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Abstracts in alphabetical order

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A HIGH-ORDER DISCONTINUOUS GALERKIN METHOD FOR UNSTEADY ADVECTION-DIFFUSION PROBLEMS

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A high-order Discontinuous Galerkin Method with Lagrange Multipliers (DGLM) is presented for the solution of the unsteady advection-diffusion equation in the high Péclet number regime. In this regime, this equation models transport problems for which the standard Finite Element Method (FEM) is typically inadequate at practical mesh resolutions, as it produces non-physical oscillations in the numerical solution.

Like a Discontinuous Enrichment Method (DEM), the DGLM method described in this presentation overcomes the issue of spurious oscillations near boundary or internal layers by attempting to resolve them using appropriate shape functions. Specifically, these are chosen as polynomials that are additively enriched with free-space solutions of the governing differential equation. Also like a DEM, the DGLM method presented herein enforces a weak continuity of the solution across inter-element boundaries using Lagrange multipliers. It operates directly on the second-order form of the advection-diffusion equation and does not require any stabilization.

DGLM approximates the solutions of both homogeneous and non-homogeneous instances of the unsteady advection-diffusion equation using carefully constructed combinations of discontinuous polynomials enriched by free-space solutions of the advection-diffusion-reaction equation. Time-integration is performed using an implicit family of schemes based on the Backward Differential Formula whose numerical stability for the resulting differential-algebraic equations is rigorously proven. A theoretical analysis of the well-posedness of the proposed overall DGLM method and optimal performance results are also presented.

DISPERSION ANALYSIS OF PLANE WAVE DISCONTINUOUS GALERKIN METHODS

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The plane wave discontinuous Galerkin (PWDG) method for the Helmholtz equation was introduced and analyzed in [Gittelson, C., Hiptmair, R., and Perugia, I. Plane wave discontinuous Galerkin methods: Analysis of the h-version. *Math. Model. Numer. Anal. 43* (2009), 297–331] as a generalization of the so-called ultra-weak variational formulation, see [O. Cessenat and B. Després, *Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz equation*, SIAM J. Numer. Anal., 35 (1998), pp. 255–299]. The method relies on Trefftz-type local trial spaces spanned by plane waves of different directions, and links cells of the mesh through numerical fluxes in the spirit of discontinuous Galerkin methods.

We conduct a partly empirical dispersion analysis of the method in a discrete translation-invariant setting by studying the mismatch of wave numbers of discrete and continuous plane waves travelling in the same direction. We find agreement of the wave numbers for directions represented in the local trial spaces. For other directions the PWDG methods turn out to incur both phase and amplitude errors. This manifests itself as a *pollution effect* haunting the h-version of the method. Our dispersion analysis allows a quantitative prediction of the strength of this effect and its dependence on the wave number and number of plane waves.

References

[1] Claude J. Gittelson and Ralf Hiptmair. Dispersion analysis of plane wave discontinuous Galerkin methods. *Internat. J. Numer. Methods Engrg.*, 98(5):313–323, 2014.

HYBRID NUMERICAL-ASYMPTOTIC METHODS FOR WAVE SCATTERING PROBLEMS

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Linear wave scattering problems (e.g. for acoustic, electromagnetic and elastic waves) are ubiquitous in science and engineering applications. However, conventional numerical methods for such problems (e.g. FEM or BEM with piecewise polynomial basis functions) are prohibitively expensive when the wavelength of the scattered wave is small compared to typical lengthscales of the scatterer (the so-called "high frequency" regime). This is because the solution possesses rapid oscillations which are expensive to capture using conventional approximation spaces. In this talk we outline recent progress in the development of "hybrid numerical-asymptotic" methods. These methods use approximation spaces containing oscillatory basis functions, carefully chosen to capture the high frequency asymptotic behaviour, leading to a significant reduction in computational cost.

SOLVING TIME-DEPENDENT HEAT TRANSFER PROBLEMS WITH ENRICHED FINITE ELEMENTS

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The finite element method (FEM) presents many advantages when solving time-dependent heat transfer problems, in comparison to other domain based methods. However, challenging aspects such as the presence of high heat gradients or multi-physics heat transfer may pose difficulties to efficiently solve practical problems. Enriching the FEM proved to be a successful approach to overcome this type of difficulties and leads to a significant reduction of the computational effort in spite of some numerical issues. In this presentation recent research progress in this area will be discussed.

PLANE WAVE DISCONTINUOUS GALERKIN METHODS FOR SCATTERING PROBLEMS

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Plane Wave Discontinuous Galerkin (PWDG) methods can be used to approximate the Helmholtz equation on a bounded domain. To approximate a scattering problem, the PWDG can be used on a bounded region of free space around the scatterer provided a suitable truncation condition is imposed on the artificial boundary. I shall present error estimates for using the Dirichlet to Neumann map to supply the truncating boundary conditions and show numerical results that demonstrate the use of this approach.

A PARTITION-OF-UNITY BOUNDARY ELEMENT METHOD WITH SPACE-TIME ENRICHMENT FOR THE WAVE EQUATION

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This talk considers a time-domain partition-of-unity boundary element method for wave propagation problems at high frequency. Travelling plane waves are included as enrichment functions into a space-time boundary element Galerkin scheme. We present some first numerical experiments with this method for high-frequency scattering problems in $\mathbb{R}^3$, discuss algorithmic aspects and comment on relevant engineering applications.

THE SIMULATION OF FRACTURE MECHANICS PROBLEMS IN ANISOTROPIC MEDIA USING THE EXTENDED BOUNDARY ELEMENT METHOD

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The computation of the stress intensity factors governing the behaviour of cracked bodies is complicated by the presence of stress singularities at crack tips. One method of overcoming this difficulty is to use an enriched form of a discrete numerical method. The eXtended Finite Element Method (XFEM) has become a popular research topic, allowing accurate results from coarse finite element discretisations, and freeing the meshing from the constraint to follow the geometry of the crack. The similar type of enrichment can also be applied to the Boundary Element Method, as shown in [Alatawi and Trevelyan (2015), Engineering Analysis with Boundary Elements, 52:56-63], allowing accurate evaluation of the stress intensity factors directly in the solution vector and without the requirement for postprocessing such as the J-integral. This approach has come to be known as the eXtended Boundary Element Method (XBEM).

In the current work we extend the XBEM to consider anisotropic media. The enrichment functions based on the Williams expansions for isotropic media are replaced by the corresponding anisotropic expressions found from the Stroh formalism approach. We present results that, with very small numbers of degrees of freedom, correspond well with XFEM solutions. Finally we show how the matrices governing these enriched systems are amenable to low rank approximation using Adaptive Cross Approximation, accelerating the matrix vector product embedded in each iteration of an iterative solver.

SOLVING PDES WITH RADIAL BASIS FUNCTIONS

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Radial basis functions (RBFs) are a popular meshfree discretisation method. They are used in various areas comprising, for example, scattered data approximation, computer graphics, machine learning, aeroelasticity and the geosciences.

The approximation space is usually formed using the shifts of a fixed basis function. This simple approach makes it easy to construct approximation spaces of arbitrary smoothness and in arbitrary dimensions. It is also possible to incorporate physical features like incompressibility into the approximation space.

Multiscale RBFs employ radial basis functions with compact support. In contrast to classical RBFs they do not only use the shifts of a fixed basis function but also vary the support radius in an orderly fashion. If done correctly, this leads to an extremely versatile and efficient approximation method.

In this talk, I will introduce various ways of solving PDEs numerically using (multiscale) RBFs. I will address collocation and Galerkin techniques for elliptic and parabolic problems. I will discuss error and stability estimates and give several examples.