$$d\mathcal{P}^{p,k}(u)(\sigma) = \overline{\nabla}^{p,k-p}_u(\sigma) = [i(v)(d\Omega)^{p-1,k-p+1}_{|\mathcal{X}_b}].$$

The closed form $i(v)(d\Omega)_{|\mathcal{X}_b}$ lies in $F^{p-1}\mathcal{A}^k_{\mathcal{X}_b}$, and the Dolbeault cohomology class of the component of type $(p-1, k-p+1)$ lies in $H^{k-p+1}(\mathcal{X}_b, \Omega^{p-1}_{\mathcal{X}_b})$.

Since $v_{|\mathcal{X}_b}$ is of type $(1,0)$, we have the following equalities for the component of type $(p-1, k-p+1)$ of $i(v)(d\Omega)_{|\mathcal{X}_b}$:

$$i(v)(d\Omega)^{p-1,k-p+1}_{|\mathcal{X}_b} = i(v)(\overline{\partial}\Omega)^{p-1,k-p+1}_{|\mathcal{X}_b} = i(v)(\overline{\partial}\Omega^{p,k-p})_{|\mathcal{X}_b}.$$

Since $i(v)(\overline{\partial}\Omega^{p,k-p}) - i(\overline{\partial}v)(\Omega^{p,k-p}) = -\overline{\partial}(i(v)(\Omega^{p,k-p}))$, and $i(v)(\Omega^{p,k-p})$ is a $\overline{\partial}$-closed form, restricting to $\mathcal{X}_b$ gives:

$$d\mathcal{P}^{p,k}(u)(\sigma) = [i(\overline{\partial}v)(\Omega^{p,q}_{|\mathcal{X}_b})] = [\rho(u) \cup \sigma].$$

□

Remark 5.37. Recall that $\mathcal{H}_b^{p,k-p} = H^{k-p}(\mathcal{X}_b, \Omega^p_{\mathcal{X}_b})$. Theorem 5.35 shows that:

$$\overline{\nabla}_b^{p,k-p} : \mathcal{H}_b^{p,k-p} \to \mathcal{H}_b^{p-1,k-p+1} \otimes \Omega_{B,b} = \mathrm{Hom}(T_{B,b}, \mathcal{H}_b^{p-1,k-p+1})$$

can be identified with the composition of

$$H^{k-p}(\mathcal{X}_b, \Omega^p_{\mathcal{X}_b}) \to \mathrm{Hom}(H^1(\mathcal{X}_b, T_{\mathcal{X}_b}), H^{k-p+1}(\mathcal{X}_b, \Omega^{p-1}_{\mathcal{X}_b})$$

given by cup product with

$$\rho^* \colon \mathrm{Hom}(H^1(\mathcal{X}_b, T_{\mathcal{X}_b}), H^{k-p+1}(\mathcal{X}_b, \Omega^{p-1}_{\mathcal{X}_b}) \to \mathrm{Hom}(T_{B,b}, H^{k-p+1}(\mathcal{X}_b, \Omega^{p-1}_{\mathcal{X}_b}).$$

The Torelli theorem for curves We admit some results on the deformation theory of compact complex manifolds.

Definition 5.38. Let X be a projective manifold. A *deformation of X over a germ $(S,0)$ of complex manifold* is a family of complex manifolds $\pi \colon \mathcal{X} \to S$, with $\pi^{-1}(0) = X$ (here π is proper, flat and submersive).

A family $\pi \colon \mathcal{X} \to S$ is a *universal family of deformations of X* if for all deformation $\pi' \colon \mathcal{X}' \to S'$, there is a unique map $S' \to S$ such that the diagram

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ S' & \longrightarrow & S \end{array}$$

where $\mathcal{X}' \to \mathcal{X}$ is the map induced by $S \to S'$. The base S is the *Kuranishi space* $\mathrm{Def}(X)$ of X; $\mathcal{X}$ is the Kuranishi family of X.

