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AN ANISOTROPIC A PRIORI ERROR ANALYSIS FOR A CONVECTION DIFFUSION PROBLEM USING AN HDG METHOD

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In this talk we present an a priori error analysis for a convection diffusion problem, considering an HDG method and a family of anisotropic triangulation. As result, we deduce that when diffusion is dominant, the behaviour of the method (considering k as degree of approximation for every unknown) is such that the global L^2-norm of the error of the scalar and vector unknowns converge with order $k+1$, while the unknown related to the trace of scalar unknown, on the skeleton of the mesh, does with order $k+2$. For convection dominated diffusion equation, isotropic triangulations are not suitable. However, the use of anisotropic meshes let us to recover the convergence of the method, once the boundary or inner layer is solved. Numerical examples confirm these theoretical results.

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HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR TIME-DOMAIN ELECTROMAGNETICS

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Discontinuous Galerkin (DG) methods have been the subject of numerous research activities in the last 15 years and have been successfully developed for various physical contexts modeled by elliptic, mixed hyperbolic-parabolic and hyperbolic systems of PDEs. Despite many advantages, one major drawback of high order DG methods is their intrinsic cost due to the very large number of globally coupled degrees of freedom as compared to classical high order conforming finite element methods. This in particular the case when one consider the possibility of using an implicit scheme for the time integration of an hyperbolic system of equations such as the system of Maxwell equations in the time-domain. Different attempts have been made in the recent past to improve this situation and one promising strategy has been recently proposed by Cockburn et al. [1] in the form of so-called hybridizable DG (HDG) formulations. The distinctive feature of these methods is that the only globally coupled degrees of freedom are those of an approximation of the solution defined only on the boundaries of the elements of the discretization mesh. Since then, this kind of methods has been developed for various physical models [3, 4]. In the case of Maxwell's equations, HDG methods have been mainly developed for time-harmonic problems [2, 5]. Thereby, the present work is concerned with the study of such a HDG method for the solution of the three-dimensional Maxwell equations in time-domain. On one hand, we are interested in designing a high order HDG method that can handle efficiently locally refined unstructured meshes by considering the possibility of using a fully implicit time scheme or a hybrid implicit-explicit (IMEX) time scheme. On the other hand, we are concerned with applications involving the interaction of light with matter at the nanoscale which possibly requires solving the system of time-domain Maxwell PDEs coupled to a system of ODEs modeling the dispersive properties of metallic nanostructures.

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BRIDGING HYBRID HIGH-ORDER METHODS AND HYBRIDIZABLE DISCONTINUOUS GALERKIN METHODS

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We consider here the application of the recently introduced Hybrid High-Order (HHO) method [3] to the model problem: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \boldsymbol{\kappa}\nabla u \cdot \nabla v = \int_\Omega fv \qquad \forall v \in H_0^1(\Omega), \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded, connected polyhedral domain and $\boldsymbol{\kappa}$ a piecewise constant, bounded, symmetric, uniformly positive-definite matrix-valued function.

The HHO method supports general polyhedral meshes and delivers an arbitrary-order accurate approximation using face-based discrete unknowns that are polynomials of degree at most $k \geq 0$ on each face. The construction hinges on two key ingredients: (i) a polynomial reconstruction of the potential of degree $(k+1)$ in each mesh cell and (ii) a face-based stabilization consistent with the high-order provided by the reconstruction. The design relies on intermediate cell-based discrete unknowns in addition to the face-based ones (hence, the term hybrid), which can be locally eliminated by static condensation. Besides the original method with cell-based unknowns of degree k, we consider here some new variants with cell unknowns of degree $(k-1)$ and $(k+1)$.

The main contribution of this work is to recast the HHO method into an equivalent mixed formulation and to identify the corresponding conservative numerical flux. We show, in particular, how the solution provided by the HHO method can be characterized as the solution of local problems which are then matched by a single global equation. Such equation can be interpreted as a discrete version of a transmission condition.

This new reformulation enables a comparison to Hybridizable Discontinuous Galerkin (HDG) methods within the general framework introduced in [2]. We show, in particular, that both the local spaces and numerical trace of the flux are novel, distinctive choices which enrich the family of HDG methods. In particular, the spaces for the flux are much smaller than the ones previously known, and the stabilization function displays a rich structure that allows for optimal convergence of both the potential u (with order $(k+2)$) and its flux $\boldsymbol{q} := -\boldsymbol{\kappa}\nabla u$ (with order $(k+1)$) on general meshes composed of polyhedral cells. We also show that one of the novel variants of the method bears relations with the recently introduced High-Order Mimetic method.

