MAFELAP 2019 abstracts for the mini-symposium Numerical Methods for Nonvariational PDEs

Organisers: Omar Lakkis and Ricardo Nochetto

Efficient discretizations of non-linear and anisotropic PDEs on cartesian grids
Jean-Marie Mirebeau, Guillaume Bonnet and Frédéric Bonnans 2

A conforming C^1 finite element method for PDEs of Monge-Ampère type
Elisa Friebel, Wolfgang Dahmen and Siegfried Müller 3

Numerical resolution through optimization of $\det(\mathrm{D}^2 u) = f(u)$
Quentin Mérigot, Bo'az Klartag and Filippo Santambrogio 4

Finite element approximation of the Isaacs equation
Abner J. Salgado and Wujun Zhang 5

Semismooth Newton Methods for HJB equations
Iain Smears and Endre Süli 5

A rate of convergence of numerical optimal transport problem with quadratic cost
Wujun Zhang 6

EFFICIENT DISCRETIZATIONS OF NON-LINEAR AND ANISOTROPIC PDES ON CARTESIAN GRIDS

Jean-Marie Mirebeau[1], Guillaume Bonnet[1] and Frédéric Bonnans[2]

[1]Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, FR.

[2]CMAP, École Polytechnique, FR.

I will describe Voronoi's first reduction, a tool coming from the field of additive lattice geometry, which turns out to be particularly efficient for the discretization of anisotropic PDEs on cartesian grids.

This approach is versatile, and yields monotone and second order consistent finite difference schemes for various PDEs, ranging from anisotropic elliptic PDEs to the Monge-Ampère equation, and more. In turn, these applications raise new questions on Voronoi's reduction, related to its continuity or its extension to inhomogeneous forms. Numerical results illustrate the method's robustness and accuracy.

A CONFORMING C^1 FINITE ELEMENT METHOD FOR PDES OF MONGE-AMPÈRE TYPE

Elisa Friebel[1a], Wolfgang Dahmen[2b] and Siegfried Müller[1c]

[1]Institute of Geometry and Practical Mathematics, RWTH Aachen, Germany
[a]friebel@igpm.rwth-aachen.de, [c]mueller@igpm.rwth-aachen.de

[2]Department of Mathematics, University of South Carolina, USA
[b]dahmen@math.sc.edu

PDEs of Monge-Ampère type arise in the context of optimal transport (OT) problems and related problems in optics. To numerically solve such a fully nonlinear partial differential equation (PDE) usually two processes are involved, a linearisation, e.g. a Newton method, and a discretisation procedure. The question is, how to intertwine these processes? Using a finite element discretisation with C^1 elements renders this question redundant, as the order of performing linearisation and discretisation is commutable [2].

In this talk we present such a C^1 conforming finite element method applicable to equations like

$$\det(D^2u) - \frac{f}{g(\nabla u)} = 0 \qquad \text{in } \Omega, \tag{1}$$

where Ω is a bounded convex domain and $f, g > 0$ are functions such that $\int f = \int g$.

Hereto, we test the PDE (1) in an L^2 sense, linearise and discretise it and finally evaluate it on the C^1-continuous S-spline basis. These trial elements originally were developed in a computer graphics context [1] and are constructed as macro elements, i.e. they are defined piecewise on the reference cell. Thus, their degree can be chosen lower than that of the standard quintic Argyris elements.

Reformulating OT problems yields (1) complemented by nonlinear, so-called transport boundary conditions. We discuss how to enforce these boundary conditions in the context of the C^1 method. The resulting algorithm provides geometric flexibility and high order of approximation for smooth solutions.

In order to solve practically relevant OT problems, we identify the critical parts of the input data and introduce stabilisation techniques within a nested iteration process. We conclude by applying the algorithm to practically relevant mirror and lens design problems.

References

[1] Elaine Cohen, Tom Lyche, and Richard F Riesenfeld. A B-spline-like Basis for the Powell-Sabin 12-Split Based on Simplex Splines. *Mathematics of computation*, 82(283):1667–1707, 2013.

[2] Klaus Böhmer. On Finite Element Methods for Fully Nonlinear Elliptic Equations of Second Order. *SIAM J. Numer. Anal.*, 46(3):1212–1249, 2008.

