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Abstracts in alphabetical order
## Contents

Agglomeration-based multigrid algorithms for high order Discontinuous Galerkin methods
Paola F. Antonietti, Paul Houston, Marco Sarti and Marco Verani
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes ........................................... 1

Divergence free Virtual Elements for the Stokes problem
Lourenço Beirão da Veiga, C. Lovadina and G. Vacca
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes ........................................... 1

The Virtual Element Method for Darcy flows in complex geometries
Andrea Borio, Stefano Berrone and Matías Fernando Benedetto
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes ........................................... 2

Recent variants of Mixed VEM spaces
Franco Brezzi
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes ........................................... 3

A vertex-based scheme on polyhedral meshes for advection-reaction equations with sub-mesh stabilization
Pierre Cantin, Jérôme Bonelle, Erik Burman and Alexandre Ern
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes ........................................... 3

A Hybrid High-Order method for Leray–Lions elliptic equations on general meshes
Daniele A. Di Pietro and Jérôme Droniou
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes ........................................... 4

hp–Version space-time discontinuous Galerkin methods for parabolic problems on prismatic meshes
Andrea Cangiani, Zhaonan Dong and Emmanuil H. Georgoulis
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes ........................................... 5

Super-convergence for modified mimetic and finite volume methods
Jérôme Droniou and Neela Nataraj
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes ........................................... 6

A posteriori error estimates for a discontinuous Galerkin method for interface problems on general domains
Andrea Cangiani, Emmanuil H. Georgoulis and Younis A. Sabawi
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes .......................................................... 7

The Serendipity Pyramid Finite Element
Andrew Gillette
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes .......................................................... 7

High order exactly divergencefree HDG methods for incompressible flows
Christoph Lehrenfeld and Joachim Schöberl
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes .......................................................... 9

The mimetic finite difference method for the Landau-Lifshitz equation
Konstantin Lipnikov and Eugenia Kim
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes .......................................................... 10

The $hp$ version of Virtual Element Methods for the Poisson problem: approximation of corner singularities
Lourenço Beirão da Veiga, Alexey Chernov, Lorenzo Mascotto and Alessandro Russo
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes .......................................................... 11

A Nyström-based finite element method on polygonal elements
Jeffrey S. Ovall, Akash Anand and Steffen Weißer
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes .......................................................... 12

A plane wave virtual element method for the Helmholtz problem
Ilaria Perugia, Paola Pietra and Alessandro Russo
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes .......................................................... 13

A posteriori error estimates for the Virtual Element Method
Andrea Cangiani, Emmanuil H. Georgoulis, Tristan Pryer and O. J. Sutton
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes .......................................................... 14

Virtual element method and topology optimization on polygonal meshes
Marco Verani, Paola F. Antonietti, Matteo Bruggi and Simone Scacchi
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes .......................................................... 15

Primal-Dual Weak Galerkin Finite Element Methods for PDEs
Junping Wang
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes .......................................................... 16

Weak Galerkin methods and applications
Xiu Ye, Junping Wang and Lin Mu
Mini-Symposium: PDE discretisation methods for polygonal and polyhedral meshes

iii
AGGLOMERATION-BASED MULTIGRID ALGORITHMS FOR HIGH ORDER DISCONTINUOUS GALERKIN METHODS

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We present multigrid algorithms for the efficient solution of the linear system of equations arising from high-order discontinuous Galerkin discretizations of second-order elliptic problems on polygonal/polyhedral meshes. We prove that, under suitable assumptions on the agglomerated coarse grid, the two-level version of the method converges uniformly with respect to the granularity of the grid and the polynomial approximation degree $p$, provided that the number of smoothing steps, which depends on $p$, is chosen sufficiently large. An analogous result is then obtained for the multigrid algorithms, under an additional assumption on the maximum number of levels. Numerical experiments confirm the effectiveness of the proposed schemes. Moreover, we also demonstrate numerically that the proposed solvers are convergent in practice, even when some of the theoretical assumptions are not fully satisfied.

