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Constantin Bacuta and Hengguang Li

Abstracts in alphabetical order

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SADDLE POINT LEAST SQUARES APPROACHES TO MIXED FORMULATIONS

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We investigate new PDE discretization approaches for solving variational formulations with different types of trial and test spaces. The general mixed formulation we consider assumes a stability LBB condition and a data compatibility condition at the continuous level. For our proposed discretization method a discrete $\inf - \sup$ condition is automatically satisfied by natural choices of test spaces (first) and corresponding trial spaces (second). For the proposed iterative method, nodal bases for the trial space are not required, and a cascadic multilevel algorithm can be adopted to speed up the approximation process. The level change criterion is based on matching the order of the the iteration error with the the order of the expected discretization error. Applications of the new approach include discretization of second order PDEs with oscillatory or rough coefficients and first order systems of PDEs, such as $div - curl$ systems and time-hamonic Maxwell equations.

FAST AUXILIARY SPACE PRECONDITIONER FOR LINEAR ELASTICITY IN MIXED FORM

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A block diagonal preconditioner with the minimal residual method and a block triangular preconditioner with the generalized minimal residual method are developed for Hu-Zhang mixed finite element methods of linear elasticity. They are based on a new stability result of the saddle point system in mesh-dependent norms. The mesh-dependent norm for the stress corresponds to the mass matrix which is easy to invert while the displacement it is spectral equivalent to Schur complement. A fast auxiliary space preconditioner based on the H^1 conforming linear element of the linear elasticity problem is then designed for solving the Schur complement. For both diagonal and triangular preconditioners, it is proved that the conditioning numbers of the preconditioned systems are bounded above by a constant independent of both the crucial Lamé constant and the mesh-size. Numerical examples are presented to support theoretical results. As a byproduct, a new stabilized low order mixed finite element method is proposed and analyzed and a superconvergence of Hu-Zhang element is obtained.

MULTIGRID METHODS FOR BOUNDARY CONTROL OF ELLIPTIC EQUATIONS

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The purpose of this project is to devise and analyze efficient multigrid algorithms for boundary control of elliptic equations. Using a reduced formulation, our focus is on designing optimal order multigrid preconditioners for the Hessian of the reduced cost functional. Ideally, the preconditioners should approximate the reduced Hessian with optimal order with respect to the discretization of the elliptic equation. We show that for Dirichlet boundary control of elliptic equations the preconditioner is of suboptimal quality, though still efficient. Instead, for Neumann boundary control, the preconditioner proves to be of optimal order. We contrast these two problems with the case of distributed optimal control, where similarly defined multigrid preconditioners are of optimal order.

PARALLEL PRECONDITIONERS FOR H(div) AND RELATED SADDLE-POINT PROBLEMS

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We present a scalable parallel solver for H(div) problems discretized by arbitrary order finite elements on general unstructured meshes. The solver is based on hybridization and algebraic multigrid (AMG). Unlike some previously developed H(div) solvers, the hybridization solver does not require discrete curl and gradient operators as additional input from the user. Instead, only fine-grid element information is needed in the construction of the solver. The hybridization results in a H^1-equivalent symmetric positive definite system, which is then rescaled and solved by AMG solvers designed for H^1 problems. Weak and strong scaling of the method are examined through several numerical tests. Our numerical results show that the proposed solver provides a competitive alternative to ADS, a state-of-the-art solver for H(div) problems from the LLNL parallel solvers library HYPRE. In fact, it outperforms ADS for high order elements.

The presentation is based on joint works with C. S. Lee (Texas A & M University), V. Dobrev (LLNL), and Tz. Kolev (LLNL).

A BLOCK-DIAGONAL PRECONDITIONER FOR A FOUR-FIELD MIXED FINITE ELEMENT METHOD FOR BIOT'S EQUATIONS

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In this talk, we explore an efficient preconditioning method for the saddle point system resulting from a four-field mixed finite element method applied to Biot's consolidation model. The proposed preconditioner is a block diagonal preconditioner based on the Schur complement. We obtain bounds on the eigenvalues of the preconditioned matrix that are clustered away from 0. To reduce the computational expense, this preconditioner is inverted approximately. Some numerical results are provided to show the efficiency of our preconditioning strategy when applied to a poroelasticity problem in a layered medium.

AUXILIARY SPACE PRECONDITIONER FOR LINEAR ELASTICITY EQUATIONS WITH WEAKLY IMPOSED SYMMETRY

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In recent years, there are considerable works on developing stable mixed finite element approximation for the linear elasticity equations with weakly imposed symmetry. However, it is still open how to efficiently solve the resulting (large-scaled) saddle point system. In this talk, we present an auxiliary space preconditioner for the mixed finite element approximation of the linear elasticity equations with weakly imposed symmetry. We apply the augmented Lagrangian Uzawa iteration for the saddle point system, which reduces to solving a nearly singular system. We then design an efficient preconditioner for solving this nearly singular equation. The preconditioner consists of a fast Poisson solver, and d copies of (vector) $H(div)$ solvers (such as HX-precoditioner) where d is the space dimension. We show that the preconditioner is uniform with respect to the mesh size and parameters in the equation. This preconditioner also provides an efficient solver for the pseudo-stress formulation of the Stokes equation.

A NEW APPROACH TO MIXED METHODS FOR BIHARMONIC PROBLEMS IN 2D AND 3D AND EFFICIENT SOLVERS FOR THE DISCRETIZED PROBLEMS

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A new variant of a mixed variational formulation for a biharmonic problem is presented, which involves a non-standard Sobolev space for the Hessian of the original unknown. This allows to rewrite the fourth-order problem as a sequence of three (consecutively to solve) second-order problems. In 2D this decomposition relies on the Hilbert complex

$$\hat{H}^1(\Omega)^2 \xrightarrow{\operatorname{sym}\mathbf{curl}} \mathbf{H}(\operatorname{div}\mathbf{div};\Omega,\mathbb{S}) \xrightarrow{\operatorname{div}\mathbf{div}} L^2(\Omega),$$

in 3D on the Hilbert complex

$$\hat{H}^1(\Omega)^3 \xrightarrow{\operatorname{dev}\nabla} \mathbf{H}(\operatorname{sym}\mathbf{curl};\Omega,\mathbb{T}) \xrightarrow{\operatorname{sym}\mathbf{curl}} \mathbf{H}(\operatorname{div}\mathbf{div};\Omega,\mathbb{S}) \xrightarrow{\operatorname{div}\mathbf{div}} L^2(\Omega),$$

which both are exact for bounded and topologically simple domains, and on a Helmholtz-like decomposition, which is different from the Helmholtz decomposition associated to the Hilbert complexes from above.

On the discrete level this approach can be exploited in 2D either to reformulate the well-known Hellan-Herrmann-Johnson method or to construct a new class of mixed finite element methods for biharmonic problems in such a way that, in both cases, the assembling of the discretized equations involves only standard Lagrangian elements. Similar to the continuous level a decomposition of the discretized problem into three discretized second-order problems is available, which substantially simplifies the construction of efficient solution techniques on the discrete level. Possible extensions to 3D on the discrete level as well as extensions to more general classes of fourth-order problems will also be shortly discussed.