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Abstracts in alphabetical order

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STABILITY ANALYSIS OF THE SPACE-TIME DISCONTINUOUS GALERKIN METHOD FOR NONSTATIONARY PROBLEMS IN TIME-DEPENDENT DOMAINS

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In this talk we will present our results concerning the stability of the space-time discontinuous Galerkin method (STDGM) for the solution of nonstationary, linear convection-diffusion-reaction problem in time-dependent domains. In the first part of the talk we formulate the continuous problem using the arbitrary Lagrangian-Eulerian (ALE) method, which replaces the classical partial time derivative with the so called ALE-derivative and an additional convective term. After that we discretize our problem using the space-time discontinuous Galerkin method. In the formulation of the numerical scheme we use the nonsymmetric, symmetric and incomplete versions of the space discretization of the diffusion term and interior and boundary penalization. Finally in the third part of the talk we present our results concerning the unconditional stability of the method.
The results were obtained in cooperation with M. Feistauer.
The contribution will be presented in the mini-symposium "Finite Element methods for PDEs in time-dependent domains".

A HAMILTONIAN FINITE ELEMENT METHOD FOR NONLINEAR POTENTIAL FLOW FREE SURFACE WAVES

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An important mathematical model to describe nonlinear water waves is provided by the potential flow equation in combination with nonlinear free surface boundary conditions. This model assumes that the flow field is inviscid and irrotational and is suitable for large amplitude, non-breaking water waves away from boundary layers, where the Navier-Stokes equations are required to account for viscous effects. Computing large amplitude potential flow water waves is, however, non-trivial since the mesh deformation, necessary to accommodate for the free surface motion, and the nonlinearities can easily result in numerical instabilities. Many numerical discretisations include therefore additional stabilisation terms, but this results in unphysical wave damping that seriously reduces the numerical accuracy.

In this presentation we will present an alternative approach. The nonlinear free surface potential flow equations, when written in terms of the free surface potential and wave height, have a Hamiltonian structure. Preserving this Hamiltonian structure in

the finite element discretisation results in an energy preserving numerical discretisation with superior (long time) accuracy and no artificial wave damping.

The finite element discretisation is based on Luke's variational formulation [J. Fluid. Mech. 27(02):395–397, 1967], expressed in terms of the Lagrangian functional

$$\mathcal{L}(\phi, \eta) = -\int_0^T \int_{\Omega_t} g \cdot x + \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \, \mathrm{d}\Omega \, \mathrm{d}t,$$

where g denotes gravity, x is the coordinate vector, ϕ the potential velocity. The free surface height η appears only implicitly as part of the boundary of the domain Ω_t. The Lagrangian functional can be rewritten as a Hamiltonian, where the restriction of ϕ to the free surface takes the role of generalised momentum and η takes the role of generalised position.

The Lagrangian functional is used to obtain a system of ordinary differential equations for the nodal values in the finite element discretisation. After introducing a suitable transformation and a lengthy computation we can rewrite these ordinary differential equations as a Hamiltonian system. This Hamiltonian structure can be proven even for time-dependent, unstructured, moving and deforming meshes, including a wave maker and general bottom surface. Combined with a symplectic time integrator this results in a numerical discretisation with extraordinary stability properties, no artificial wave damping, and very good long time accuracy. We will demonstrate the preservation of the discrete energy and the accuracy of the finite element discretisation, including simulations of strongly interacting waves resulting in a large, highly nonlinear splash.

AN ASSESSMENT OF TIME DISCRETIZATIONS FOR SCALAR PDES IN TIME-DEPENDENT DOMAINS

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Many engineering and industrial processes are modeled using partial differential equations (PDEs) in time-dependent domains. Since the analytical solutions of these PDEs are almost impossible to obtain, the numerical approximation of these solutions is the only viable option, especially when the deformation of the domain is large. Apart from other challenges associated with the simulations of industrial processes, the presence of moving boundaries/interfaces makes the computation more complex. Moreover, the computational domain becomes a part of the numerical solution. Even though several approaches have been proposed to track/capture the moving boundaries, arbitrary Eulerian-Lagrangian (ALE) approach is preferred when the application demands accurate numerical solution and/or sharp moving boundaries/interfaces.

In ALE finite element schemes, PDEs can be written in (i) conservative and (ii) non-conservative ALE forms. Although both forms are equivalent in continuous level, these forms are no longer equivalent in (time) discrete level. Further, appropriate quadrature formula needs to be used for the time integration when different time discretizations are applied. This quadrature requirement is imposed as the geometric conservation law (GCL) in numerical schemes. Moreover, standard time discretizations, e.g., implicit Euler, Crank-Nicolson, discontinuous Galerkin, need to be modified in order to satisfy the GCL condition and to derive stability estimates. In this talk, an assessment of these modified time-discretizations applied to both conservative and non-conservative ALE forms will be presented. Further, the application of these methods in practical applications will also be discussed.