NUMERICAL RESOLUTION THROUGH OPTIMIZATION OF $\det(\mathrm{D}^2 U) = F(U)$

Quentin Mérigot[1], Bo'az Klartag[2] and Filippo Santambrogio[3]

[1]Laboratoire de mathématiques d'Orsay, Université Paris-Sud, France
quentin.merigot@math.u-psud.fr

[2]Department of Mathematics, Weizmann Institute, Israel

[3]Institut Camille Jordan, Universit de Lyon, France

Several problems in geometry lead to equations of the form

$$g(\nabla u)) \det(\mathrm{D}^2 u) = a(u), \tag{1}$$

where $g : \mathbf{R}^d \to \mathbf{R}$ is a probability density and $a : \mathbf{R} \to \mathbf{R}$ decays sufficiently fast at infinity:

- For $a(t) = \exp(-t)$ this equation is related to the moment measure problem studied in [D. Cordero-Erausquin, B. Klartag, Journal of functional analysis, **268** (12), 3834–3866, 2015];
- For $a(t) = t^{-d+2}$ this equation appears in the construction of $(d-1)$-dimensional affine hemispheres in convex geometry [B. Klartag, arXiv:1508.00474, 2015].

As in optimal transport, one can define a Brenier solution to equation (1) as a convex function $u : \mathbf{R}^d \to \mathbf{R} \cup \{+\infty\}$ which satisfies $\nabla u_{\#} a(u) = \mu$, where μ is the measure on $\mathbf{R}^d$ with density g with respect to the Lebesgue measure. When $a(t) = \exp(-t)$ or $a(t) = t^{-d+2}$, Brenier solutions to (1) maximize a concave functional similar to the one appearing in Kantorovich duality. We will show that this leads to efficient numerical methods when the measure μ is finitely supported. In the moment measure case, we will deduce the convergence of a Newton algorithm from a discrete version of the differential Brascamp-Lieb inequality.

FINITE ELEMENT APPROXIMATION OF THE ISAACS EQUATION

Abner J. Salgado[1a] and Wujun Zhang[2]

[1]Department of Mathematics, University of Tennessee
[a]asalgad1@utk.edu

[2]Department of Mathematics, Rutgers University

We propose and analyze a two-scale finite element method for the Isaacs equation. The fine scale is given by the mesh size h whereas the coarse scale ε is dictated by an integro-differential approximation of the partial differential equation. We show that the method satisfies the discrete maximum principle provided that the mesh is weakly acute. This, in conjunction with weak operator consistency of the finite element method, allows us to establish convergence of the numerical solution to the viscosity solution as $\varepsilon, h \to 0$, and $\varepsilon \geq C(h|\log h|)^{1/2}$. In addition, using a discrete Alexandrov Bakelman Pucci estimate we deduce rates of convergence, under suitable smoothness assumptions on the exact solution.

SEMISMOOTH NEWTON METHODS FOR HJB EQUATIONS

Iain Smears[1] and Endre Süli[2]

[1]University College London
i.smears@ucl.ac.uk

[2]Mathematical Institute, University of Oxford, OX2 6GG, United Kingdom.

The numerical solution of fully nonlinear Hamilton–Jacobi–Bellman (HJB) partial differential equations (PDE) leads to the task of solving large systems of equations, with a nonlinearity that is often not differentiable in a classical sense. Nevertheless, a well-established iterative method, commonly called policy iteration or Howard's algorithm, often converges rapidly in practice, as well as constituting an essential ingredient for constructive proofs of the existence of discrete numerical solutions for monotone discretization schemes. More recently, it has become apparent that the notions of semismoothness and of semismooth Newton methods provide a suitable framework for establishing the superlinear convergence of the algorithm. In this talk, we present the proof of semismoothness of HJB differential operators posed on Sobolev spaces, with a possibly infinite control set that is merely assumed to be a compact metric space. In particular, we will show how the measurable selection theorem of Kuratowski and Ryll-Nardzewski plays a central role in guaranteeing the existence of a generalized differential for the HJB operator. We illustrate the theory with numerical experiments showing the performance of the semismooth Newton method in applications to high-order discretizations of HJB equations with Cordes coefficients.

A RATE OF CONVERGENCE OF NUMERICAL OPTIMAL TRANSPORT PROBLEM WITH QUADRATIC COST

Wujun Zhang

110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA
wujun@math.rutgers.edu

In recent years, optimal transport has many applications in evolutionary dynamics, statistics, and machine learning. The goal in optimal transportation is to transport a measure $\mu(x)$ into a measure $\nu(y)$ with minimal total effort with respect to a given cost function $c(x, y)$. On way to approximate the optimal transport solution is to approximate the measure μ by the convex combination of Dirac measure μ_h on equally spaced nodal set and solve the discrete optimal transport between μ_h and ν. If the cost function is quadratic, i.e. $c(x, y) = |x - y|^2$, the optimal transport mapping is related to an important concept from computational geometry, namely Laguerre cells. In this talk, we study the rate of convergence of the discrete optimal mapping by introducing tools in computational geometry, such as Brun-Minkowski inequality. We show that the rate of convergence of the discrete mapping measured in W_1^1 norm is of order $O(h^2)$ under suitable assumptions on the regularity of the optimal mapping.