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Abstracts in alphabetical order

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## FINITE ELEMENT APPROXIMATIONS FOR A FRACTIONAL LAPLACE EQUATION

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In this talk we deal with the integral version of the Dirichlet homogeneous fractional Laplace equation. For this problem, weighted and fractional Sobolev a priori estimates are provided in terms of the Hölder regularity of the data. By relying on these results, optimal order of convergence for the standard linear finite element method is proved for adapted meshes designed to handle the singular behavior of solutions near the boundary. Some numerical examples are given showing results in agreement with the theoretical predictions.

## ELLIPTIC PROBLEMS IN A NON-LIPSCHITZ DOMAIN

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In this work we review and analyze the approximation, by standard piecewise linear finite elements, of some elliptic problems in the plane domain $\Omega = \{(x;y) : 0 < x < 1; 0 < y < x^\alpha\}$; which gives, for $\alpha > 1$, the simplest model of an external cusp. The focus of interest resides in the fact that, since the domain is curved and non- Lipschitz, the problems under consideration had not been covered by the standard literature which only had dealt with polygonal or smooth domains.

First, since many of the results on Sobolev spaces, which are fundamental in the usual error analysis, do not apply to cusp domains [5], we had to develop trace and extension theorems in weighted Sobolev spaces, with the weight being a power of the distance to the cuspidal. These estimates allowed us to prove, for the Poisson problem, that the optimal order, with respect to the number of nodes, could be recovered by using appropriate graded meshes [3, 4, 1].

Then, we studied the Laplacian eigenvalue problem, in which the classical spectral theory could not be applied directly, and in consequence, this eigenvalue problem had to be reformulated in a proper setting in order to obtain quasi optimal order of convergence for the eigenpairs [2].

At present, we are studying a Steklov eigenvalue problem and the particular difficulties that arise in this problem.

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## FINITE ELEMENT APPROXIMATION FOR THE FRACTIONAL EIGENVALUE PROBLEM

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Given $s \in (0,1)$, the fractional Laplacian of order $s$ of a smooth function $u$ is defined by

$$(-\Delta)^s u(x) = C(n,s)\ \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy,$$

where $C(n,s)$ is a normalization constant. In this talk we address the equation

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded set. Even if $\Omega$ is an interval, it is very challenging to obtain closed analytical expressions for the eigenvalues and eigenfunctions of the fractional Laplacian. This motivates the utilization of discrete approximations of this problem; we study a conforming, piecewise linear finite element method. The main advantage of such an approximation is that it provides upper bounds for the eigenvalues, regardless of the regularity of the domain $\Omega$.

Unlike the classical Laplacian, eigenfunctions of the fractional Laplacian in $\Omega$ are not smooth up to the boundary; in particular, the first eigenfunction behaves as $d(x,\partial\Omega)^s$, and therefore it should not be expected to be more regular than $H^{s+1/2-\epsilon}(\Omega)$ for $\epsilon > 0$. We study the order of convergence for eigenvalues and eigenfunctions, both in the energy and the $L^2$-norm, and perform numerical experiments that illustrate the optimality of our theoretical findings. The eigenvalue estimates we provide are in good agreement with previous work by other authors.

# DOMAIN DECOMPOSITION METHODS WITH LOW-REGULARITY SOLUTION FOR NUCLEAR CORE REACTOR SIMULATIONS

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The behaviour of a nuclear core reactor depends on the nuclear chain reaction, which is described by the neutron transport equation. This equation is a balance statement that conserves neutrons. It governs the neutron flux density, which depends on 7 variables: 3 for the space, 2 for the motion direction, 1 for the energy (or the speed), and 1 for the time. In the steady-state case, one must solve an eigenvalue problem. The energy variable is discretized using the multigroup theory ($G$ groups). Concerning the motion direction, an inexpensive approach to approximate the transport equation is to solve the simplified $PN$ equations ($\frac{N+1}{2}$ coupled diffusion equations). It can be shown that the basic building block which allows to solve the general multigroup simplified $PN$ equations, is the so-called neutron diffusion equation set in a bounded domain $\Omega$ of $\mathbf{R}^3$ ($G = 1$, $N = 1$), which reads:
Find $\phi \in H^1(\Omega) \setminus \{0\}, \lambda \in \mathbf{R}^+$ such that:

$$\left\{ \begin{array}{rcll} -\mathrm{div}\, D\,\mathbf{grad}\,\phi + \Sigma_a\,\phi & = & \lambda\,\nu\Sigma_f\phi & \text{in } \Omega \\ \phi & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1}$$

