MAFELAP 2019 abstracts for the mini-symposium
Recent developments in the numerical approximation
of transport equations

Organisers: Matthais Schlottbom and Herbert Egger

A regularized entropy-based moment method for kinetic equations
Graham W. Alldredge, Martin Frank and Cory D. Hauck ......................... 2

Structure-aware Taylor time stepping for tent pitching schemes
Jay Gopalakrishnan, Joachim Schöberl and Christoph Wintersteiger .......... 3

A spatial discontinuous Galerkin method with rescaled velocities for the Boltzmann equation
Gerhard Kitzler and Joachim Schöberl ................................................. 4

An Adaptive Nested Source Term Iteration for Radiative Transfer Equations
Olga Mula ............................................................................................... 5

Tensor-product discretization for the spatially inhomogeneous and transient Boltzmann equation in two dimensions
Philipp Grohs, Ralf Hiptmair and Simon Pintarelli ............................... 6

A convergent Lagrangian discretization for $p$-Wasserstein and flux-limited diffusion equations
Benjamin Söllner and Oliver Junge ........................................................ 7

Adaptive strategies for transport equations
W. Dahmen and R.P. Stevenson ............................................................. 7

Guaranteed and robust $L^2$ a posteriori error estimates for 1D linear advection problems
Alexandre Ern, Martin Vohralik and Mohammad Zakerzadeh ................. 8
A REGULARIZED ENTROPY-BASED MOMENT METHOD
FOR KINETIC EQUATIONS

Graham W. Alldredge\textsuperscript{1a}, Martin Frank\textsuperscript{2} and Cory D. Hauck\textsuperscript{3}

\textsuperscript{1}Freie Universität Berlin,
\textsuperscript{a}graham.alldredge@fu-berlin.de
\textsuperscript{2}Karlsruhe Institute of Technology,
\textsuperscript{3}Oak Ridge National Laboratory

We present a new entropy-based moment method for the velocity discretization of kinetic equations. This method is based on a regularization of the optimization problem defining the original entropy-based moment method, and this gives the new method the advantage that the moment vectors of the solution do not have to take on realizable values. We show that this equation still retains many of the properties of the original equations, including hyperbolicity, an entropy-dissipation law, and rotational invariance. The cost of the regularization is mismatch between the moment vector of the solution and that of the ansatz returned by the regularized optimization problem. However, we show how to control this error using the parameter defining the regularization. This suggests that with proper choice of the regularization parameter, the new method can be used to generate accurate solutions of the original entropy-based moment method, and we confirm this with numerical simulations.
Many ideas have been put forth to advance numerical solutions of wave and other hyperbolic problems in time through local operations in spacetime regions. Tent-shaped spacetime regions appear to be natural for solving hyperbolic systems because one can ensure causality by constraining the height of the tent pole. Specifically, the domain of dependence of all points within a tent can be guaranteed to be contained within the tent by constraining the tent pole height. Moreover, a spacetime simulation region can be covered by advancing fronts of such tents. Building on this idea, we have introduced new schemes, called Mapped Tent Pitching (MTP) schemes, which proceed by transforming tents into domains where space and time are separated, allowing standard methods to be used within tents. This technique also allows, for the first time, the use of fully explicit schemes within tents. After highlighting certain difficulties that arise with naive use of standard explicit Runge-Kutta time stepping algorithms in this context, we propose an alternative structure-aware Taylor time-stepping technique. New explicit methods are thus built that allow variable time steps and local refinements without compromising high-order accuracy in space and time. They lead to highly parallel algorithms, which utilize modern computer architectures extremely well.
A SPATIAL DISCONTINUOUS GALERKIN METHOD WITH RESCALED VELOCITIES FOR THE BOLTZMANN EQUATION

Gerhard Kitzler\textsuperscript{1} and Joachim Schöberl\textsuperscript{2}

\textsuperscript{1}Universität Wien, gerhard.kitzler@tuwien.ac.at
\textsuperscript{2}Technische Universität Wien, joachim.schoeberl@tuwien.ac.at

In the talk we present a numerical method for the Boltzmann equation. It is a spectral discretization in the velocity and a discontinuous Galerkin discretization in physical space. To obtain uniform approximation properties in the mach number, we shift the velocity by the (smoothed) bulk velocity and scale it by the (smoothed) temperature, both extracted from the density distribution. The velocity trial functions are polynomials multiplied by a Maxwellian. Consequently, an expansion with a low number of trial functions already yields satisfying approximation quality for nearly equilibrated solutions. By a polynomial test space, density, velocity and energy are conserved on the discrete level. To stabilize the free flow operator we use upwind fluxes in phase space. Several numerical results are presented to justify our approach.
In this talk, we present a new approach to the numerical solution of radiative transfer equations with certified a posteriori error bounds. We formulate a fixed-point iteration in a suitable, infinite dimensional function space that is guaranteed to converge with a fixed error reduction per step. The numerical scheme is then based on approximately realizing this outer iteration within dynamically updated accuracy tolerances that still ensure convergence to the exact solution. To guarantee that these error tolerances are met, we employ rigorous a posteriori error bounds based on a Discontinuous Petrov–Galerkin (DPG) scheme. These a posteriori bounds are also used to generate adapted angular dependent spatial meshes to significantly reduce overall computational complexity. The scheme also requires the evaluation of the global scattering operator at increasing accuracy at every iteration and its computation is accelerated through low-rank approximation and matrix compression techniques. We will illustrate the theoretical findings with numerical experiments involving non-trivial scattering kernels.