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SUPERCONVERGENT HDG METHODS FOR THIRD-ORDER EQUATIONS IN ONE-SPACE DIMENSION

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We design and analyze the first hybridizable discontinuous Galerkin methods for stationary, third-order linear equations in one-space dimension. The methods are defined as discrete versions of characterizations of the exact solution in terms of local problems and transmission conditions. They provide approximations to the exact solution u and its derivatives $q := u'$ and $p := u''$ which are piecewise-polynomials of degree k_u, k_q and k_p, respectively. We consider the methods for which the difference between these polynomial degrees is at most two. We prove that all these methods have superconvergence properties which allows us to prove that their numerical traces converge at the nodes of the partition with order at least $2\,k+1$, where k is the minimum of k_u, k_q, k_p. This allows us to use an element-by-element post-processing to obtain new approximations for u, q and p converging with order at least $2k+1$ uniformly. Numerical results validating our error estimates are displayed.

HDG METHODS FOR DIFFUSION PROBLEMS

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We present an HDG formulation for a model diffusion equation on a polygonal/polyhedral mesh. We then show how to obtain optimal and superconvergent HDG methods by carefully choosing the approximate finite element spaces; see [1, 2, 3]. We also briefly discuss another approach to superconvergence by carefully choosing the stabilization operator; see [4, 5].

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THE HDG METHOD FOR IMPLICIT LARGE EDDY SIMULATION OF TRANSITIONAL TURBULENT FLOWS

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We present a high-order Implicit Large-Eddy Simulation (ILES) approach for simulating transitional turbulent flows. The approach consists of hybridized Discontinuous Galerkin (DG) methods for the discretization of the Navier-Stokes (NS) equations and a parallel preconditioned Newton-GMRES method for the resulting nonlinear system of equations. The combination of hybridized DG methods with an efficient solution procedure leads to a high-order accurate NS solver that is competitive with finite volume codes in terms of computational cost. The proposed approach is applied to transitional turbulent flows over a NACA 65-(18)10 compressor cascade and an Eppler 387 wing at Reynolds numbers up to 300,000. Grid convergence studies are presented and the required resolution to capture transition at different Reynolds numbers is investigated. Numerical results show rapid convergence and excellent agreement with experimental data. This work aims to demonstrate the potential of high-order ILES for transition prediction and present a rationale for this approach through empirical observations.

A HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR THE P-LAPLACIAN

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We propose the first hybridizable discontinuous Galerkin (HDG) method for the p-Laplacian equation. When using polynomials of degree $k \geq 0$ for the approximation spaces of u, ∇u, and $|\nabla u|^{p-2}\nabla u$, the method exhibits optimal $k+1$ order of convergence for all variables in L^1- and L^p-norms in our numerical experiments. For $k \geq 1$, an element-wise computation allows us to obtain a new approximation u_h^* that converges to u with order $k+2$. We rewrite the scheme as discrete minimization problems in order to solve them with nonlinear minimization algorithms. The unknown of the first problem is the approximation of u on the *skeleton* of the mesh but requires solving nonlinear local problems. The second problem has the approximation on the elements as an additional unknown but it only requires solving linear local problems. We present numerical results displaying the convergence properties of the methods, demonstrating the utility of using frozen-coefficient preconditioners, and indicating that the second method is superior to the first one even though it has more unknowns.

REDUCED ORDER HDG METHODS BASED ON GENERAL POLYGONAL MESHES

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Recently in a series of papers, we developed a class of reduced order HDG methods for various linear and nonlinear problems. A main feature of this approach is to apply different polynomial spaces for the unknowns. It was first discovered in 2009 by Lehrenfeld in his thesis for diffusion problem. Under the standard HDG framework, if we apply P_{k+1} polynomial space for the pressure while we still use P_k spaces for the other two unknowns, by a simple modification of the numerical flux we can obtain optimal order of convergence for all unknowns. The analysis is valid for general polygonal meshes. In this talk, we will present this general framework for linear elasticity, convection-diffusion and steady Navier-Stokes equations.