ALE FINITE ELEMENT METHOD FOR TWO-PHASE FLOWS WITH SURFACTANTS.

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Surface active agents (surfactants) are of importance in numerous practical applications. The influence of surfactants on the deformation of droplets and the structure of the surrounding flow field is an active research area. Local accumulation of surfactants and the resulting Marangoni forces may lead to a destabilization of the interface with essential consequences on the flow structure. This is a complex process whose tailored use in applications requires a fundamental understanding of the mutual interplay.

We present a finite element method for the flow of two immiscible, incompressible fluids in two and three dimensions. Thereby the presence of soluble and insoluble surfactants is considered. The finite element method uses the Arbitrary Lagrangian Eulerian (ALE) technique, which tracks the interface by moving grids. We use second order finite elements and a second order interface approximation, which allows precise incorporation of surface tension forces and Marangoni forces.

We consider a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, filled with two incompressible, immiscible fluids, which occupy at time t the domains $\Omega^i(t)$, $i = 1, 2$. Let $\partial\Omega^i(t)$ denote the boundary of $\Omega^i(t)$, $i = 1, 2$, and $\Gamma(t) = \partial\Omega^1(t) \cap \partial\Omega^2(t)$ the interface of $\Omega^1(t)$ and $\Omega^2(t)$. Our model consists of the Navier-Stokes equations for the flow fields u^i and pressure fields p^i in the phases $\Omega^i(t)$, $i = 1, 2$, a convection diffusion equation for the bulk surfactant c^i in $\Omega^i(t)$, $i = 1, 2$ and a convection diffusion equation on the moving manifold $\Gamma(t)$ for the surface surfactant c_Γ, completed with various coupling terms. The full model reads:

$$\varrho_i(\partial_t u^i + (u^i \cdot \nabla)u^i) - \nabla \cdot \mathbb{S}(u^i, p^i) = f, \quad \nabla \cdot u^i = 0 \quad \text{in } \Omega^i(t), \tag{1}$$

$$[-\mathbb{S}]n = \sigma(c_\Gamma)\kappa n + \nabla_\Gamma \sigma(c_\Gamma), \quad [u] = 0, \quad V = u \cdot n \quad \text{on } \Gamma(t), \tag{2}$$

$$\partial_t c^i - D^i \Delta c^i + (u^i \cdot \nabla)c^i = 0 \quad \text{in } \Omega^i(t), \tag{3}$$

$$[D\partial_n c] = -S(c^1, c^2, c_\Gamma) \quad \text{on } \Gamma(t), \tag{4}$$

$$\partial_t c_\Gamma - D_\Gamma \Delta_\Gamma c_\Gamma + \nabla_\Gamma \cdot (c_\Gamma u|_\Gamma) = S(c^1, c^2, c_\Gamma) \quad \text{on } \Gamma(t). \tag{5}$$

for $i = 1, 2$. Where, $\mathbb{S}$ is the usual stress tensor for Newtonian fluids, f describes gravitational forces, $[h] := h^1 - h^2$ denotes a jump of quantity h across the interface, $\sigma(c_\Gamma)$ is the surface tension coefficient, κ denotes the mean curvature of the interface, D^i is the diffusion coefficient for the bulk $\Omega^i(t)$, D_Γ is the surface diffusion coefficient, ∇_Γ and Δ_Γ are the surface version of the corresponding differential operators, S describes ad- and absorption of surfactant at the interface and while ∂_t denotes the time derivative, ∂_n denotes the spatial derivative in normal direction and ϱ_i the fluid density in phase i.

We study the influence of surfactants on the dynamics of two-phase flows. In several numerical tests we compare cases of soluble and insoluble surfactants and surfactant free settings.

