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Abstracts in alphabetical order

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THE DOUBLE ADAPTIVITY ALGORITHM

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The ideal DPG method [2] reproduces the stability of the continuous problem and guarantees optimal convergence for any well posed problem. The broken test spaces methodology makes it computationally efficient and can be applied to any well posed variational formulation [2]. The practical DPG method approximates the Riesz (error) representation function ψ using an enriched test space. Needless to say, the ultimate success of the practical DPG method hinges on controlling the error in resolving ψ. For standard, "mathematician" test norms, the resolution of ψ is relatively easy and the damage due to the error in ψ can be estimated via the construction of appropriate Fortin operators [2,3]. For challenging singular perturbation problems, and test norms involving the perturbation parameter, resolution of ψ is challenging but not because of stability (as for the original problem) bur rather approximability issues.

The double adaptivity idea of Cohen, Dahmen and Welper [1] calls for introducing an inner adaptivity loop to control the error in ψ. The adaptively determined enriched test space is "custom made" for the particular load and the trial space, and it does not imply the discrete stability. And yet the ultimate method converges.

I will present a series of 1D and 2D double adaptivity experiments for convection dominated diffusion. Out of many possible variational formulations, the ultraweak formulation stands out as the corresponding optimal test norm is known explicitly, and it is robustly equivalent to the adjoint graph norm (with a properly scaled L^2-term). Consequently, the DPG method delivers an orthogonal projection in an energy norm robustly equivalent to the trial L^2-norm. The adjoint graph norm, however, is difficult to resolve, and the double adaptivity comes in as a natural means to cope with the problem.

The inner adaptivity loop requires a robust a-posteriori error estimate for the discretization of Riesz representation function ψ. A residual estimate seems to be a natural (if not the only possible) option. For a broken test space, the residual is equal to the sum of element residuals, so the residual estimation is naturally reduced to a single element K. Cumbersome construction of Clement-like interpolation operators, necessary for standard conforming methods, reduces to a simple orthogonal projection in the test norm. The element residual estimate leads to a number of multiscale generalized eigenvalue problems involving the test norm, $L^2(K), L^2(\partial K)$ and $H^{-1}(\partial K)$ norms. The eigenvalue problems are solved off line, harvesting appropriate "interpolation" constants for different values of diffusion ϵ, element size h, enriched element order r, and advection vector components. The precomputed constants enter then the residual estimate.

Ideally, one should use two independent meshes, one for the original unknown u, and the second for Riesz representation ψ. The dynamically determined mesh for ψ depends upon approximate solution u_h (and, therefore, the first mesh). For practical reasons, we attempt to use the same mesh for both unknowns, enriching only the order of approximation for ψ. If the maximum order is reached, we force h-refinements and restart the whole problem. 1D and 2D numerical experiments indicate that, for small diffusion, the adaptivity process is driven entirely by the resolution of ψ, i.e. the inner adaptivity loop. This is rather disappointing as we would like to see a robust solution for very coarse meshes (which is critical for nonlinear problems).

In the end, we will present experiments based on the ideas of Broersen and Stevenson [4] based on evolving a pure convection to a convection-diffusion problem. With a proper selection of a variational formulation, the underresolved Riesz representation function ψ for the confusion problem, represents a perfect approximation for the corresponding Riesz representation function for the pure convection problem. The game involves also relaxing the full stop outflow boundary conditions which must evolve with the mesh. The numerical results are promising but, at the moment, we do not have a full understanding of the underlying mathematics. We hope to understand it better by the time of the conference.

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ROBUST COUPLING OF DPG AND BEM FOR A SINGULARLY PERTURBED TRANSMISSION PROBLEM

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In this talk we present our recent work [Führer, Heuer: Robust coupling of DPG and BEM for a singularly perturbed transmission problem, arXiv:1603.05164], in which we consider a transmission problem consisting of a singularly perturbed reaction diffusion equation on a bounded domain and the Laplacian in the exterior, connected through standard transmission conditions. We establish a DPG scheme coupled with Galerkin boundary elements for its discretization, and prove its robustness for the field variables in so-called *balanced norms*. Our coupling scheme is the one from [Führer, Heuer, Karkulik: On the coupling of DPG and BEM, arXiv:1508.00630], adapted to the singularly perturbed case by using the scheme from [Heuer, Karkulik: A robust DPG method for singularly perturbed reaction diffusion problems, arXiv:1509.07560]. Essential feature of our method is that optimal test functions have to be computed only locally. We report on various numerical experiments in two dimensions.