Theorem 5.39 (Kuranishi theorem)**.** *Let X be a compact complex manifold. If $H^0(X, T_X) = (0)$, there is a universal deformation space* $\mathrm{Def}(X) = S$ *with bijective Kodaira-Spencer map. The base S is the fibre over the origin of a holomorphic map between neighbourhoods of the origin in* $\{0\} \in H^1(X, T_X) \to \{0\} \in H^2(X, T_X)$ *with vanishing differential at* 0*. The base S is smooth precisely when this map is zero.*

Let X be a compact Riemann surface (a compact complex manifold of dimension 1) and denote $g = \dim H^0(X, \Omega^1_X)$. When $g \geq 2$, $H^0(X, T_X) = H^2(X, T_X) = (0)$, and Kuranishi's theorem shows that there is a universal deformation family $\pi\colon \mathcal{X} \to S$ over a smooth germ $(S, 0)$. Since the Kodaira-Spencer map is bijective, $T_S \simeq H^1(X, T_X)$.

Remark 5.40. It is known that there also exists a universal family of deformations $\mathcal{X} \to S$ when the genus is 0 or 1 (Compare with ES 2, ex.3 for the case of $g = 1$)).

Question 5.41 (Infinitesimal Torelli problem). Assume that $\mathcal{X} \to S$ is a family of complex manifolds over a contractible base, with injective Kodaira-Spencer map ρ (we will focus on the universal family of deformations of X in what follows).

The infinitesimal Torelli problem is concerned with knowing when the period map $\mathcal{P}\colon S \to \mathcal{D}$ is an immersion. This amounts to asking whether infinitesimal changes of the complex structure at 0 are faithfully reflected in the changes of the Hodge Decomposition near the central fibre.

By Theorem 5.35, finding a criterion for $\mathcal{P}$ to be an immersion is equivalent to finding a criterion for the cup-product

$$H^1(X, T_X) \to \oplus \operatorname{Hom}(H^q(X, \Omega^p_X), H^{q+1}(X, \Omega^{p-1}_X))$$

to be injective on $T_{S,0}$ (recall that the Kodaira-Spencer map $T_{S,0} \to H^1(X, T_X)$ is assumed to be injective).

Assume that X is a compact complex curve, and let $\pi\colon \mathcal{X} \to S$ be its universal family of deformations of X. The study of the period map $\mathcal{P} = (\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2)$ is equivalent to the study of $\mathcal{P}^1\colon S \to \operatorname{Gr}(g, H^1(X, \mathbb{C}))$. We want to determine criteria for $\mathcal{P}^1$ to be an immersion near $s \in S$, that is for

$$d\mathcal{P}^1\colon T_{S,s} \to \operatorname{Hom}(H^{1,0}(\mathcal{X}_s), H^{0,1}(\mathcal{X}_s))$$

to be injective. By Theorem 5.35, this is equivalent to determining when the map

$$H^1(\mathcal{X}_s, T_{\mathcal{X}_s}) \to \operatorname{Hom}(H^0(\mathcal{X}_s, \Omega_{\mathcal{X}_s}), H^1(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}))$$

given by cup product and the contraction

$$H^0(\mathcal{X}_s, \Omega_{\mathcal{X}_s}) \otimes H^1(\mathcal{X}_s, T_{\mathcal{X}_s}) \to H^1(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}) \tag{25}$$

is injective.

