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MULTIRATE METHODS FOR COUPLED FREE FLOW WITH POROUS MEDIA FLOW PROBLEMS

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We present a decoupled multiple time step finite element scheme for the Stokes/Darcy model. The scheme utilizes small time steps for the fast moving free flow and a relatively larger time step for the slow moving porous media flow. The spatial discretization combines standard inf-sup continuous elements for the Stokes problem and the primal Discontinuous Galerkin method for the porous media flow. We present stability and convergence analysis results for the decoupling scheme. In addition, we compare the fully coupled scheme to the multiple time step decoupling scheme in-terms of CPU time and accuracy. We demonstrate long term stability of the scheme and robustness under realistic parameter regimes.

ANALYSIS OF A SPACE–TIME HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR THE ADVECTION–DIFFUSION PROBLEM ON TIME-DEPENDENT DOMAINS

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A viable candidate for the solution of partial differential equations on time-dependent domains is the space-time discontinuous Galerkin (DG) method, wherein the problem is fully discretized in space and time using discontinuous finite elements. The resulting scheme is well suited to handle moving and deforming domains but at a significant increase in computational cost in comparison to traditional time-stepping methods. Attempts to rectify this situation have led to the pairing of space-time DG with the hybridizable DG (HDG) method. The combination of the two methods retains the higher-order accuracy and geometric flexibility of space-time DG while mitigating the computational burden through the use of static condensation.

In this work, we introduce and analyze a space-time HDG method for the time-dependent advection-diffusion problem on moving domains. We discuss well-posedness of the discrete system, and we derive theoretical rates of convergence in a mesh-dependent norm.

A DIVERGENCE FREE GALERKIN NUMERICAL SCHEME FOR DOUBLE-DIFFUSION EQUATIONS IN POROUS MEDIA

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A stationary Navier-Stokes-Brinkman model coupled to a system of advection-diffusion equations serves as a model for so-called double-diffusive viscous flow in porous media in which both heat and a solute within the fluid phase are subject to transport and diffusion. The solvability analysis of these governing equations results as a combination of compactness arguments and fixed-point theory. In addition an $\boldsymbol{H}(\mathrm{div})$-conforming discretisation is formulated by a modification of existing methods for Brinkman flows. The well-posedness of the discrete Galerkin formulation is also discussed, and convergence properties are derived rigorously. Computational tests confirm the predicted rates of error decay and illustrate the applicability of the methods for the simulation of bacterial bioconvection and thermohaline circulation problems.

A STABLE AND CONSERVATIVE DG DISCRETIZATION FOR COUPLED SHALLOW WATER–DARCY FLOW

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Interaction between surface flow and subsurface systems is important for a variety of environmental and industrial applications. Mathematical models for such coupled surface/subsurface flows generally express the conservation of mass and momentum, and pose substantial challenges for mathematicians and computational scientists due to the complexity of interface driven processes that are coupled, non-linear, and evolving on various spatial and temporal scales.

We propose a coupled surface/subsurface flow model that relies on hydrostatic (shallow water) equations with free surface in the free flow domain and on the Darcy model in the subsurface part. The interface conditions are motivated by the continuity of the normal flux, the continuity of the total head (different terms of the total head are neglected in the Darcy and the shallow water models, respectively), and a standard friction law. The model is discretized using the local discontinuous Galerkin method and implemented within the fully-vectorized MATLAB / GNU Octave toolbox FESTUNG. Moreover, a discrete energy stability analysis for the coupled system is provided.

POSITIVITY PRESERVING LIMITERS FOR TIME-IMPLICIT HIGHER ORDER DISCONTINUOUS GALERKIN DISCRETIZATIONS

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In the numerical solution of partial differential equations it is frequently necessary to ensure that certain variables, e.g. density, pressure, or probability density distribution, remain within strict bounds. Strict observation of these bounds is crucial otherwise unphysical or unrealistic solutions will be obtained that might result in the failure of the numerical algorithm. Bounds on certain variables are generally ensured using positivity preserving limiters, which locally modify the solution to ensure that the constraints are satisfied. For discontinuous Galerkin methods, in combination with explicit time integration methods, this approach works well and many accurate positivity preserving limiters are available, see e.g. [1]. The combination of (positivity preserving) limiters and implicit time integration methods results, however, in serious problems. Many limiters have a complicated, non-smooth formulation, which hampers the use of standard Newton methods to solve the nonlinear algebraic equations of the implicit time discretization.