Above, $D$, $\Sigma_a$, $\nu$ and $\Sigma_f$ denote respectively the diffusion coefficient, the macroscopic absorption cross section, the fission yield and the fission cross section. More precisely, we look for the criticality factor: $1/\min_\lambda \lambda$, together with the associated $\phi$ which corresponds to the averaged neutron flux density. Special attention is paid to the case where the solution $\phi$ to problem (1) is of low regularity. Such a situation commonly arises in the presence of three or more intersecting material components with different characteristics. As a matter of fact, the reactor cores often have a Cartesian geometry and the cross sections are averaged in every cell of the discretization. They may be constant or piecewise polynomial, and can differ from one cell to its neighbor by a factor of order 10.
We analyze matching and non-matching domain decomposition methods for the numerical approximation of the dual-mixed equations. The domain decomposition method can be non-matching in the sense that the meshes of the subdomains, and more generally the finite elements spaces, may not fit at the interface between subdomains. We prove well-posedness of the discrete problems with the help of a uniform discrete inf-sup

condition, and we provide optimal a priori convergence estimates. To improve the convergence rate, one can use a coarse grid correction based on the singular complement method. Numerical experiments illustrate the accuracy of the method.

## ON POSITIVITY OF THE DISCRETE GREEN'S FUNCTION AND DISCRETE HARNACK INEQUALITY FOR PIECEWISE LINEAR ELEMENTS

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In this talk we discuss some recent results obtained for the finite element discrete Green's function and its positivity. The first result shows that on smooth two-dimensional domains the discrete Green s function with singularity in the interior of the domain must be strictly positive throughout the computational domain once the mesh is sufficiently refined. As an application of this result, we establish a discrete Harnack inequality for piecewise linear discrete harmonic functions. In contrast to the discrete maximum principle, the result is valid for general quasi-uniform shape regular meshes except for a condition on the layer near the boundary.

# ADAPTED NUMERICAL METHODS FOR THE POISSON EQUATION WITH $L^2$ BOUNDARY DATA IN NON-CONVEX DOMAINS

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This talk is concerned with adapted numerical methods for the Poisson equation with $L^2$ boundary data and emphasis on non-convex domains. Due to the rough boundary data, the equation needs to be understood in the very weak sense. For a standard finite element discretization with regularized boundary data, a convergence order of 1/2 in the $L^2(\Omega)$-norm can be proved provided that the domain is convex. However, in non-convex domains the convergence rate is reduced although the solution remains to be contained in $H^{1/2}(\Omega)$. The reason is a singularity in the solution of the dual problem. In this talk, as a remedy, both a standard finite element method with mesh grading and a dual variant of the singular complement method are proposed and analyzed in order to retain a convergence rate of 1/2 also in non-convex domains. Finally, numerical experiments are presented in order to illustrate the theoretical results.

# ENERGY-CORRECTION METHOD FOR DIRICHLET BOUNDARY CONTROL PROBLEM

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Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with a re-entrant corner, i.e. corner with an angle $\Theta > \pi$, with disjoint boundary parts $\Gamma_D$ and $\Gamma_C$, satisfying $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_C$. In this talk we consider the optimal Dirichlet control problem in the energy space [2]. This problem is defined as a minimization of the following tracking-type functional

$$\mathcal{J}(u,z) = \frac{1}{2}\|u - \overline{u}\|^2_{L^2(\Omega)} + \frac{\rho}{2}\|z\|^2_{H^{\frac{1}{2}}(\Gamma_C)},$$

subject to the constraint

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ u &= z \quad \text{on } \Gamma_C, \end{aligned}$$

and the control constraints

$$z_a \leq z \leq z_b \quad \text{a.e. on } \Gamma_C.$$

We present the saddle-point structure of the problem and investigate the behaviour of the piecewise linear finite element approximation. Its convergence order is lower due to the reduced regularity in the presence of re-entrant corner. Recently, an effective method of recovering the full second-order convergence for elliptic equations on domains with re-entrant corners, when measured in locally modified $L_2$ and $H^1$ norms, known as energy-correction, has been proposed [1]. This method is based on a modification of a fixed number of entries in the system's stiffness matrix. We show how energy-correction method can be successfully applied to regain optimal convergence in weighted norms for optimal control problems. All theoretical results are confirmed by numerical test.

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# FINITE ELEMENT APPROXIMATION OF GRADIENT CONSTRAINT ELLIPTIC OPTIMIZATION PROBLEMS ON NON-SMOOTH DOMAINS

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In this talk, we are concerned with the discretization of PDE constrained optimization problems with pointwise constraints on the gradient of the state. Particular emphasis will be given to the case of non smooth domains, where the control to state mapping does not assert the gradient of the PDE solution to be Lipschitz. Nonetheless, convergence of the finite element approximation can be shown.