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VARIATIONAL TIME DISCRETIZATION OF HIGHER ORDER AND HIGHER REGULARITY WITH APPLICATION TO WAVES AND INCOMPRESSIBLE FLOW

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We present families of variational space-time finite element discretisations which we combine with collocation conditions in order to get solutions of higher-order regularity in time of the fully discrete approximations. By condensation, linear systems show up that have less degrees of freedom than the corresponding pure variational formulations but lead to the same order of convergence, cf. [3].

Firstly, we apply this approach to the hyperbolic wave equation, written as a first-order in time system

$$\partial_t u - v = 0\,, \quad \partial_t v - \nabla \cdot (c\nabla u) = f$$

and equipped with appropriate initial and boundary conditions. This system is studied as a prototype model for elastic waves with applications in, e.g., non-destructive material inspection. We derive the linear system that leads to a globally C^1-regular in time solution. Optimal order error estimates for the fully discrete scheme are given and illustrated by challenging numerical experiments, cf. [1, 2].

Next, we extend our investigations to the incompressible nonstationary Navier–Stokes equations

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} - \nu\Delta\vec{v} + \nabla p = \vec{f}\,, \quad \nabla \cdot \vec{v} = 0\,.$$

We solve the resulting nonlinear system by Newton's method and analyze the convergence properties of the C^1-regular in time discrete solution numerically.

The subproblems of wave propagation and fluid flow are studied as building blocks for fluid-structure interaction which will be our work for the future.

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OPTIMAL ORDER ERROR ANALYSIS FOR GALERKIN–COLLOCATION APPROXIMATION OF WAVE PROBLEMS AND RELATED SCHEMES

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The elucidation of many physical problems in science and engineering is subject to the accurate numerical modelling of complex wave propagation phenomena. In multi-physics, time derivatives of unknowns can be involed as coefficient functions in some of the subproblems. Over the last decades high-order numerical approximation schemes for partial differential equations have become well-established tools.

Here we propose the combined higher-order variational and collocation approximation in time of wave equations. Continuous and discontinuous finite element approaches are used for the discretization in space. The conceptual basis is the establishment of a direct connection between the Galerkin method for the time discretization and the classical collocation methods, with the perspective of achieving the accuracy of the former with reduced computationals costs provided by the latter in terms of less complex linear algebraic systems. Higher order regularity in time of the discrete solution is ensured. Moreover, a related class of schemes that is based on a computationally cheap post-processing in time is proposed. The key ingredient is the construction of a special lifting operator that improves the discrete solution's regularity and approximation quality in time. A posteriori error control and adaptive mesh refinement mechanisms become feasible.

A unified framework for the derivation of optimal order a priori error estimates in time and space and in various norms is presented. For stabilty reasons, optimal-order error estimates for the time derivative are derived first and, then, used to bound the approximation error of the solution itself. For homogeneous right-hand side terms the conservation of energy is proved for the proposed classes of schemes which is a key feature in the numerical approximation of wave phenomena.

The results are a joint work with S. Becher, G. Matthies, F. Radu and F. Schieweck.

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HP-VERSION SPACE-TIME DISCONTINUOUS GALERKIN METHODS FOR PARABOLIC PROBLEMS ON GENERAL PRISMATIC MESHES.

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We present the *hp*-version space-time interior penalty discontinuous Galerkin (dG) finite element method for parabolic equations on general spatial meshes, giving rise to prismatic space-time elements. A key feature of the proposed method is the use of space-time elemental polynomial bases of total degree defined in the physical coordinate system. This approach leads to a fully discrete hp-dG scheme using fewer degrees of freedom for each time step compared to standard dG time-stepping schemes whereby spatial elemental bases are tensorized with temporal basis functions. A second key feature of the new space-time dG method is the incorporation of very general spatial meshes consisting of polygonal/polyhedral elements with arbitrary number of straight/curved faces.

SPACE-TIME DISCONTINUOUS GALERKIN METHOD FOR THE NUMERICAL SOLUTION OF THE RICHARDS EQUATION

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We deal with the numerical solution of the Richards equation describing time-dependent variably-saturated flow though porous media. It can be written in the form

$$\frac{\partial \vartheta(\Psi - z)}{\partial t} - \nabla \cdot (\boldsymbol{K}(\Psi - z))\nabla\Psi) = 0,$$

where Ψ is the hydraulic head [L], z is the geodetic head [L] (distance from the reference level), $\boldsymbol{K}$ is the unsaturated hydraulic conductivity tensor of the second order [L.T^{-1}] and ϑ [-] represents the volumetric water content and the term of a possible storativity of the porous material. The nonlinear function ϑ and $\boldsymbol{K}$ are given by constitutive relations and they can degenerate, i.e., vanish or blow up.