Recall that $T_{\mathcal{X}_s}$ is dual to $K_{\mathcal{X}_s} = \Omega_{\mathcal{X}_s}$. Further, by Serre Duality,

$$H^1(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}) \simeq H^0(\mathcal{X}_s, K_{\mathcal{X}_s})^* \text{ and } H^1(\mathcal{X}_s, T_{\mathcal{X}_s}) \simeq H^0(\mathcal{X}_s, K_{\mathcal{X}_s} \otimes K_{\mathcal{X}_s})^* \tag{26}$$

Lemma 5.42. *The map*

$$H^0(\mathcal{X}_s, K_{\mathcal{X}_s}) \otimes H^0(\mathcal{X}_s, K_{\mathcal{X}_s}) \to H^0(\mathcal{X}_s, K_{\mathcal{X}_s} \otimes K_{\mathcal{X}_s})$$

obtained by dualising (25) *and applying Serre Duality coincides with the multiplication of sections* μ*.*

Proof. Let $\eta = \alpha \otimes \beta$ be a global section of $K_{\mathcal{X}_s}^{\otimes 2}$ (i.e. $\eta \in H^0(\mathcal{X}_s, K_{\mathcal{X}_s} \otimes K_{\mathcal{X}_s})$). Consider a class $[u] \in H^1(\mathcal{X}_s), T_{\mathcal{X}_s}$, where the representative u is a section $u \in \mathcal{A}^{0,1}(T_{\mathcal{X}_s})$. We have:

$$< \mu(\eta), u >= [u\alpha\beta] \in H^1(\mathcal{X}_s, K_{\mathcal{X}_s}) \simeq \mathbb{C},$$

where the class $[u\alpha\beta]$ is obtained by contracting $u \in \mathcal{A}^{0,1}(T_{\mathcal{X}_s})$ with the section $\alpha \otimes \beta$ of $K_{\mathcal{X}_s} \otimes K_{\mathcal{X}_s}$. We also have $[u\alpha\beta] = [(u\alpha)\beta]$, where the class of $u\alpha$ is in $H^1(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s})$ and the class of β is in $H^0(\mathcal{X}_s, K_{\mathcal{X}_s})$. This proves the Lemma. □

Definition 5.43. A complete curve X is *hyperelliptic* if there exists a $2:1$ rational map $X \to \mathbb{P}^1$; X is *trigonal* if there is a $3:1$ rational map $X \to \mathbb{P}^1$.

We admit the following results, which are consequences of the classical study of algebraic curves.

Theorem 5.44.

1. *A generic complete curve X of genus $g \geq 3$ (resp. $g \geq 5$) is not hyperelliptic (resp. not trigonal). A generic curve of genus $g = 6$ is not a planar quintic (i.e. a curve of the form $X = \{f_5 = 0\} \subset \mathbb{P}^2$ where f_5 is a homogeneous polynomial of degree 5).*
2. *(Noether) If X is a non-hyperelliptic curve, the multiplication map μ is surjective.*
3. *(Petri) If X is non-hyperelliptic, non-trigonal, and not a planar quintic, X is determined by* $\ker \mu$*.*

Corollary 5.45 (Infinitesimal Torelli theorem for curves)**.** *Let X be a non-hyperelliptic curve. The local period map*

$$\mathcal{P}^1 \colon S \to \mathrm{Gr}(g, H^1(X, \mathbb{C}))$$

is an embedding at the point $0 \in S$ corresponding to X.

Proof. By Lemma 5.42, when μ is surjective, the period map $\mathcal{P}^1$ is an embedding. The result follows from Noether's theorem. □

As a result, when X is a generic complete curve of genus $g \geq 3$, $\mathcal{P}^1$ is an embedding, so that variations of the complex structure on X are faithfully represented by variations of the Hodge structure.

One can show that when X is not hyperelliptic, the canonical morphism (associated to the linear system K_X)

$$\Phi_{|K_X|}\colon X \to \mathbb{P}^{g-1} = \mathbb{P}(H^0(X, \Omega_X))$$

is an embedding. Petri's result claims that if X is neither trigonal nor a planar quintic, symmetric elements of $\ker \mu$ are precisely the homogeneous polynomials of degree 2 on $\mathbb{P}^{g-1}$ that vanish on the image of X. In other words, the image of X in $\mathbb{P}^{g-1}$ is the algebraic subscheme of $\mathbb{P}^{g-1}$ defined as the intersection of quadric hypersurfaces that contain it (this is a highly non-trivial statement).