STABILIZED CUTFEM FOR THE DISCRETIZATION OF TWO-PHASE INCOMPRESSIBLE FLOWS IN 3D

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For the mathematical modeling of two-phase flow phenomena we use the incompressible (Navier-)Stokes equations in each phase. The coupling of the phases is achieved by a surface tension force at the interface (coupling of the stresses normal to the interface) and a no-slip condition on the velocity tangential to the interface. For the evolution of the interface a level set technique is used.
For the finite element discretization of *one*-phase flow problems the Taylor-Hood P_2-P_1 pair is a popular choice due to the quadratic convergence and LBB-stability. For two-phase flow problems however, the P_2-P_1 discretization with unfitted meshes leads to a rather poor approximation quality of $\mathcal{O}(\sqrt{h})$ as P_1 elements are not able to represent discontinuities in the solution. Enriching the P_1 space with Heaviside jump functions one can recover the optimal approximation property, but numerical experiments indicate that the P_2-P_1X velocity-pressure pair is not LBB stable.
In [3] the enriched pressure space has been reduced by omitting the extended basis functions with small supports, which cause the instability. Introducing the so-called ghost penalty stabilization [1] for the pressure space results in a discrete inf-sup stability for a modified bilinear form. As opposed to the reduced XFEM space, the ghost penalty method does not need to reduce the approximation space and thus may lead to smaller errors in the solution. The added stability terms lead to a modified Schur complement and therefore the preconditioners have to be adapted in order to solve the system matrix efficiently. New preconditioning strategies developed in [2] are presented here. For a constructed Stokes model problem with an analytical solution both stabilization methods are compared with respect to the discretization errors and convergence rates. For a realistic, fully coupled Navier-Stokes rising droplet problem the stabilization methods are compared with respect to the resulting droplet position and velocity.

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PARAMETRIC FINITE ELEMENT METHODS FOR THE DYNAMICS OF FLUIDIC MEMBRANES AND VESICLES

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A parametric finite element approximation of a fluidic membrane, whose evolution is governed by a surface Navier–Stokes equation coupled to bulk Navier–Stokes equations, is presented. The elastic properties of the membrane are modelled with the help of curvature energies of Willmore and Helfrich type. Forces stemming from these energies act on the surface fluid, together with a forcing from the bulk fluid.

We introduce a stable parametric finite element method to solve this complex free boundary problem. Local inextensibility of the membrane is ensured by solving a tangential Navier–Stokes equations, taking surface viscosity effects of Boussinesq–Scriven type into account. In our approach the bulk and surface degrees of freedom are discretized independently, which leads to an unfitted finite element approximation of the underlying free boundary problem. Bending elastic forces resulting from an elastic membrane energy are discretized using an approximation introduced by Dziuk. The obtained numerical scheme can be shown to be stable and to have good mesh properties.

LOCAL PROJECTION STABILIZATION WITH DISCONTINUOUS GALERKIN METHOD IN TIME APPLIED TO TRANSIENT SCALAR EQUATION IN TIME DEPENDENT DOMAINS

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In this work, a stabilized finite element scheme combined with the discontinuous Galerkin(dG) discretization in time for the solution of a transient convection diffusion reaction equation in a time-dependent domain is analyzed. In particular, the local projection stabilization(LPS) and the higher order dG in time discretization is considered. Further, arbitrary Lagrangian Eulerian (ALE) is used to handle the domain deformation. The stability and error estimates are presented for the proposed numerical scheme.

The stabilization term in local projection method is based on a projection $\pi_h : V_h \to D_h$ of finite element approximation space V_h into a discontinuous space D_h. LPS was originally given as a two level method in which projection space D_h lies on a coarser grid, but this approach increases the discretization stencil [1]. In this work, we use the one level approach, in which the approximation space Y_h and projection space D_h are defined on the same mesh, with enrichment of the approximation space Y_h [3].

The analysis is mainly based on a quadrature formula for approximating the integrals in time. We consider exact integration in time which is impractical to implement, and the Radau quadrature in time, which can be used in practical [2]. The stability and error estimates are given for both the time integration methods. Analysis reveals that the numerical scheme for exact in time integration is unconditionally stable, while Radau quadrature in time is conditionally stable only with time step restriction depending on ALE map. The validation of the proposed local projection stabilization scheme with higher order discontinuous Galerkin time discretization is demonstrated with appropriate numerical examples.

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ON A PHASE FIELD APPROACH TO PDES ON BUBBLE CLUSTERS

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Modelling surface active agents (surfactants) in multi-phase flow leads to coupled bulk and interface advection-diffusion equations. Aiming for a phase field description we present a Cahn-Hilliard-Navier-Stokes system which is coupled to suitable PDEs for surfactant(s). The challenge is to correctly recover the conditions in the triple junctions if the fluid interfaces are given by thin layers rather than by hypersurfaces as in classical approaches. We will therefore focus on a diffuse interface approach to partial differential equations on evolving bubble clusters and network-like structures. Thanks to smoothing the problem by replacing the interfaces with thin layers we can use standard bulk finite elements though mesh adaptation is mandatory to make simulations feasible. Simulation results will be presented which support the theory behind the modelling approach.