Algebraic Curves - Problem Sheet 1

The starred questions are assessed coursework, please hand in your solutions on 28/10/2013.

Exercise 1. Let $[1,1],[2,5]$ and $[3,4]$ be points in the projective line $\mathbb{P}^1$. Find representative vectors v_1, v_2, v_3 for these points which satisfy $v_1+v_2+v_3=0$.

Exercise 2. (*) Show that the following 2 sets define complex plane curves.

1. $\{(t^2-1, t^4+1) \in \mathbb{C}^2 \mid t \in \mathbb{C}\}$,
2. $\{(t^2, t^3+1) \in \mathbb{C}^2 \mid t \in \mathbb{C}\}$

Exercise 3. Show that there is precisely one affine line through two distinct points $P, Q \in \mathbb{C}^2$.

Let $L = \{ax+by+c=0\} \subset \mathbb{C}^2$ and $L' = \{a'x+b'y+c'=0\} \subset \mathbb{C}^2$, where $(a,b),(a',b') \neq (0,0)$

(i) Show that L and L' meet at exactly one point if and only if $ab'-a'b \neq 0$.

(ii) Show that $L = L'$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $a' = \lambda a,\ b' = \lambda b$, and $c' = \lambda c$.

Exercise 4. (*) Let $p, q, r \in \mathbb{C}[t]$ be pairwise coprime polynomials [1] such that $p(t)^3+q(t)^3+r(t)^3 \equiv 0$. Let $\omega = \exp(2i\pi/3)$, then the equation $p(t)^3+q(t)^3+r(t)^3=0$ can be written:

$$(p(t)+q(t))(\omega p(t)+\omega^2 q(t))(\omega^2 p(t)+\omega q(t)) = (-r(t))^3.$$

(i) Show that the polynomials $p(t)+q(t), \omega p(t)+\omega^2 q(t)$ and $\omega^2 p(t)+\omega q(t)$ are pairwise coprime.

(ii) Show that there exist pairwise coprime polynomials $\alpha, \beta, \gamma \in \mathbb{C}[t]$ such that $p(t)+q(t) \equiv \alpha(t)^3$, $\omega p(t)+\omega^2 q(t) \equiv \beta(t)^3$ and $\omega^2 p(t)+\omega q(t) \equiv \gamma(t)^3$.

(iii) Deduce that $\alpha(t)^3+\beta(t)^3+\gamma(t)^3 \equiv 0$.

(iv) By considering the degrees of p, q, r and α, β, γ, deduce that the polynomials p, q, r are constant.

(v) Deduce that there are no non-constant rational functions $x(t), y(t)$ such that $x(t)^3+y(t)^3 \equiv 1$.

Exercise 5. Show that the number of points in $\mathbb{P}(\mathbb{F}_2^{n+1})$ is $2^{n+1}-1$. How many projective lines are there in this space? What are the answers if $\mathbb{F}_2$ is replaced by $\mathbb{F}_q$?

Exercise 6. Show that the function $\Psi\colon \mathbb{P}^1 \to \mathbb{R}^3$:

$$\Psi\colon ([z_0,z_1]) \mapsto \left(\frac{z_0\overline{z}_1+\overline{z}_0 z_1}{|z_0|^2+|z_1|^2},\ i\frac{z_0\overline{z}_1-\overline{z}_0 z_1}{|z_0|^2+|z_1|^2},\ \frac{|z_0|^2-|z_1|^2}{|z_0|^2+|z_1|^2} \right)$$

is well defined and defines a homeomorphism between $\mathbb{P}^1_{\mathbb{C}}$ and the sphere $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$. *(Hint: Consider $\Phi\colon S \to \mathbb{P}^1$ defined by $\Phi(x,y,z) = [x-iy, 1-z]$.)*

Exercise 7. Let $S^{2n+1} = \{(x_0,\cdots,x_n) \in \mathbb{C}^{n+1}\colon |x_0|^2+\cdots+|x_n|^2 = 1\}$. Show that S^{2n+1} is compact and that the restriction of $\Pi\colon \mathbb{C}^{n+1}\{0\} \to \mathbb{P}^n$ to S^{2n+1} is continuous and surjective onto $\mathbb{P}^n$. Deduce that $\mathbb{P}^n$ is compact. (Recall that the image of a compact space under a continuous map is compact).

Exercise 8. (*) Let V be a K-vector space, $T\colon V \to V$ a linear isomorphism, and $\tau\colon \mathbb{P}(V) \to \mathbb{P}(V)$ the associated projective transformation. When is $[v] \in \mathbb{P}(V)$ a fixed point of τ? Show that every projective transformation of $\mathbb{P}(\mathbb{R}^3)$ has a fixed point and give an example of a projective transformation of $\mathbb{P}(\mathbb{R}^4)$ with no fixed point. When does a projective transformation of $\mathbb{P}(\mathbb{C}^{n+1})$ have fixed points?

Exercise 9. Find explicitly the projective transformation of $\mathbb{P}^2$ that takes the points $[1,0,0],[1,1,0],[0,0,1]$ and $[0,1,1]$ to $[1,2,3],[1,0,1],[0,1,0]$ and $[1,0,-1]$ respectively.

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[1] Two polynomials $p, q \in K[t]$ are coprime if they have no common factor of degree ≥ 1.