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Abstracts in alphabetical order

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A FULLY CONSERVATIVE AND SUPERCONVERGENT DISCONTINUOUS GALERKIN METHOD FOR THIRD-ORDER LINEAR EQUATIONS IN ONE SPACE DIMENSION

Yanlai Chen[1a], Bernardo Cockburn[2] and Bo Dong[1b]

[1]Department of Mathematics, University of Massachusetts Dartmouth,
285 Old Westport Road, North Dartmouth, MA 02747, USA
[a]yanlai.chen@umassd.edu, [b]bdong@umassd.edu

[2]School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
cockburn@math.umn.edu

We introduce a Bassi-Rebay type discontinuous Galerkin method for both stationary and time-dependent third-order linear equations. This method is *the first* discontinuous Galerkin method which conserves the mass and the L^2-norm of the approximations of the solution *and* that of its first and second derivatives. For the stationary case, L^2-projections of the errors (in the approximation of the solution, its first and second derivatives) are proven to have optimal convergence rates when the polynomial degree k is even and the mesh is uniform, and to converge sub-optimally, but sharply, with order k when k is odd or the mesh is non-uniform. We show that suitably defined projections of the errors superconverge with order $k+1+\min\{k, \frac{1}{2}\}$ on uniform meshes and converge optimally on non-uniform meshes. The numerical traces are proven to superconverge with order $2k$ if k is odd or the mesh is non-uniform. For even k and uniform meshes, we show that the numerical traces superconverge with order $2k+\frac{3}{2}$. If in addition, the number of intervals is odd, the convergence order is improved to $2k+\frac{3}{2}+\min\{k, \frac{1}{2}\}$. This allows us to use an element-by-element postprocessing to construct new approximations that superconverge with the same orders as the numerical traces. For the time-dependent case, the errors are proven to be of order $k+1$ for even k on uniform meshes, and of order k when k is odd or the mesh is nonuniform. Numerical results are displayed which verify all of the above-mentioned theoretical orders of convergence as well as the conservation properties of the method. We also show that the orders of convergence of the the stationary case also hold for the time-dependent case.

GEOMETRY OF ROTATED SMOOTHNESS-INCREASING-ACCURACY-CONSERVING (SIAC) FILTERS

Julia Docampo Sánchez[1a], Jennifer K. Ryan[1b] and Robert M. Kirby[2]

[1]School Of Mathematics, University of East Anglia, Norwich, UK.
[a]J.Docampo@uea.ac.uk, [b]jennifer.ryan@uea.ac.uk

[2]Scientific Computing and Imaging Institute,
University of Utah, Salt Lake City, UT, USA.
kirby@sci.utah.edu

Over the past decades there has been a strong effort on developing a family of filters, Smoothness-Increasing-Accuracy-Conserving (SIAC) filters, designed to extract superconvergence from discontinuous Galerkin (DG) solutions. The filtering technique is a point-wise convolution of a B-Spline kernel with the DG solution at final time, resulting in a smoother solution and in many cases, of higher order acuracy. These advantages can be exploited during flow visualization of Partial Differential Equations (PDEs). For example, introducing the filter between the underlying Discontinuous Field and a streamline solver, produces locally a high order smooth solution, allowing implementation of relatively simple schemes whilst obtaining satisfactory curves. In this work we concentrate on the filter geometry and its impact on error. We introduce the rotated SIAC filter and discuss the kernel orientation, support size and splines configuration to obtain maximum error reduction from the DG solution for its applications to streamline visualization.

DISCONTINUOUS GALERKIN METHODS: TIME EVOLUTION OF SUPERCONVERGENCE PROPERTIES

Daniel Frean

Department of Mathematics, University of East Anglia, UK
d.frean@uea.ac.uk

Numerical solutions to hyperbolic conservation laws whose spatial discretisation is completed by the discontinuous Galerkin (DG) method often profit from a superconvergence property. In this talk, we consider how superconvergence properties are affected by the pairing of DG with a time-stepping method as well as by the choice of flux function. We present theoretical results for linear equations and illustrate nonlinear cases with numerical experiments.

This is joint work with Jennifer Ryan.