In this presentation, we will discuss a different approach to ensure that a higher order accurate numerical solution satisfies the positivity constraints [2]. Instead of using a limiter, we impose the positivity constraints directly on the algebraic equations resulting from a higher order accurate time implicit discontinuous Galerkin discretization by reformulating the DG equations with constraints using techniques from mathematical optimization theory [3]. The resulting algebraic equations are then solved using a semi-smooth Newton method that is well suited to deal with the resulting nonlinear complementarity problem [3]. This approach allows the direct imposition of constraints in higher order accurate discontinuous Galerkin discretizations combined with Diagonally Implicit Runge-Kutta methods and results in more efficient solvers for time-implicit discretizations. We will demonstrate the novel algorithm on a number of model problems, namely the advection, Burgers, Allen-Cahn, Barenblatt, and Buckley-Leverett equations, both in 1D and 2D, using time-implicit DG-DIRK discretizations with order of accuracy ranging between 2 and 5.

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EMBEDDED/HYBRIDIZED DISCONTINUOUS GALERKIN METHOD FOR INCOMPRESSIBLE FLOWS

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We present an embedded/hybridized discontinuous Galerkin method for velocity-pressure formulations of incompressible flows. The method maintains the attractive features of hybridized methods, namely a H(div)-conforming velocity field, pointwise solenoidal velocity field (without post-processing), local momentum conservation, and energy stability when applied to the Navier-Stokes equations. By using an 'embedded' approach for the velocity field with the facet fields being continuous, the method is significantly faster than a full hybridized method to reach a specified error tolerance. Moreover, the embedded/hybridized method is suited to preconditioners that are known to be effective for continuous problems. We present analysis results, including a result showing that error estimates for the velocity are independent of the pressure. Analysis results are supported by a range of numerical examples.

A MIXED ELASTICITY FORMULATION FOR FLUID POROELASTIC STRUCTURE INTERACTION

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We study a mathematical model and its finite element approximation for solving the coupled problem arising in the interaction between a free fluid and a fluid in a poroelastic material. The free fluid flow is governed by the Stokes equations, while the poroelastic material is modeled using the Biot system of poroelasticity. The model is based on a mixed stress-displacement-rotation elasticity formulation and mixed velocity-pressure Darcy and Stokes formulations. The mixed finite element approximation provides local mass and momentum conservation in the poroelastic media. We discuss well posedness of the mathematical model as well as stability, accuracy, and robustness of the numerical method. Applications to flows in fractured poroelastic media and arterial flows are presented.

A POSTERIORI ERROR ANALYSIS OF A CONFORMING FOUR-FIELD FORMULATION FOR BIOT'S CONSOLIDATION MODEL IN POROELASTICITY

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In this talk we present the *a priori* and *a posteriori* error analyses for the conforming approximation of the four-field formulation recently proposed in [2] for Biot's consolidation in poroelasticity, in which the solid displacement, the fluid pressure, the fluid flux and the total pressure, are the primal variables. For the *a priori* error analysis we provide suitable hypotheses on the corresponding finite dimensional subspaces ensuring that the associated Galerkin scheme becomes well posed. A feasible choice of subspaces is given by Raviart–Thomas elements of order $k \geq 0$ for the fluid flux, discontinuous polynomials of degree k for fluid pressure, and any stable pair of Stokes elements, such as Hood–Taylor elements, for the solid displacements and total pressure. In turn, we develop a reliable and efficient residual-based *a posteriori* error estimator. The proof of the realibilty is based on a stability result providing global inf-sup conditions on each one of the spaces involved, and stable Helmholtz decompositions. On the other hand, the usual localization technique of bubble functions and inverse inequalities, are the main tools behind the efficiency estimate. Both estimates are shown to be independent of the moludi of dilatation. Numerical examples in 2D and 3D illustrate the performance of the conforming scheme and show the good behaviour of the adaptive algorithm associated to the proposed *a posteriori* error indicator.

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