MAFELAP 2019 abstracts for the mini-symposium Novel adaptive discretization schemes for variational inequalities

Organisers: Gerhard Starke, Andreas Schröder

hp-adaptive FEM for VIs in optimal control problems
Lothar Banz, Michael Hintermüller and Andreas Schröder 2

Equilibrated stress approximation and error estimation with application to solid mechanics
Fleurianne Bertrand 3

A least-squares finite element method for the obstacle problem
Thomas Führer 3

Discontinuous Skeletal Methods for the Obstacle Problem
Matteo Cicuttin, Alexandre Ern and Thirupathi Gudi 4

Reconstruction-based a-posteriori error estimation in stress-based FEM for frictional contact problems
B. Kober and G. Starke 4

A framework for applying the DWR method on variational inequalities
Andreas Rademacher and Ullrich Heupel 6

On quasi-optimality of finite element methods for the obstacle problem
Andreas Veeser 7

Residual-type a posteriori estimators for the singularly perturbed variational inequality in quasi-static fracture phase-field models
Mirjam Walloth, Katrin Mang, Thomas Wick and Winnifried Wollner 8

AFEM for elliptic-H(curl) variational inequalities of the second kind
Malte Winckler, Irwin Yousept and Jun Zou 9

HP-ADAPTIVE FEM FOR VIS IN OPTIMAL CONTROL PROBLEMS

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A distributed elliptic control problem with control constraints is typically formulated as a three field problem and consists of two variational equations for the state and the co-state variables as well as of a variational inequality for the control variable. In view of a flexible adaptive scheme, the three variables are discretized independently by (non-conforming) *hp*-finite elements. While that discrete formulation is well suited for computations, it is less favorable for numerical analysis. We therefore rewrite the weak three field formulation as an equivalent variational inequality with an operator of the type $B = I + \alpha^{-1}A^{-2}$ formulated in the control variable only. Analogously, we condense out the state and the co-state variables from the discrete three field formulation. This implicates to deal with an approximation of the operator B, as A^{-2} cannot be realized numerically, depending on the independent discretization spaces for the eliminated state and co-state variables.

We provide sufficient conditions for the unique existence of a discrete solution triple. Also a priori error estimates and guaranteed convergence rates are derived, which are stated in terms of the mesh size as well as of the polynomial degree. Moreover, reliable and efficient a posteriori error estimates are presented. The efficiency estimate rests on an analytic solution of an optimization problem to find the best conforming approximation of the non-conforming discrete solution.

Several numerical experiments demonstrate the applicability of the discretization with *hp*-finite elements, the efficiency of the a posteriori error estimates and the improvements with respect to the order of convergence resulting from the application of *hp*-adaptivity. In particular, the *hp*-adaptive schemes show superior convergence rates.

EQUILIBRATED STRESS APPROXIMATION AND ERROR ESTIMATION WITH APPLICATION TO SOLID MECHANICS

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A stress equilibration procedure for linear elasticity is presented with emphasis on the behavior for nearly incompressible materials. It is based on the displacement-pressure approximation computed with a stable finite element pair and constructs an H(div)-conforming, weakly symmetric stress reconstruction. The construction leads then to reconstructed stresses by Raviart- Thomas elements of degree k which are weakly symmetric. The computation is performed locally on a set of vertex patches. The resulting error estimator constitute a guaranteed upper bound for the error with a constant that depends only on the shape regularity of the triangulation. The extension to the Signorini contact problem is straightforward.

A LEAST-SQUARES FINITE ELEMENT METHOD FOR THE OBSTACLE PROBLEM

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In this talk we present recent results on a least-squares finite element method for a first-order reformulation of the obstacle problem. A priori error estimates including the case of non-conforming convex sets are given and optimal convergence rates are shown for the lowest-order case. We provide a posteriori bounds that can be used as error indicators in an adaptive algorithm. Numerical studies are presented.

DISCONTINUOUS SKELETAL METHODS FOR THE OBSTACLE PROBLEM

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Discontinuous-skeletal methods are introduced and analyzed for the elliptic obstacle problem in two and three space dimensions. The methods are formulated in terms of face unknowns which are polynomials of degree $k = 0$ or $k = 1$ and in terms of cell unknowns which are polynomials of degree $l = 0$. The discrete obstacle constraints are enforced on the cell unknowns. A priori error estimates of optimal order (up to the regularity of the exact solution) are shown. Specifically, for $k = 0$, the method employs a local linear reconstruction operator and achieves an energy-error estimate of order h, where h is the mesh-size, whereas for $k = 1$, the method employs a local quadratic reconstruction operator and achieves an energy-error estimate of order $h^{\frac{3}{2}-\epsilon}$, $\epsilon > 0$. Numerical experiments in two and three space dimensions illustrate the theoretical results

RECONSTRUCTION-BASED A-POSTERIORI ERROR ESTIMATION IN STRESS-BASED FEM FOR FRICTIONAL CONTACT PROBLEMS

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The use of stress-based finite element methods for the treatment of contact problems admits locking free performance in the incompressible limit as well as direct access to the surface forces at the contact zone. Consequently we are studying the application of the stress-based FEM described in [1] featuring next-to-lowest order Raviart-Thomas-Elements to the Signorini contact problem with Coloumb friction using a dual variational formulation similar to the one studied in [2].
Since frictional contact problems tend to feature singularities, adaptive refinement strategies are to be considered and reliable a-posteriori error estimation is needed. We therefore extend the a-posteriori error estimator in [5] to frictional contact and reconstruct a H^1-conforming displacement following the ideas in [3] and [4]. We prove reliablity of our error esitmator under similar assumptions as those made in [6] for uniqueness and test its efficiency by numerical experiments in two and three dimensions.

