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GOAL-ORIENTED ADAPTIVITY FOR ELLIPTIC PDES WITH PARAMETRIC OR UNCERTAIN INPUTS

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In this talk, we present a goal-oriented adaptive algorithm for approximating linear quantities of interest derived from solutions to elliptic partial differential equations (PDEs) with parametric or uncertain inputs. Specifically, we consider a class of elliptic PDEs where the underlying differential operator has affine dependence on a countably infinite number of uncertain parameters and employ the stochastic Galerkin finite element method to approximate the solutions to the corresponding primal and dual problems.

Our algorithm follows the standard adaptive loop: $\mathsf{SOLVE} \implies \mathsf{ESTIMATE} \implies \mathsf{MARK} \implies \mathsf{REFINE}$. Here, the error in the goal functional (e.g., the expectation of the quantity of interest) is *estimated* by the product of computable estimates of the energy errors in Galerkin approximations of the primal and dual solutions. Drawing information from the spatial and parametric contributions to these error estimates, the *marking* is performed by extending the strategy proposed in [4]. Finally, following the methodology developed in [3, 2], a balanced adaptive *refinement* of spatial and parametric components of the approximation space is performed by combining the associated error reduction indicators computed for the primal and dual solutions (see [1]).

We will discuss the results of numerical experiments that demonstrate the effectiveness of our goal-oriented adaptive strategy for a representative model problem with parametric coefficients and for various quantities of interest (including the approximation of pointwise values).

References

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SOME THOUGHTS ON ADAPTIVE STOCHASTIC GALERKIN FEM

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The numerical treatment of random PDEs as examined in Uncertainty Quantification is challenged by the high dimensionality of the problem. The data of the model is commonly characterised by a countable infinite sequence of random variables. However, it is well-known that for many model problems the solution manifold exhibits favourable properties, in particular sparsity and low-rank approximability. In order to practically construct surrogate models, these properties have to be exploited adequately. A successful approach for this task is the stochastic Galerkin FEM steered by an a posteriori error estmation, which has been shown to achieve optimal rates when an affine parameter dependence is assumed.

In this talk we consider two recent extensions with adaptive stochastic Galerkin methods. First, more general coefficients (e.g. lognormal) represented in hierarchical tensor formats are examined. Although the complexity increases substantially, the tensor compression still allows for an efficient computation of an adaptive algorithm. Second, a statistical learning framework (Variational Monte Carlo) for random PDEs is presented. It enables the treatment of very general problems and computes the Galerkin solution with high probability.

STOCHASTIC GALERKIN MIXED FINITE ELEMENT APPROXIMATION FOR LINEAR POROELASTICITY WITH UNCERTAIN INPUTS

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Over the last couple of decades, mathematical models of fluid flow through deformable porous media have gained a lot attention due to their wide applicability in science and engineering. In particular, Biot's consolidation model arises in applications ranging from geoscience to medicine. In such applications, we encounter scenarios where there is uncertainty about the model inputs.

In this talk, we discuss a novel locking-free stochastic Galerkin mixed finite element method for the Biot consolidation model with uncertain Young's modulus and hydraulic conductivity field. After introducing a five-field mixed variational formulation of the standard Biot consolidation model, we discuss stochastic Galerkin mixed finite element approximation, focusing on the issue of well-posedness and efficient linear algebra for the discretized system. We introduce a new preconditioner for use with MINRES and establish eigenvalue bounds. Finally, we present specific numerical examples to illustrate the efficiency of our numerical solution approach.

