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LOW-ORDER DIVERGENCE-FREE FINITE ELEMENT METHODS IN FLUID MECHANICS

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It is a well-known fact that the finite element approximation of equations in fluid mechanics (e.g., the Navier-Stokes equations) is simpler, and more accurate, if the finite element method delivers an exactly divergence-free velocity. In fact, in such a case the convective term remains antisymmetric in the discrete setting, without the need to rewrite it, and the stability analysis can be greatly simplified. Now, when the finite element spaces used are the conforming $\mathbf{P}_1$ for velocity and $\mathbf{P}_0$ for pressure, then, in addition to add appropriate terms to stabilise the pressure, some extra work needs to be done in order to compensate for the lack of incompressibility of the discrete velocity.

In this talk I will present two recent applications of a technique developed previously in [2] to overcome the limitations described in the previous paragraph. The first example presented is the application of this idea to the steady-state Boussinesq equation [1]. Then, the transient Navier-Stokes equations will be analysed, where some error estimates independent of the Reynolds number will be shown.

References

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A POSTERIORI ERROR ANALYSIS FOR THE MIXED LAPLACE EIGENVALUE PROBLEM

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We review some a posteriori estimators for the Laplace eigenvalue problem in mixed form and show how to extend them to the corresponding eigenvalue problem.

The standard residual error estimator is known to work well and it has been proved to provide an optimally convergent adaptive scheme when used in combination with the usual Dörfler marking strategy [D. Boffi, D. Gallistl, F. Gardini, L. Gastaldi, Math. Comp., 86(307) (2017) 2213-2237].

On the other hand, non residual error estimators are more challenging to be analyzed. Using the Prager-Synge hypercircle approach with local flux reconstructions, a fully computable upper bound for the flux error in the L2-norm is derived when the problem is discretized using Raviart–Thomas finite elements of arbitrary polynomial degree. Efficiency of the local error estimators is proved and numerical experiments with optimal convergence rates are provided [F. Bertrand, D. Boffi, R. Stenberg, arXiv:1812.11203]. The extension of the result to other mixed finite elements, such as BDM spaces, is not straightforward.

ALGEBRAIC FLUX CORRECTION FOR ADVECTION PROBLEMS AND ITS EXTENSION TO SYMMETRIC TENSOR FIELDS

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The work to be presented in this talk extends the algebraic flux correction (AFC) methodology to advection(-reaction) equations and symmetric tensor fields [3]. The new theoretical results are used to design bound-preserving finite element methods for steady and unsteady model problems. The proposed approaches add algebraically defined artificial diffusion operators to the Galerkin discretization. Then limited antidiffusive fluxes are incorporated into the residual of the resulting low order method to remove redundant diffusivity and to improve the accuracy of the approximation.

In the case of the steady state advection problem with a scalar solution, convergence with order $\frac{1}{2}$ is shown by adapting an a priori error estimate derived in [1]. Existence of a unique solution is proved under suitable assumptions. Furthermore, sufficient conditions for the validity of generalized discrete maximum principles (DMPs) are formulated. In addition to guaranteeing boundedness of the function values in terms of weakly imposed boundary conditions, they provide local L^∞ estimates for subsets of degrees of freedom. These DMP results are extended to the transient advection equation discretized in time by the θ-scheme. A priori time step restrictions are derived for $\theta \in [0, 1)$. The analysis of the forward Euler time discretization implies that bound-preserving approximations of higher order can be obtained without solving nonlinear systems if explicit strong stability preserving (SSP) Runge-Kutta time integrators are employed. Based on the results of theoretical studies, new definitions of correction factors are proposed which potentially facilitate the development of more efficient solvers and/or implementations.

Using the concept of Löwner ordering, the scalar AFC framework is extended to the numerical treatment of symmetric tensor quantities. In tensorial versions of the methods under consideration, antidiffusive fluxes are limited to constrain the eigenvalue range of the tensor field by imposing discrete maximum principles on the extremal eigenvalues [2]. This criterion is shown to be an appropriate frame invariant generalization of scalar maximum principles. It leads to a family of robust property-preserving limiters based on the same design principles as their scalar counterparts.

The potential of the presented methods is illustrated by numerical examples.

References

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THE UNSTEADY, INCOMPRESSIBLE STOKES EQUATIONS WITH COMPATIBLE DISCRETE OPERATOR SCHEMES

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We investigate the extension of Compatible Discrete Operator schemes (CDO) [1] to the unsteady, incompressible Stokes equations.