MINIMUM RESIDUAL METHODS APPLIED TO LINEAR THERMOVISCOELASTICITY

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The motivation is to study void formation inside thermoset polymers used as matrices for composite materials that act as electrical insulators inside form-wound coils of large medium-voltage electromachinery. A full derivation of the linear first order system of thermoviscoelastic equations in the time and frequency domain is presented. Compatible variational formulations with unbroken test spaces and broken test spaces are deduced for the thermoviscoelasticity equations in the frequency domain. A minimum residual method with broken test spaces, i.e. the discontinuous Petrov-Galerkin (DPG) methodology, is applied to the "broken" variational formulation to solve the equations. Expected convergence rates for $p = 1, 2, 3$ are observed for a manufactured setting with a smooth solution. Preliminary results used to validate experimental data are also shown.

A DPG METHOD FOR THE HEAT EQUATION

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We present and analyse a time-stepping DPG method for the heat equation. Motivation of this work is to develop a DPG framework that can lead to robust approximations of singularly perturbed parabolic problems.

We use the backward Euler scheme as time discretisation and propose a DPG space approximation of the time-discrete scheme. Well-posedness and stable approximation properties are obtained from a precise analysis of the underlying time-discrete variational formulation at every time step. Appropriate convergence properties for field variables are proved. We present numerical experiments that underline our theoretical results.

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SOME RECENT PROGRESS WITH THE DPG METHOD

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A growing interest for the DPG method is developing in our community. In this talk we reformulate the method as the approximate solution of a convex optimization problem. We then demonstrate some recent discoveries which stem from the generality of this formulation.

Topics discussed for the linear theory will include the solution of problems with more than one variational formulation in the same domain (e.g. mixed + primal + ultraweak), inequality constraints, and optimal test norms of primal linear elasticity. We will also illustrate the built-in adaptivity and stability of the method with a nonlinear viscoelastic fluid flow benchmark problem.

THE NONLINEAR PETROV–GALERKIN METHOD IN BANACH SPACES: YET ANOTHER IMPROVEMENT OF BABUŠKA'S *A PRIORI* ERROR ESTIMATE

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In a recent 2015 paper by Stern [1], the author has sharpened the classical Babuška's *a priori* error estimate for Petrov–Galerkin methods in Banach spaces (cf. [2], 1971). The estimate had been previously sharpened only for the case of Hilbert spaces in a 2003 paper by Xu & Zikatanov [3] (more than 30 years after Babuška's result!). All of these estimates rely on a compatibility condition between the discrete *trial* and *test* spaces, known a the *discrete inf–sup condition*.

From a different point of view, inspired in the residual minimization approach [4] and the Hilbert-space theory of optimal Petrov-Galerkin methods [5], we address the question of how to inherit discrete stability from continuous stability in a Banach space setting. As a result, we deduce the nonlinear Petrov–Galerkin method in [6], whose implementable (inexact) version consists in a monotone mixed method.

In this talk, we show in detail the error estimates of the method proposed in [6], which depend explicitly on geometrical constants of the involved Banach spaces.

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GEOMETRIC MULTIGRID FOR SCALABLE DPG SOLVES IN CAMELLIA

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The discontinuous Petrov-Galerkin finite element methodology of Demkowicz and Gopalakrishnan (DPG) [1, 2] offers a host of appealing features, including automatic stability and minimization of the residual in a user-controllable energy norm. DPG is, moreover, well-suited for high-performance computing, in that the extra work required by the method is embarrassingly parallel; the use of a discontinuous test space allows the computation of optimal test functions to be done element-wise. Additionally, the approach gives almost total freedom in the choice of basis functions, so that high-order discretizations can be employed to increase *computational intensity* (the number of floating point operations per unit of communication). Finally, since the method is stable even on a coarse mesh and comes with a built-in error measurement, it enables robust adaptivity which in turn means less human involvement in the solution process, a desirable feature when running large-scale computations.