This is a joint work with Wolfgang Dahmen (University of South Carolina) and Felix Gruber (RWTH Aachen).
In this talk we present a numerical method for the spatially inhomogeneous and nonlinear Boltzmann equation for the variable hard spheres model. The distribution function is discretized by a tensor-product ansatz by combining Maxwellian modulated Laguerre polynomials in velocity with continuous, linear finite elements in the spatial domain. The advection problem in phase space is discretized through a Galerkin least squares technique and yields an implicit formulation in time. The discrete collision operator can be evaluated with an asymptotic effort of $O(K^5)$, where $K$ is the number of velocity degrees of freedom in a single direction. Numerical results in 2D are presented for rarefied gases with different Mach and Knudsen numbers.

References

A CONVERGENT LAGRANGIAN DISCRETIZATION FOR P-WASSERSTEIN AND FLUX-LIMITED DIFFUSION EQUATIONS

Benjamin Sölner and Oliver Junge

Zentrum Mathematik, Technische Universität München, Germany

soellneb@math.tum.de

A Lagrangian numerical scheme for solving a non-linear drift diffusion equation of the form

$$\partial_t u = \partial_x (u \cdot (c^*(\partial_x (h'(u) + V))))$$

on an interval with no-flux boundary conditions is presented. Here $h$ is the internal energy density, $V$ an external energy potential and $c^*$ the Legendre-transform of a transport cost $c$ from a family of $L^p$-like cost functions, but possibly with discontinuities. Examples include the Fokker-Planck equation, the Porous medium equation, the parabolic $p$-Laplace equation and the so-called relativistic heat equation.

Our numerical scheme is based on the equation’s gradient flow formulation in a Wasserstein-like metric for the cost function $c$. Specifically, we use the representation in terms of the inverse distribution function and perform a spatio-temporal discretization in these coordinates.

The discretization inherits various properties from the continuous flow, like entropy monotonicity, mass preservation, a minimum/maximum principle and flux-limitation in the case of flux-limiting cost. Our main result is that of convergence in the spatio-temporal continuous limit.

ADAPTIVE STRATEGIES FOR TRANSPORT EQUATIONS

W. Dahmen and R.P. Stevenson

1 Mathematics Department, University of South Carolina, Columbia, SC 29208, USA

2 Korteweg-de Vries Institute for Mathematics, University of Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands

r.p.stevenson@uva.nl

We consider linear transport equations with a variable convection field. A discontinuous Petrov-Galerkin (DPG) discretization is presented that is uniformly inf-sup stable whilst applying a test search space with a dimension that is proportional to that of the trial space (‘practical’ DPG method). Applying the same test search space we show that the natural a posteriori error estimator is efficient and, modulo a data-oscillation term, reliable. Finally we discuss a bulk chasing adaptive scheme driven by this estimator. For the one-dimensional case we show a fixed error reduction in each step, where a corresponding result in multiple dimensions is currently partly based on a conjecture.
GUARANTEED AND ROBUST $L^2$ A POSTERIORI ERROR ESTIMATES FOR 1D LINEAR ADVECTION PROBLEMS

Alexandre Ern$^{1a}$, Martin Vohralík$^{2b}$ and Mohammad Zakerzadeh$^{2c}$

$^1$Université Paris-Est, CERMICS (ENPC), 77455 Marne-la-Vallée 2, France & Inria, 2 rue Simone Iff, 75589 Paris, France
alexandre.ern@enpc.fr

$^2$Inria, 2 rue Simone Iff, 75589 Paris, France & Université Paris-Est, CERMICS (ENPC), 77455 Marne-la-Vallée 2, France
$^b$martin.vohralik@inria.fr, $^c$seyed-mohammad.zakerzadeh@inria.fr

We propose a reconstruction-based a posteriori error estimate for linear advection problems in one space dimension. In our framework [1], a stable variational ultra-weak formulation is adopted, and the equivalence of the $L^2$-norm of the error with the dual graph norm of the residual is established. This dual norm is showed to be localizable over patch subdomains of the computational domain under the condition of the orthogonality of the residual to the piecewise affine hat functions. We show that this condition is valid for some well-known numerical methods including continuous/discontinuous Petrov–Galerkin and discontinuous Galerkin methods. Consequently, a well-posed local problem on each patch is identified, which leads to a global reconstruction of the solution belonging to the graph space. We prove that this reconstruction provides a guaranteed upper bound on the $L^2$ error. Moreover, up to a constant, it also gives local lower bounds on the $L^2$ error, where the generic constant is proven to be independent of mesh-refinement, polynomial degree of the approximation, and the advective velocity. This leads to robustness of our estimates with respect to the advection as well as the approximation polynomial degree. All the above properties are verified in a series of numerical experiments, additionally yielding asymptotic exactness.

Motivated by these results, we finally propose a heuristic extension of our methodology to any space dimension, achieved by solving local least squares problems on the patches. Though not anymore guaranteed, the resulting error indicator is numerically still robust with respect to both advection and polynomial degree, for a collection of two-dimensional test cases including discontinuous solutions.

References