We discretize this problem by an adaptive higher-order space-time discontinuous Galerkin method which employs piecewise polynomial discontinuous approximation with respect to the space and time coordinates. The discretization leads to a system of nonlinear algebraic equations which are solved by a Newton-like method in the combination with the Anderson acceleration. Convergence problems related to the transition between unsaturated and saturated flow are eliminated by a regularization of the constitutive formulas. A special attention is paid to the realization of the seepage boundary condition (or outflow boundary condition) which is treated by a variable penalty in the context of the discontinuous Galerkin method.

We develop an adaptive technique which optimizes accuracy and efficiency by balancing the errors that arise from the space and time discretizations and from the resulting nonlinear algebraic system. Additionally, we adopt a *hp*-anisotropic mesh adaptation technique capable of generating unstructured triangular elements with optimal sizes, shapes, and polynomial approximation degrees. Several numerical experiments are presented to demonstrate the accuracy, efficiency, and robustness of the numerical method.

ON THE STABILITY OF THE SPACE-TIME DISCONTINUOUS GALERKIN METHOD FOR NONLINEAR PARABOLIC PROBLEMS IN TIME DEPENDENT DOMAINS

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The paper is devoted to the analysis of the space-time discontinuous Galerkin method (STDGM) applied to the numerical solution of nonstationary nonlinear convection-diffusion initial-boundary value problem in a time-dependent domain. The problem is reformulated using the arbitrary Lagrangian-Eulerian (ALE) method, which replaces the classical partial time derivative by the so-called ALE derivative and an additional convective term. The problem is discretized with the use of the ALE-space time discontinuous Galerkin method (ALE-STDGM). In the formulation of the numerical scheme we use the non-symmetric, symmetric and incomplete versions of the space discretization of diffusion terms and interior and boundary penalty. The nonlinear convection terms are discretized with the aid of a numerical flux. The goal is to prove the unconditional stability of the method. An important step is the generalization of a discrete characteristic function associated with the approximate solution and the derivation of its properties. It is important that the ALE technique can use different meshes on different time levels and different ALE mappings are prescribed for time slabs separately.

APOSTERIORI ANALYSIS OF HP-DISCONTINUOUS GALERKIN TIMESTEPPING FOR FULLY DISCRETIZED PARABOLIC PROBLEMS

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We derive aposteriori error bounds in time-maximum-space-squared-sums and time-mean-squares-of-spatial-energy norms for a class of fully-discrete methods for linear parabolic partial differential equations (PDEs) on the space-time domain based on hp-version discontinuous Galerkin time-stepping scheme combined with conforming spatial Galerkin finite element method. The proofs are based on a novel space-time reconstructions, which combines the elliptic reconstruction Georgoulis, Lakkis & Virtanen (2011), Lakkis & Makridakis (2006), and Makridakis & Nochetto (2003) of and the time reconstruction for discontinuous time-Galerkin schemes Makridakis & Nochetto (2006), Sch tzau & Wihler (2010) into a novel tool, allows for the user's preferred choice of aposteriori error estimates in space and careful analysis of mesh-change effects.

VARIATIONAL TIME DISCRETISATIONS OF HIGHER ORDER AND HIGHER REGULARITY

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Starting from the well-known discontinuous Galerkin (dG) and continuous Galerkin-Petrov (cGP) methods we will present a two-parametric family of time discretisation schemes which combine variational and collocation conditions. The first parameter corresponds to the ansatz order while the second parameter is related to the global smoothness of the numerical solution. Hence, higher order schemes with higher order regularity can be obtained by adjusting the family parameters in the right way.

All members of the considered family show the same stability properties as either dG or cGP. Furthermore, the considered schemes provide a cheap post-processing to achieve better convergence orders in integral-based norms. In addition, the post-processing could be used for adaptive time-step control.

Error estimates and numerical results will be given.

A SPACE-TIME DPG METHOD FOR ACOUSTIC WAVES IN HETEROGENEOUS MEDIA

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We apply the discontinuous Petrov-Galerkin method (DPG) to linear acoustic waves in space and time using the framework of first-order Friedrichs systems. Based on results for operators and semigroups of hyperbolic systems, we show that the ideal DPG method is well-posed on a suitable subset of the space-time cylinder. Therefore, we use the graph norm of the space-time differential operator, and traces are implicitly defined as distributions. Then, the practical DPG method is analyzed by constructing a Fortin operator numerically, and nonconforming traces are considered by comparison with an equivalent conforming scheme.

For our numerical experiments we introduce a simplified DPG method with discontinuous ansatz functions on the faces of the space-time skeleton, where the error is bounded by an equivalent conforming DPG method. Examples for a plane wave configuration confirms the numerical analysis, and the computation of a diffraction pattern illustrates a first step to applications. Finally, we present results for a benchmark configuration in seismic imaging with a point source and a small region with measurement points, where we show that the computation on a truncated space-time cylinder allows for a substantial reduction of degrees of freedom.

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