Algebraic Curves - Problem Sheet 4

The starred questions are assessed coursework, please hand in your solutions on 09/12/2013.

Exercise 1. (*) Let $L = \{P = 0\} \subset \mathbb{P}^2$ be a projective line, $C = \{Q = 0\} \subset \mathbb{P}^2$ an irreducible conic, and let $p_1, \cdots, p_n$ be distinct points of $\mathbb{P}^2$.

(i) Show that if $p_1, \cdots, p_k \in L$ for $k > d$, then '

$$\mathcal{S} = \{D \subset \mathbb{P}^2 | \deg D = d; p_1, \cdots, p_n \in D\} \simeq \{D' \subset \mathbb{P}^2 | \deg D = d - 1; p_{k+1}, \cdots, p_n \in D'\}.$$

(ii) Show that if $p_1, \cdots, p_k \in C$ for $k > 2d$, then

$$\mathcal{S} = \{D \subset \mathbb{P}^2 | \deg D = d; p_1, \cdots, p_n \in D\} \simeq \{D' \subset \mathbb{P}^2 | \deg D' = d - 2; p_{k+1}, \cdots, p_n \in D'\}.$$

Exercise 2. Show that every holomorphic map $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ is defined by a rational function, i.e. if φ, ψ are holomorphic coordinates on $\mathbb{P}^1$, $\varphi \circ f \circ \psi^{-1}$ is a rational function.

Exercise 3. (*) In lectures, we proved that a smooth plane cubic curve has the topology of a torus. In this question, we consider the topology of singular irreducible cubic curves.

(i) Let $C_1 = \{y^2 = x^3 + x^2\} \subset \mathbb{C}^2$ be the affine nodal cubic curve. Show that $f_1\colon t \mapsto (t^2 - 1, t - t^3)$ defines a map from $\mathbb{C}$ to C_1 and describe its fibres (that is $f^{-1}(x, y)$ for $(x, y) \in C_1$).

(ii) Let $C_2 = \{y^2 = x^3\} \subset \mathbb{C}^2$ be the affine cuspidal cubic curve. Show that $f_2\colon t \mapsto (t^2, t^3)$ defines a map from $\mathbb{C}$ to C_2 and describe its fibres.

(iii) What can you deduce about the topology of $\overline{C}_1 = \{x_1^2 x_2 = x_0^2(x_0 + x_2)\} \subset \mathbb{P}^2$ and of $\overline{C}_2 = \{x_1^2 x_2 = x_0^3\} \subset \mathbb{P}^2$?

Exercise 4. (*) Let C be a Riemann surface, and $f, g\colon C \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ be meromorphic functions such that $f, g \not\equiv \infty$ and $\lambda \in \mathbb{C}$. Show that $fg, f + g$ and λf are also meromorphic functions on C (or at least can be extended to meromorphic functions). Deduce that $\{f\colon C \to \mathbb{C} \cup \{\infty\} \mid f \text{ meromorphic }, f \not\equiv \infty\}$ is a commutative algebra over $\mathbb{C}$.

Note: addition, multiplication and scalar multiplication are defined on $\mathbb{C}$, but not on $\mathbb{C} \cup \{\infty\}$. For example, $f(p)g(p)$ is not well-defined if $f(p) = \infty$ and $g(p) = 0$. You need to prove $fg, f + g$ and λf are well defined except at a finite number of points and extend uniquely over these to give meromorphic functions.

Exercise 5. (*) Let $C \subset \mathbb{P}^2$ be a smooth cubic. Let $p_1, p_2, p_0 \in C$ be three non collinear points on C, and assume neither p_1 nor p_2 is an inflection point. Construct a meromorphic function f on C whose only zeroes are at p_1, p_2 and with a pole at p_0. (Hint: Consider a stereographic projection from p_3, the third point of intersection of L_{p_1,p_2} with C).

Show that if g is any other meromorphic function whose only zeroes are at p_1, p_2 and with a pole at p_0, then f/g is constant. (* Optional: Conclude that all meromorphic functions of degree 2 on the cubic arise via stereographic projection.)

Exercise 6. Let $\wp$ be the Weierstrass $\wp$-function. Consider the meromorphic function $\wp'(z)$ on $X = \mathbb{C}/\Lambda$. By considering $\wp'$ as a map to $\mathbb{P}^1$, determine its degree and the number and indices of its ramification points. Is there a meromorphic function f on X such that $f'(z) = \wp(z)$? (Hint: What would its poles look like?)

Exercise 7. (i) Show that $C = \{x_0^2 + x_1^2 - x_2^2 = 0\} \subseteq \mathbb{P}^2$ is rational and that there are infinitely many rational points on C, i.e. points $[a, b, c]$ with $(a, b, c) \in \mathbb{Q}^3$. Find a smooth conic $C = \{P = 0\}$ which does not admit infinitely many rational points.

(iii) Let $C = \{P = 0\} \subseteq \mathbb{P}^2$ be a curve that does not contain $\{x_2 = 0\}$. Assume that there are non-constant rational functions $r_1, r_2 \in \mathbb{C}(t)$ such that $P(r_1(t), r_2(t), 1) = 0$ for all $t \in \mathbb{C} - \{t_1, \cdots, t_N\}$. Show that there is a finite field extension $\mathbb{Q} \subseteq K$ such that C contains infinitely points $[a, b, c]$ with $(a, b, c) \in K^3$.