Camellia [3] is a software framework for DPG with the aim of enabling rapid development of DPG solvers both for running on a laptop and at scale. Camellia supports spatial meshes in 1D through 3D; initial support for space-time elements is also available. Camellia supports h- and p-adaptivity, and offers distributed computation of essentially all the algorithmic components of a DPG solve. (One exception, which we plan to address, is the generation and storage of the mesh geometry; at present, this happens redundantly on each MPI rank.) Camellia supports static condensation for reduction of the global problem, and has a robust, flexible interface for using third-party direct and iterative solvers for the global solve.

Until recently, we have almost always solved the global DPG system matrix using parallel direct solvers such as SuperLU_Dist. This is not a scalable strategy, particularly for 3D and space-time meshes. Both memory and time costs therefore motivate our recent work, developing and studying iterative solvers in the context of a range of example problems. Since Camellia's adaptive mesh hierarchy provides us with rich geometric information, we focus on hp-geometric multigrid preconditioners with additive Schwarz smoothers of minimal or small overlap. Preconditioning a conjugate gradient solve using such preconditioners, we are able to solve much larger problems within the same memory footprint.

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FIRST-ORDER SYSTEM $LL^\star$ USING NONCONFORMING TEST FUNCTIONS

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The first-order system $LL^\star$ formulation is based on the ultra weak formulation

$$\langle U, L^* V\rangle = F(V) \; \forall \; V$$

of some first-order system of differential equations $LU = F$ and closely related to the DPG methodology. It is obtained by setting $U = L^* W$ with W being in the test space, therefore leading to a self-adjoint coercive variational problem. We consider the $H(\text{div}) \times H^1$ first-order system $LL^\star$ formulation studied in [Z. Cai, R. Falgout and S. Zhang, SIAM J. Numer. Anal. **53** (2015), 405–420] for Poisson-type equations. The local conservation properties of the method using next-to-lowest-order Raviart-Thomas spaces for $H(\text{div})$ combined with quadratic nonconforming elements for H^1 are investigated in this contribution. This will also be discussed in the context of conservation of momentum in a stress-velocity formulation of the Stokes system.

A STABLE DPG FORMULATION OF TRANSPORT EQUATIONS

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We formulate and analyze a Discontinuous Petrov Galerkin formulation of linear transport equations with variable convection fields. We show that a corresponding *infinite dimensional* mesh-dependent variational formulation, in which besides the principal field also its trace on the mesh skeleton is an unknown, is uniformly stable with respect to the mesh, where the test space is a certain product space over the underlying domain partition.

Our main result states then the following. For piecewise polynomial trial spaces of degree m, we show under mild assumptions on the convection field that piecewise polynomial test spaces of degree $m+1$ over a refinement of the primal partition with uniformly bounded refinement depth give rise to uniformly (with respect to the mesh size) stable Petrov-Galerkin discretizations.

Finally we show how rigorously computable a posteriori error bounds can drive a convergent adaptive algorithm.

THE NONLINEAR PETROV–GALERKIN METHOD IN BANACH SPACES: ELIMINATING THE GIBBS PHENOMENA

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Is it possible to obtain near-best approximations to solutions of linear operator equations in a general Banach-space setting? Can this be done with guaranteed stability?

In this talk we address these questions by considering nonstandard, nonlinear Petrov–Galerkin discretisations, proposed in [1], which aim to guarantee stability in general Banach-space settings, and builds on ideas of residual minimisation [2] and the recent Hilbert-space theory of optimal Petrov-Galerkin methods [3].

We demonstrate that the inexact (implementable) version is naturally related to a mixed method with a monotone nonlinearity. For this method, optimal a priori error estimates hold (a la Céa / Babuška), with constants depending on the geometry of the involved Banach spaces.

As an elementary, but important, application of the nonlinear Petrov–Galerkin method, we consider the advection equation in dual Sobolev spaces (of integrability p). It is demonstrated that in the approximation of solutions with discontinuities, the Gibbs phenomena, which is inherently present in the Hilbert case ($p = 2$), is eliminated as $p \searrow 1$.

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