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A FRAMEWORK FOR APPLYING THE DWR METHOD ON VARIATIONAL INEQUALITIES

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In this talk, we consider variational inequalities of first and second kind including a smooth nonlinear differential operator. We rewrite the variational inequalities with the help of nonlinear complementarity (NCP) functions as a nonlinear problem. However, due to the nonsmoothness of the NCP functions, the resulting problem is not smooth and we have potentially to assume additional properties of the analytic solution to do the reformulation. We want to apply the dual weighted residual (DWR) method to estimate the discretization error in a user defined quantity of interest. The classic DWR approach, see, for instance, [1], relies on the fact that the semi linearform defining the problem is optimally three times directional differentiable. Due to the nonsmoothness of our formulation, we cannot directly apply the classic DWR approach, where the problem is linearized on the basis of the directional derivative of the underlying semi linearform. Instead, we use the active sets provided by the NCP functions to do the linearization w.r.t. them, while the differential operator is treated in the classic way. The arising dual problem is closely connected to the linear system of equations, which has to be solved in the last step of a semi smooth Newton method applied to the original problem. We derive an error identity, which consists in the primal residual and the dual residual plus a remainder term. The remainder term includes the error due to the linearization and is usually of higher order in the error. Hence, it is neglected. The primal residual and the dual residual are weighted by the dual and the primal discretization error, respectively. Thus, their evaluation has to be approximated numerically using suitable techniques. Finally, the error estimate is localized to the mesh cells by a filtering approach to utilize it in an adaptive strategy. A comparatively simple example, Signorini's problem, for the application of this framework is discussed first, cf. [3]. Then, the more complex case of frictional contact is considered, cf. [2]. Finally, we give an outlook on elasto-plastic problems and thermoplastic contact problems.

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ON QUASI-OPTIMALITY OF FINITE ELEMENT METHODS FOR THE OBSTACLE PROBLEM

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A finite element method is quasi-optimal if the error of its approximate solution is bounded in terms of the best error in the trial space and a multiplicative constant for any exact solution. Quasi-optimality is a very useful property, in particular in an adaptive context. In fact, adaptive decision are usually based upon the approximate solution and affect the approximation properties of the underlying trial space.

The elliptic obstacle problem may be viewed as a model case for variational inequalities. In this talk, we shall discuss the quasi-optimality of selected finite element methods for this model case.

RESIDUAL-TYPE A POSTERIORI ESTIMATORS FOR THE SINGULARLY PERTURBED VARIATIONAL INEQUALITY IN QUASI-STATIC FRACTURE PHASE-FIELD MODELS

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We consider a quasi-static fracture phase-field model which is given by a coupled system of an equation and a singularly perturbed variational inequality. The unknowns of the system are the displacements $\boldsymbol{u}$ of a linear elastic body and the phase-field variable φ which describes a diffusive transition zone between the broken ($\varphi = 0$) and the unbroken material ($\varphi = 1$). This zone has a half bandwidth ϵ, which is a model regularization parameter. The phase-field variable is constrained by the irreversibility condition, i.e. $\varphi^{n+1} \leq \varphi^n$ where n denotes the time step.

In the numerical simulation we are especially interested in a good approximation of the fracture growth. Therefore we use adaptive finite element methods which are based on a residual-type a posteriori estimator which we present here. Due to the irreversibility constraint standard residual a posteriori estimators are inappropriate. They would cause an overestimation where the constraints are active which is for example in the unbroken area. Thus, in order to derive a residual-type a posteriori estimator under the aspects of reliability, efficiency and robustness we follow a new approach [A. Veeser, Efficient and reliable a posteriori error estimators for elliptic obstacle problems. SIAM J. Numer. Anal. 39, 2001. pp. 146–167] for variational inequalities. We use an ϵ-dependent energy norm to measure the error in the phase-field variable and the corresponding dual norm to measure the error in the constraining forces. The resulting estimator [M. Walloth, Residual-type A Posteriori Estimators for a Singularly Perturbed Reaction-Diffusion Variational Inequality – Reliability, Efficiency and Robustness. Preprint 2018, arXiv:1812.01957] generalizes well-known and approved estimators such as standard residual estimators for linear elliptic problems, residual-type a posteriori estimators for obstacle problems and robust a posteriori estimators for singularly perturbed reaction-diffusion equations [R. Verfürth, Robust a posteriori error estimators for a singularly perturbed reaction-diffusion equation. Numer. Math. 78, 1998, pp. 479–493]. Numerical results complement the theoretical analysis.

AFEM FOR ELLIPTIC-H(CURL) VARIATIONAL INEQUALITIES OF THE SECOND KIND

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In this talk we discuss the analysis of an adaptive mesh refinement strategy for elliptic **curl**-**curl** variational inequalities of the second kind. We present a posteriori error estimators based on a careful combination between the Moreau-Yosida regularization approach and Nédélec's first family of edge elements. The reliability and efficiency of the estimators are proved through the use of a specific linear auxiliary problem involving the discrete Moreau-Yosida-regularized dual variable in combination with a local regular decomposition for **H**(**curl**)-functions and the well-known bubble functions. Hereafter, we propose an AFEM algorithm and discuss the convergence analysis thereof by using a limiting-space approach. Under a certain condition on the regularization parameter depending on the adaptive mesh size, we derive convergence results for the maximal error indicator and the corresponding residual. Finally, the strong convergence of the sequence of the adaptive solutions generated by the AFEM algorithm follows by combining all these mathematical findings. In the last part of the talk, the implementation of the algorithm is presented and applied to a problem stemming from the type-II (high-temperature) superconductivity.