REVISITING ACCURACY PRESERVING PROPERTIES OF SIAC FILTERING FROM AN APPROXIMATION THEORY PERSPECTIVE

Mahsa Mirzargar[1a], Robert M. Kirby[1b] and Jennifer K. Ryan[2]

[1]Scientific Computing and Imaging Institute, University of Utah,
Salt Lake City, UT, USA.
[a]mirzargar@sci.utah.edu, [b]kirby@sci.utah.edu

[2]School Of Mathematics, University of East Anglia, Norwich, UK.
jennifer.ryan@uea.ac.uk

Filtering plays a crucial role in various scientific and engineering applications where the accuracy and efficiency of the filtering scheme varies based on the application. SIAC filtering is a class of B-spline-based techniques that aim to increase the smoothness of the DG approximation while conserving the inherent order of accuracy of the DG solution (i.e., superconvergence). Accuracy-order *conserving* properties of SIAC filter have been well-established in the literature. From an approximation theory perspective, accuracy *preserving* properties of SIAC kernel has a close connection with spline approximation of polynomial spaces that has not been thoroughly investigated before. In this talk, we summarize our theoretical results that establish this connection and provide a unified view of SIAC filtering. This unified view bridges the analysis gap between accuracy-order conserving properties of SIAC filtering and its accuracy-preserving properties. Our results broaden the mathematical analysis tools available for analyzing and designing new filtering schemes using variations of B-splines with desirable approximation properties and kernel supports.

DISCONTINUOUS GALERKIN METHODS FOR NONLINEAR SCALAR HYPERBOLIC CONSERVATION LAWS: DIVIDED DIFFERENCE ESTIMATES AND ACCURACY ENHANCEMENT

Xiong Meng[a] and Jennifer K. Ryan[b]

School of Mathematics, University of East Anglia, UK
[a]x.meng@uea.ac.uk, [b]Jennifer.Ryan@uea.ac.uk

In this talk, we present an analysis of the accuracy-enhancement for the discontinuous Galerkin (DG) method applied to one-dimensional scalar nonlinear hyperbolic conservation laws. This requires analyzing the divided difference of the errors for the DG solution. We therefore first prove that the α-th order ($1 \leq \alpha \leq k+1$) divided difference of the DG error in the L^2 norm is of order $k + \frac{3}{2} - \frac{\alpha}{2}$ when upwind fluxes are used, under the condition that $|f'(u)|$ possesses a uniform positive lower bound. By the duality argument, we then derive superconvergence results of order $2k + \frac{3}{2} - \frac{\alpha}{2}$ in the negative-order norm, demonstrating that it is possible to extend the Smoothness-Increasing Accuracy-Conserving filter to nonlinear conservation laws to obtain at least $(\frac{3}{2}k + 1)$th order superconvergence for post-processed solutions. As a by-product, for variable coefficient hyperbolic equations, we provide an explicit proof for optimal convergence results of order $k + 1$ in the L^2 norm for the divided differences of DG errors and thus $(2k + 1)$th order superconvergence in negative-order norm holds. Numerical experiments are given that confirm the theoretical results.

NON-UNIFORM FILTERS VIA SHIFT AND SCALE FOR DISCONTINUOUS GALERKIN OUTPUT

Jörg Peters[a] and Dang-Manh Nguyen[b]

Department of Computer & Information Science & Engineering,
University of Florida, USA.
[a]jorg@cise.ufl.edu, [b]dmNguyen@cise.ufl.edu

Convolving the output of Discontinuous Galerkin computations with symmetric Smoothness-Increasing Accuracy-Conserving (SIAC) filters can improve both smoothness and accuracy. To extend convolution to the boundaries, several one-sided spline filters have recently been developed. This paper interprets these filters as instances of a general class of position-dependent (PSIAC) spline filters that can have non-uniform knot sequences and skip B-splines of the sequence.

PSIAC filters with rational knot sequences have rational coefficients. For prototype knot sequences, such as integer sequences that may have repeated entries, PSIAC filters can be expressed in symbolic form. Based on the insight that filters for shifted or scaled knot sequences are easily derived by non-uniform scaling of one prototype filter, a single filter can be re-used in different locations and at different scales. Computing a value of the convolution then simplifies to forming a scalar product of a short vector with the local output data. Restating one-sided filters in this form improves both stability and efficiency compared to their original formulation via numerical integration. PSIAC filtering is demonstrated for several established and one new boundary filter.