DIVERGENCE FREE VIRTUAL ELEMENTS FOR THE STOKES PROBLEM

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We present a Virtual Element Method for the Stokes problem that, with respect to a more standard Virtual approximations of the same problem, holds two different advantages: (1) the discrete solution is exactly divergence free (2) the discrete problem is equivalent to a reduced version with much less degrees of freedom. This two advantages sum up with the traditional Virtual Element benefits of polygonal meshes and potentially high order polynomial degree. In the talk we present the construction of the method, theoretical convergence results and related numerical tests.
The simulation of flows in fractured media is a very challenging issue in applications and is often tackled by considering stochastically generated Discrete Fracture Networks (DFN) as models for the medium. These are sets of planar polygons, representing fractures, intersecting each other in the three dimensional space in such a way that the physical properties of the medium are well represented. The polygon sizes span many orders of magnitude and the stochastic nature of intersections between polygons causes infeasibility when generating meshes that require some kind of conformity. The Virtual Element Method (VEM) [1] was recently developed to enable the use of arbitrarily shaped polygons to discretize the spatial domain. This flexibility can be exploited to handle the issue of discretizing DFNs using elements which are conforming to intersections, thus enabling the application of domain decomposition techniques to compute the distribution of hydraulic head [2, 3, 4]. From this distribution, it is possible to obtain the Darcy velocity, that can then be used as input for the simulation of the transport of a passive scalar, e.g. the density of a pollutant. This requires to solve an additional advection-diffusion problem that is naturally advection dominated, which is known to lead to instabilities when discretised by the standard Galerkin approach. In order to tackle this issue, in [2] we show that a Streamline Upwind Petrov-Galerkin stabilization can be derived for the VEM, in a consistent way such that the rate of convergence of the method is preserved. This approach can be applied to DFN simulations to obtain the steady state solution of the transport problem in the DFN.

References


We recall the definitions and the basic properties of the original $H(\text{div})$ and $H(\text{curl})$-conforming Virtual Element spaces. Then we propose a new, simpler, presentation, and the Serendipity variants of them. Some applications (to Darcy flows or to magnetostatic problems) are also briefly discussed.

A VERTEX-BASED SCHEME ON POLYHEDRAL MESHES FOR ADVECTION-REACTION EQUATIONS WITH SUB-MESH STABILIZATION

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We devise and analyze vertex-based schemes on general meshes, i.e. with polyhedral cells or with non-matching interface, to approximate advection-reaction equations. Error estimates of order $O(h^{3/2})$ in the discrete inf-sup stability norm are established. The two key ingredients are a local polyhedral reconstruction map leaving affine polynomials invariant, and a local design of stabilization whereby gradient jumps are only penalized across some sub-faces in the interior of each mesh cell. Numerical results are presented on three-dimensional general meshes.
A HYBRID HIGH-ORDER METHOD FOR LERAY–LIONS ELLIPTIC EQUATIONS ON GENERAL MESHES

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technical content:

We study here the numerical approximation of the steady Leray–Lions equation

\[- \nabla \cdot (a(\cdot, u, \nabla u)) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, \quad (1)\]

where \( \Omega \subset \mathbb{R}^d \), \( d \geq 1 \), is a polytopal bounded connected domain of boundary \( \partial \Omega \), while \( a : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a (possibly nonlinear) function of its arguments. This model, which contains the \( p \)-Laplace equation, appears, e.g., in the modelling of glacier motion, of incompressible turbulent flows in porous media, and in airfoil design. Our goal is to design and analyze a discretization method for problem (1) inspired by the Hybrid High-Order (HHO) of [2] in the context of a linear diffusion model problem. The proposed method offers several assets: (i) The construction is dimension-independent; (ii) general polytopal meshes are supported; (iii) arbitrary polynomial orders (including \( k = 0 \)) can be considered; (iv) it is efficiently parallelisable (the local stencil only connects a mesh element with its faces), and it has reduced computational cost (the element-based unknowns can be eliminated locally).

The design of a HHO method for the nonlinear problem (1) entails several new ideas. A first difference with respect to the linear case is that a more natural choice is to seek the gradient reconstruction in the full space of vector-valued polynomials of degree \( \leq k \) (as opposed to the space spanned by gradients of scalar-valued polynomials of degree \( \leq (k + 1) \)). The main consequence of this choice is that, when applied to the interpolates of smooth functions, the discrete gradient operator commutes with the \( L^2 \)-projector, and can thus be shown to enjoy \( L^p \)-stability properties. A second important point is the design of a high-order stabilization term with appropriate scaling. Here, we propose a generalization of the stabilization term of [2] which preserves the property of vanishing for polynomials of degree \( \leq (k + 1) \). As in the linear case, the construction hinges on the solution of small local linear problems inside each element, and the possibility of statically condense element-based DOFs remains available.

