Algebraic Curves - Problem Sheet 3

The starred questions are assessed coursework, please hand in your solutions on 25/11/2013.

Exercise 1. (*) Let $C \subset \mathbb{P}^2$ be a projective curve of degree d, and assume that there is a point $p \in C$ with $\operatorname{mult}_p C = d$. Show that C is the union of f projective lines for $f \leq d$.

Exercise 2. Write down the equation of a smooth projective curve of degree d for any $d \in \mathbb{N}^*$.

Exercise 3. (*) Let $C \subset \mathbb{P}^2$ be a projective curve of degree d with $k > d/2$ singular points lying on a line L. Show that L is a component of C. Deduce that a cubic curve $C \subset \mathbb{P}^2$ with at least two singular points is reducible.

Show that given any 5 points of $\mathbb{P}^2$, there always exists a conic that contains them. Deduce that if $C \subseteq \mathbb{P}^2_{\mathbb{C}}$ is a curve of degree 4 with 4 singular points, then C is reducible.

Exercise 4. Let C be a projective curve of degree d and p a smooth point of C. Show that p is an inflection point if and only if there is a line L through p with $I_p(C, L) \geq 3$ (L is necessarily T_pC).

Exercise 5. Let $C \subset \mathbb{P}^2$ be an irreducible projective curve of degree d, and $p \in C$ a point of multiplicity q. Show that there is a line $L \subset \mathbb{P}^2$ with $p \in L$ which meets C in exactly $d - q + 1$ points. Deduce that there always exists a line that meets C in d distinct points.

Exercise 6. (*) Let $C_\lambda = \{x_0^3 + x_1^3 + x_2^3 + \lambda x_0 x_1 x_2 = 0\} \subset \mathbb{P}^2$ be a cubic curve with $\lambda^3 + 27 \neq 0$. By computing the Hessian, show that the points of inflection of C_λ satisfy:

$$x_0^3 + x_1^3 + x_2^3 = 0 = x_0 x_1 x_2.$$

Show that C_λ has exactly 9 points of inflection that are independent of λ, and that the line through any two of them meets C again in a third point of inflection.

Exercise 7. (*) Let $C \subset \mathbb{P}^2$ be an irreducible cubic curve and assume that $[0, 0, 1] \in C$ is a singular point.

(i) Show that the equation of C can be written $C = \{a_2(x_0, x_1)x_2 = a_3(x_0, x_1)\} \subset \mathbb{P}^2$, where $a_i \in \mathbb{C}[x_0, x_1]$ is a homogeneous polynomial of degree i.

(ii) Show that up to change of coordinates, we may assume that $a_2(x_0, x_1)$ is either x_1^2 or $x_0 x_1$.

(iii) Show that after a change of coordinates of the form $x_2 \mapsto \lambda x_0 + \mu x_1 + \nu x_2$, the equation of C can be written:

$$x_1^2 x_2 = (x_0 + b x_1)^3 \text{ for some } b \in \mathbb{C} \text{ or } x_0 x_1 x_2 = (x_0 + x_1)^3.$$

Deduce that if $C \subset \mathbb{P}^2$ is an irreducible cubic, there is a projective transformation $\Psi\colon \mathbb{P}^2 \to \mathbb{P}^2$ such that the equation of $\Psi(C)$ is one of:

$$x_1^2 x_2 = x_0^3 \text{ (cuspidal) } x_1^2 x_2 = x_0^2(x_0 + x_2) \text{ (nodal) } x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2), \lambda \neq 0, 1 \text{ (smooth)}.$$

Exercise 8. Let $C = \{x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2)\} \subset \mathbb{P}^2$ be a nonsingular cubic curve and let $p_0 = [0, 1, 0]$. Show that the additive group structure on C defined in lectures is given by:

$$[x, y, 1] + [x', y', 1] = \begin{cases} [0, 1, 0] \text{ if } x = x', \text{ and } y \neq y', \\ [x", y", 1] \text{ otherwise.} \end{cases}$$

where if $x \neq x'$:

$$(x"; y") = ((\frac{y - y'}{x - x'})^2 + 1 + \lambda - x - x'; (\frac{y - y'}{x - x'})x" + (\frac{xy' - yx'}{x - x'})).$$

Find formulae for $(x", y")$ when $x = x', y = y'$.

Exercise 9. Without using the formula proved in lectures, determine the dimension N of the linear system $\mathcal{S} = \{C \in \mathcal{L}_2 | \operatorname{mult}_p C \geq 2\} \simeq \mathbb{P}^N$. Let $p \neq q \in \mathbb{P}^2$, what is the dimension N' of $\mathcal{S}' = \{C \in \mathcal{L}_2 | \operatorname{mult}_p C \geq 2, \operatorname{mult}_q C \geq 2\} \simeq \mathbb{P}^{N'}$?