CDO is a unifying framework on low-order mimetic schemes which preserve at the discrete level the structural properties of the PDEs, such as conservation laws and mathematical relations involving differential operators, while still ensuring competitive computational performances. General, polytopal and nonmatching meshes can be considered. Several discretizations are possible according to the mesh entities on which the main unknowns are defined. For the problem at hand, a full 3D face-based discretization has been selected, hence making the method close to the Hybrid High-Order method [2] for $k = 0$. A divergence operator relying on the Green theorem ensures the velocity-pressure coupling. Concerning the viscosity part, the scheme hinges on a stabilized gradient reconstruction which is piecewise constant on the sub-cell pyramids. Cell-based degrees of freedom are used as well, but eliminated before the assembly stage by means of static condensation.

In order to deal with the velocity-pressure coupling in the time-dependent case, several methods can be considered. In addition to the fully-coupled monolithic one, two well-known segregated approaches can be taken into account: the Augmented Lagrangian-Uzawa (ALU) and the Artificial Compressibility (AC) methods. Both schemes involve at least one user-parameter but, in the unsteady case, differently from the ALU, which consists in an iterative procedure, the AC framework has the advantage of requiring only one pressure update for the lowest-order time discretization, hence ensuring superior performances. Moreover, higher order time schemes can be devised as shown in [3]. The price to pay for this benefit is a non-divergence-free velocity: in fact, the method itself hinges on a small perturbation of the incompressibility constraint. We then explore the effects of such a velocity when it is used in a pure convection problem to transport a passive tracer.

Results concerning the latest developments towards the Navier-Stokes equations will be presented as well.

References

[1] J. Bonelle and A. Ern, *Analysis of Compatible Discrete Operator schemes for elliptic problems on polyhedral meshes*, ESAIM Math. Model. Numer. Anal., 48 (2014), pp. 553–581.

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WEAKLY SYMMETRIC STRESS RECONSTRUCTION AND A POSTERIORI ERROR ESTIMATION FOR HYPERELASTICITY

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By extending the techniques in [1] for the linear elastic case, an algorithm to reconstruct a H(div)-conforming weakly symmetric stress tensor for the non-linear hyperelastic case is presented. This work builds upon [2] where a local weakly symmetric stress reconstruction is derived for arbitrary conforming finite elements in linear elasticity. The reconstructed stress tensor is used as an a posteriori error estimator. Numerical results for the incompressible hyperelastic case are presented.

[1] F. Bertrand, M. Moldenhauer, G. Starke, A posteriori error estimation for planar linear elasticity by stress reconstruction, Computational Methods in Applied Mathematics (2018), *https://doi.org/10.1515/cmam-2018-0004*

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FIRST-ORDER SYSTEM LEAST SQUARES METHODS FOR SEA ICE MODELS

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In this talk a viscoplastic sea ice model is considered. Sea ice is modeled as a generalized Newtonian compressible fluid, which satisfies a power law. The nonlinearity caused by the power law for the viscosity is severe and requires careful treatment. The talk considers a first-order system least squares method (FOSLS), where the velocity, ice height, ice concentration and the stress are used as variables. This approach leads to a nonlinear system of partial differential equations and for the least-squares finite element method to a non-quadratic minimization problem.

The talk addresses the numerical challenges that arise while solving this system. After deriving a proper linearization and discussing fitting approximation spaces, numerical experiments will be presented. The talk also examines the advantages of adaptive refinement in this setting.

MOMENTUM-CONSERVATIVE STRESS APPROXIMATION METHODS IN ELASTOPLASTICITY

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In many applications in computational solid mechanics, one is primarily interested in accurate approximations of the associated stresses. This is due to the fact that stresses are responsible for effects like plastic material behavior, damage or failure due to surface forces. In particular, $H(\mathrm{div})$-conforming stress finite element approximations with local momentum conservation properties possess some advantageous properties in this context.

Two approaches for the computation of such stress approximations in the $H(\mathrm{div})$-conforming Raviart-Thomas finite element space will be treated in this talk: Directly, via the dual variational formulation known as Hellinger-Reissner principle [BBF09] and indirectly, via a stress equilibration procedure based on some primal finite element approximation [BKMS18]. Both approaches lead to weakly symmetric stresses with local momentum conservation and crucially depend on the inf-sup stability of the underlying constraint minimization problem.

The extension to elastoplasticity will turn out to be more complicated than the purely elastic case due to the additional inequality constraint. The case of a von Mises flow rule with hardening as well as perfect plasticity will be discussed. For the latter, a weakly symmetric generalization of the nonconforming stress elements of [GG11] will prove to be useful.

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