The convergence analysis is carried out using a compactness argument in the spirit of [3], and under minimal regularity assumptions on the exact solution. Adapting the compactness argument has prompted us to develop discrete functional analysis tools (stability and approximation properties for the \( L^2 \)-orthogonal projector, discrete Sobolev embeddings, compactness for bounded sequences of discrete functions, etc.) whose interest goes beyond the specific method and problem considered in this work.
References


We present a new $hp$–version space-time discontinuous Galerkin (dG) finite element method for the numerical approximation of parabolic evolution equations on general spatial meshes consisting of general polygonal/polyhedral (polytopic) elements, giving rise to prismatic space-time elements. A key feature of the proposed method is the use of space-time elemental polynomial bases of total degree, say $p$, defined in the physical coordinate system, as opposed to standard dG-time-stepping methods whereby spatial elemental bases are tensorized with temporal basis functions. This approach leads to a fully discrete $hp$–dG scheme using less degrees of freedom for each time step, compared to standard dG time-stepping schemes employing tensorized space-time, with acceptable deterioration of the approximation properties. A second key feature of the new space-time dG method is the incorporation of very general spatial meshes consisting of possibly polygonal/polyhedral elements with arbitrary number of faces or shape irregular elements with finite number of face. A priori error bounds are shown for the proposed method in various norms. An extensive comparison among the new space-time dG method, the (standard) tensorized space-time dG methods and the classical dG-time-stepping and conforming finite element method in space, is also presented in a series of numerical experiments.
SUPER-CONVERGENCE FOR MODIFIED MIMETIC AND FINITE VOLUME METHODS

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Mimetic Finite Difference (MFD) methods are numerical schemes developed for diffusion equations on generic grids. Recently, it was understood that the lowest order mixed/hybrid MFD method is similar to two finite volume methods, and that all three could be gathered into a same family of methods, the Hybrid Mimetic Mixed (HMM) family.

The construction of an HMM scheme requires one to select a point inside each cell (a “cell centre”). Usually, the centre of gravity is chosen and, in that case, an HMM method is an MFD scheme. Super-convergence, that is an $O(h^2)$ rate of convergence in the $L^2$ norm (compared with an $O(h)$ in an $H^1$-equivalent norm), is then known.

For flexibility of the methods, cell centres other than the centres of gravity can also be considered. This allows for example to include, for triangular meshes, the Two Point Flux Approximation (TPFA) finite volume in the HMM family. In this talk, we will analyse the super-convergence of HMM methods when the cell centres are moved away from the centres of gravity. Specifically, we will:

1. show that, for specific meshes, super-convergence can fail;

2. describe a modification of HMM methods (which only consists in modifying the quadrature rule for the source term) to recover the super-convergence for any choice of cell centres;

3. describe a local compensation phenomenon, occurring for many meshes, that ensure the super-convergence of HMM schemes even if the cell centres are not the centres of gravity.

Finally, using the flexibility of the choice of cell centres, we will solve a 20+-year old conjecture on TPFA schemes, namely their super-convergence on (almost all) triangular grids.
A POSTERIORI ERROR ESTIMATES FOR A DISCONTINUOUS GALERKIN METHOD FOR INTERFACE PROBLEMS ON GENERAL DOMAINS

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An interior-penalty discontinuous Galerkin (dG) method for an elliptic interface problem involving, possibly, curved, interfaces, modelling the mass transfer of solutes through semi-permeable membranes, is considered. The method allows for extremely general curved element shapes employed to resolve the interface geometry exactly. A residual-type a posteriori error estimator for this dG method is proposed and respective upper and lower bounds of the error in the respective dG-energy norm are proven. The a posteriori error bounds are subsequently used to prove a basic a priori convergence result. The theory presented is complemented by a series of numerical experiments.