Theorem 5.46 (Generic Torelli Theorem for curves)**.** *If X is a generic compact complex curve of genus $g \geq 5$, X is determined by its infinitesimal variation of Hodge structure and by the isomorphism*

$$H^{1,0}(X) \simeq (H^1(X, \mathbb{C})/H^{1,0}(X))^*.$$

In particular, let X and X' be two compact complex curves. Assume that $i\colon H^1(X, \mathbb{C}) \simeq H^1(X', \mathbb{C})$ is an isomorphism compatible with the intersection forms, that $j\colon (S, 0) \simeq (S', 0)$ is a germ isomorphism between the bases of the local universal deformations of X and X', and that j yields an identification of the variations of Hodge structures in the neighbourhood of X and X':

$$\begin{array}{ccc} \mathcal{P}^1\colon S & \longrightarrow & \mathrm{Gr}(g, H^1(X, \mathbb{C})) \\ \downarrow j & & \downarrow i \\ \mathcal{P}^1\colon S' & \longrightarrow & \mathrm{Gr}(g, H^1(X', \mathbb{C})) \end{array},$$

then the curves X and X' are isomorphic.

Proof. We prove the first statement: we have seen that the infinitesimal variation of Hodge structure $d\mathcal{P}^1$ at $0 \in S$ defines a map that is symmetric (relative to the Serre Duality $H^{1,0}(X) \simeq (H^1(X, \mathbb{C})/H^{1,0}(X))^*$):

$$T_{S,0} \to \mathrm{Hom}(H^{1,0}(X), H^1(X, \mathbb{C})/H^{1,0}(X)).$$

Given an isomorphism $H^{1,0}(X) \simeq (H^1(X, \mathbb{C})/H^{1,0}(X))^*$, we may dualise this map to a (symmetric) map:

$$\mu\colon H^{1,0}(X) \otimes H^{1,0}(X) \to (T_{S,0})^*.$$

By Theorem 5.44, when X is non-hyperelliptic, non-trigonal and not a planar quintic, X is identified with the subscheme of $\mathbb{P}(H^{1,0}(X)^*)$ defined by symmetric elements of $\ker \mu$.

The second statement follows at once. By differentiation, the diagram above gives an identification of the infinitesimal variations of Hodge structures at s and $j(s)$ for all $s \in S$, that is compatible with the Serre duality isomorphisms (because i preserves the intersection form), so that the vertical arrows in

$$\begin{array}{ccc} \mu_s^* \colon T_{S,s} & \longrightarrow & \mathrm{Hom}(H^{1,0}(\mathcal{X}_s), (H^1(\mathcal{X}_s, \mathbb{C})/H^{1,0}(\mathcal{X}_s))^* \\ \downarrow j_* & & \downarrow i_* \\ \mu_{j(s)}^* \colon T_{S',j(s)} & \longrightarrow & \mathrm{Hom}(H^{1,0}(\mathcal{X}'_{j(s)}), (H^1(\mathcal{X}'_{j(s)}, \mathbb{C})/H^{1,0}(\mathcal{X}'_{j(s)}))^* \end{array}$$

are isomorphisms. For generic $s \in S$, $\mathcal{X}_s \simeq \mathcal{X}'_{j(s)}$ because $\mathcal{X}_s$ and $\mathcal{X}'_{j(s)}$ are determined by $\ker \mu_s$ and $\ker \mu_{j(s)}$ respectively. It then can be proved that $\mathcal{X}_s \simeq \mathcal{X}'_{j(s)}$ for all $s \in S$. □

Remark 5.47. The generic Torelli theorem for curves of genus $g \geq 5$ asserts that the global period map that associates to X the polarised Hodge Structure on $H^1(X, \mathbb{Z})$ is of degree 1 on its image. This is a finer statement: it is known that the global polarised period map is injective, so that if two complex curves have isomorphic polarised Hodge structures, they are isomorphic. Note that such strong statements do not in general hold in higher dimensions.

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