CONVERGENCE OF ADAPTIVE STOCHASTIC GALERKIN FEM FOR ELLIPTIC PARAMETRIC PDES

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We present an adaptive stochastic Galerkin finite element method for a class of parametric elliptic boundary value problems. The adaptive algorithm is steered by a reliable and efficient *a posteriori* error estimator, which can be decomposed into a two-level spatial estimator and a hierarchical parametric estimator [2]. The structure of the estimated error is exploited by the algorithm to perform a balanced adaptive refinement of the spatial and parametric discretizations. The adaptive algorithm is proved to be convergent in the sense that the estimated error converges to zero [1]. Numerical experiments underpin the theoretical findings and show that the proposed adaptive strategy helps to mitigate the 'curse of dimensionality' which usually afflicts the numerical approximation of parametric PDEs.

References

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MULTILEVEL BEST LINEAR UNBIASED ESTIMATORS

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We present a general variance reduction technique to accelerate the estimation of an expectation of a scalar valued quantity of interest. We reformulate the estimation as a linear regression problem and show that the derived estimators are variance minimal within the class of linear unbiased estimators. We further decrease the cost of the estimation by solving a sample allocation problem and analyze the asymptotic complexity of the resulting estimater, which we call Sample Allocation Optimal Best Linear Unbiased Estimator, using Richardson extrapolation. We provide theoretical results that show that our framework improves upon other sampling based estimation techniques like Monte Carlo, Multilevel Monte Carlo and Multifidelity Monte Carlo. The results are illustrated by a numerical example where the underlying model of the quantity of interest is a partial differential equation.

MULTILEVEL QUADRATURE FOR ELLIPTIC PROBLEMS ON RANDOM DOMAINS BY THE COUPLING OF FEM AND BEM

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Elliptic boundary value problems which are posed on a random domain can be mapped to a fixed, nominal domain. The randomness is thus transferred to the diffusion matrix and the loading. While this domain mapping method is quite efficient for theory and practice, since only a single domain discretisation is needed, it also requires the knowledge of the domain mapping.

However, in certain applications the random domain is only described by its random boundary, i.e. the domain mapping is only known for the boundary, while the quantity of interest is defined on a fixed, deterministic subdomain. In this setting, it thus becomes necessary to extend the domain mapping from the boundary to the whole domain, such that the domain mapping is the identity on the fixed subdomain.

To overcome the necessity of computing the extension, we therefore couple the finite element method on the fixed subdomain with the boundary element method on the random boundary. We verify the required regularity of the solution with respect to the random perturbation field for the use of multilevel quadrature, derive the coupling formulation, and show by numerical results that the approach is feasible.

References

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A POSTERIORI ERROR ESTIMATION FOR THE STOCHASTIC COLLOCATION FINITE ELEMENT APPROXIMATION OF A HEAT EQUATION

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In this talk we present a residual based a posteriori error estimation for a heat equation with a random forcing term and a random diffusion coefficient which is assumed to depend affinely on a finite number of independent random variables. The problem is discretized by a stochastic collocation finite element method and advanced in time by the theta-scheme. The a posteriori error estimate consists of three parts controlling the finite element error, the time discretization error and the stochastic collocation error, respectively. These estimators are then used to drive an adaptive choice of FE mesh, collocation points and time steps. We study the effectiveness of the estimate and the performance of the adaptive algorithm on a numerical example.

References

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A POSTERIORI ERROR ESTIMATION AND ADAPTIVITY IN STOCHASTIC GALERKIN FEM FOR PARAMETRIC ELLIPTIC PDES: BEYOND THE AFFINE CASE

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We consider a linear elliptic partial differential equation (PDE) with a generic uniformly bounded parametric coefficient. The solution to this PDE problem is approximated in the framework of stochastic Galerkin finite element methods. We present a reliable and efficient a posteriori estimate of the energy error in Galerkin approximations generalising the results of [1, 2]. Practical versions of this error estimate are discussed and tested numerically for a model problem with non-affine parametric representation of the coefficient. Furthermore, we use the error reduction indicators derived from spatial and parametric error estimators to guide an adaptive solution algorithm for the given parametric PDE problem. The performance of the adaptive algorithm is tested numerically for a model problem with non-affine parametric representation of the coefficient.

References

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