THE SERENDIPITY PYRAMID FINITE ELEMENT

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Pyramid geometries are used to bridge between tetrahedral and hexahedral meshing regimes. While pyramid finite elements with tensor product degrees of freedom on the base have been well-studied, pyramid finite elements with ‘serendipity’ degrees of freedom on the base have not. In the process of describing the scalar serendipity pyramid element, we will see definitions of shape functions and degrees of freedom related to finite element exterior calculus, as well as unisolvence and polynomial reproduction results.

References


HIGH ORDER EXACTLY DIVERGENCE FREE
HDG METHODS FOR INCOMPRESSIBLE FLOWS

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In this talk we consider the discretization of the unsteady incompressible Navier-Stokes equations in a velocity-pressure formulation:

\[
\begin{align*}
\frac{\partial}{\partial t} u + \text{div}(-\nu \nabla u + u \otimes u + pI) &= f & \text{in } \Omega \\
\text{div} u &= 0 & \text{in } \Omega 
\end{align*}
\]

(1)

with boundary conditions \( u = u_D \) on \( \Gamma_D \subset \partial \Omega \) and \((\nu \nabla u - pI) \cdot n = 0\) on \( \Gamma_{\text{out}} = \partial \Omega \setminus \Gamma_D \). Here, \( \nu = \text{const} \) is the kinematic viscosity, \( u \) the velocity, \( p \) the pressure, and \( f \) is an external body force. We present an efficient and high order accurate discretization method based on the following main ingredients:

First, we make a distinction between stiff linear parts and less stiff non-linear parts with respect to their temporal and spatial treatment. We exploit this using operator-splitting time integration schemes which rely only on efficient solution strategies for two simpler sub-problems: a corresponding hyperbolic transport problem and an unsteady Stokes problem.

Secondly, for the hyperbolic transport problem a spatial discretization with an Upwind Discontinuous Galerkin (DG) method and an explicit treatment in the time integration scheme is rather natural and allows for an efficient implementation.

Thirdly, the discretization of the Stokes problems is tailored with respect to two important challenges: efficient solution of arising linear systems and exactly divergence-free solution. In order to fulfill the incompressibility constraint exactly we use an \( H(\text{div}) \)-conforming discretization of the velocity combined with discontinuous pressures. To enforce continuity of the velocity (weakly) also in tangential direction we apply a Hybrid DG formulation. For the task of solving linear systems, a discretization with Hybrid DG methods is better suited than standard DG methods. To improve the efficiency even further we apply a projection operator in the Hybrid DG formulation which allows to reduce the unknowns on element interfaces and thereby the number of globally coupled unknowns.

We present the method, discuss aspects of the temporal and spatial discretization, implementational aspects and numerical results.

References

THE MIMETIC FINITE DIFFERENCE METHOD
FOR THE LANDAU-LIFSHITZ EQUATION

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The Landau-Lifshitz equation describes the dynamics of the magnetization inside ferromagnetic materials. This equation is highly nonlinear and has a non-convex constraint (the magnitude of the magnetization is constant) which pose interesting challenges in developing numerical methods. We develop and analyze explicit and implicit mimetic finite difference schemes for this equation. These schemes work on general polytopal meshes which provide enormous flexibility to model magnetic devices with various shapes. A projection on the unit sphere is used to preserve the magnitude of the magnetization. The developed schemes are tested on general meshes that includes distorted and randomized meshes. The numerical experiments include a test proposed by the National Institute of Standard and Technology and a test showing formation of domain wall structures in a thin film.
THE $H^p$ VERSION OF VIRTUAL ELEMENT METHODS FOR THE POISSON PROBLEM: APPROXIMATION OF CORNER SINGULARITIES

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The Virtual Element Method (VEM) is a recent generalization of the Finite Element Method (FEM), see [1]. The main features of VEM are the employment of polygonal/polyhedral meshes (thus including non conforming meshes) and the possibility of building in an easy fashion global spaces of arbitrary regularity, see [4, 5].

In [3], the $hp$ version of VEM was introduced; the basic idea of $hp$ methods is that the convergence of the errors is achieved by means of mesh refinement and by increasing the dimension of local spaces.

