Algebraic Curves - Problem Sheet 2

The starred questions are assessed coursework, please hand in your solutions on 11/11/2013.

Exercise 1. Without using Bézout's Theorem, show that every projective line meets the projective curve $\{x_1^2 + x_2^2 - x_0^2 = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$.

Similarly, show that every projective curve in $\mathbb{P}^2_{\mathbb{C}}$ meets the line $\{x_0 = 0\}$.

Exercise 2. Let P_1, P_2 and P_3 be 3 points in $\mathbb{P}^2_{\mathbb{C}}$ with coordinates $P_i = [a_{i,1}, a_{i,2}, a_{i,3}]$ for $1 \leq i \leq 3$. Let $A = (a_{i,j})$ be the associated 3×3 matrix; show that P_1, P_2 and P_3 lie on a projective line L if and only if $\det A = 0$.

Exercise 3. (*) Find a projective transformation of $\mathbb{P}^2$ that takes the lines $\{x_0 + 2x_1 + 3x_2 = 0\}, \{x_0 + x_2 = 0\}, \{x_1 = 0\}$ and $\{x_0 - x_2 = 0\}$ to $\{x_0 = 0\}, \{x_0 + x_1 = 0\}, \{x_2 = 0\}$ and $\{x_1 + x_2 = 0\}$ respectively.

Exercise 4. (*) For which values of $\lambda \in \mathbb{C}$ does the curve $C_\lambda = \{y^2 - x(x-1)(x-\lambda) = 0\} \subset \mathbb{C}^2$ admit at least one singular point?

Let $f \in \mathbb{C}[x]$ be a polynomial of degree $d \geq 3$. Find the points at infinity and the singular points of the affine curve $C = \{y^2 = f(x)\} \subset \mathbb{C}^2$.

Exercise 5. The goal of this question is to define tangent lines at singular points. For ease of notation, we work with affine plane curves. Let $C = \{Q(x,y) = 0\} \subset \mathbb{C}^2$ be an affine plane curve, where $Q \in \mathbb{C}[x,y]$ is a polynomial with no repeated factor. Recall that the multiplicity of C at (a,b) is:

$$m = \operatorname{mult}_{(a,b)} C = \min\{i + j \,|\, \frac{\partial^{i+j} Q}{\partial x^i \partial y^j}(a,b) \neq 0\},$$

and (a,b) is a singular point of C if and only if $m > 1$. Consider the polynomial

$$\sum_{i+j=m} \frac{\partial^{i+j} Q}{\partial x^i \partial y^j}(a,b) \cdot \frac{(x-a)^i (y-b)^j}{i!\, j!}$$

(i) Show that this factorises as a product of m linear factors of the form $\alpha(x-a) + \beta(y-b)$.

The lines $L_{\alpha,\beta} = \{\alpha(x-a) + \beta(y-b) = 0\}$ are the m-tangent lines to C at (a,b).

(ii) Show that if $m = 1$ (so that (a,b) is a smooth point of C), this agrees with the definition of the tangent line $T_{(a,b)}C$ given in lectures.

(iii) Find the singular points, and for each singular point, find the multiplicities and tangent lines of the following affine curves:

(a) the (nodal) cubic $y^2 = x^3 + x^2$; (b) the (cuspidal) cubic $y^2 = x^3$; (c) the quartic $x^4 + y^4 = 2x^2 y$.

Exercise 6. Let $\overline{C} = \{P = 0\} \subset \mathbb{P}^2$ be a projective curve that does not contain the line $\{x_2 = 0\}$, and assume that P has no repeated factor. Let $C = \varphi(\overline{C} \cap U_2) = \{f(x,y) = P(x,y,1) = 0\}$ be the affine curve obtained by restriction to $U_2 = \{x_2 \neq 0\}$. Show that if $(a,b) \in C$, then $\operatorname{mult}_{[a,b,1]} \overline{C} = \operatorname{mult}_{(a,b)} C$.

Exercise 7. (*) Let A and B be two complex symmetric 3×3 matrices, and assume that $\det(xA - B) = 0$ has three distinct solutions $\lambda_0, \lambda_1, \lambda_2$.

(i) Show that there is an invertible matrix P with ${}^T PAP = \mathrm{Id}$ and ${}^T PBP = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2)$.

(ii) Let C_A and C_B be the conics defined by A and B. Deduce that there is a projective transformation $f \colon \mathbb{P}^2 \to \mathbb{P}^2$ such that

$$f(C_A) = \{x_0^2 + x_1^2 + x_2^2 = 0\} \text{ and } f(C_B) = \{\lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 = 0\}.$$

(iii) Show that C_A and C_B intersect in 4 distinct points.