In the present talk, based on [2], after recalling the notation and the results of [3], we discuss about the approximation properties of VEM in presence of corner singularities. As in FEM, it is possible to prove exponential convergence of the errors in terms of the number of degrees of freedom, by geometrically refining the mesh towards the singularity and increasing the local space dimension properly. VEM seems to be appropriate for solving this kind of approximation, since it allows for an automatic geometric refinement towards the corners of the domain, without the need of reshaping the mesh, as in FEM, in presence of hanging nodes. We also present a new stabilization of the method, on which explicit bounds on $p$ are proven. Finally, numerical tests are shown.

References


Standard forms of virtual element methods (VEM) and Boundary-Element based Finite Element Methods (BEM-FEM) employ local spaces that are defined implicitly in terms of solutions of Poisson problems with polynomial data. We here follow the path of BEM-FEM in term of evaluating local basis functions for quadratures via integral equation techniques, but instead employ Nyström methods, which we believe provide several practical advantages. Among these are well-conditioned local linear systems that are trivial to set up even for high-order discretizations, better resolution of singular behavior in basis functions on non-convex elements, and flexibility to allow for elements having curved edges. We will describe the key details of the proposed approach and illustrate its performance in terms of interpolation and discretization errors.
The virtual element method (VEM) is a generalisation of the finite element method recently introduced in [1], which takes inspiration from mimetic finite difference schemes, and allows to use very general polygonal/polyhedral meshes.

This talk is concerned with a new method introduced in [4], based on inserting plane wave basis functions within the VEM framework in order to construct an $H^1$-conforming, high-order method for the discretisation of the Helmholtz problem, in the spirit of the partition of unity method (PUM, see e.g., [3]).

Plane wave functions are a particular case of Trefftz functions for the Helmholtz problem, i.e., functions belonging to the kernel of the Helmholtz operator. Finite element methods based on inserting Trefftz basis functions within the approximating spaces [2] allow to obtain, compared to standard polynomial finite element methods, similar accuracy with less degrees of freedom, mitigating the strong requirements in terms of number of degrees of freedom per wavelength due to the pollution effect.

The main ingredients of the plane wave VEM scheme (PW-VEM) are: i) a low order VEM space whose basis functions, which form a partition of unity and are associated to the mesh vertices, are not explicitly computed in the element interiors; ii) a proper local projection operator onto the plane wave space, which has to provide good approximation properties for Helmholtz solutions; iii) an approximate stabilisation term. Convergence of the $h$-version of the PW-VEM was proved, and numerical results testing its performance on general polygonal meshes were presented.

References


The virtual element method is a recently introduced method for approximating solutions to partial differential equations on meshes consisting of arbitrary (convex or non-convex) polygonal elements. We present a reliable and efficient residual-type a posteriori error estimator for the virtual element method applied to linear second order elliptic problems, and demonstrate its effectiveness when used to drive an adaptive algorithm. In keeping with the usual virtual element methodology, the estimator is fully computable since it may be evaluated using just the degrees of freedom of the discrete space and element-wise projections onto the polynomial subspace. The flexibility of polygonal meshes offers many attractive features for mesh adaptation, such as the fact that hanging nodes can be handled with no mesh post-processing since elements with coplanar edges are allowed.

References


VIRTUAL ELEMENT METHOD AND TOPOLOGY OPTIMIZATION ON POLYGONAL MESHES

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Topology optimization is a fertile area of research that is mainly concerned with the automatic generation of optimal layouts to solve design problems in Engineering. The classical formulation addresses the problem of finding the best distribution of an isotropic material that minimizes the work of the external loads at equilibrium, while respecting a constraint on the assigned amount of volume. This is the so-called minimum compliance formulation that can be conveniently employed to achieve stiff truss-like layout within a two-dimensional domain. A classical implementation resorts to the adoption of four node displacement-based finite elements that are coupled with an elementwise discretization of the (unknown) density field. When regular meshes made of square elements are used, well-known numerical instabilities arise, see in particular the so-called checkerboard patterns. On the other hand, when unstructured meshes are needed to cope with geometry of any shape, additional instabilities can steer the optimizer towards local minima instead of the expected global one. Unstructured meshes approximate the strain energy of truss-like members with an accuracy that is strictly related to the geometrical features of the discretization, thus remarkably affecting the achieved layouts. In this talk we will consider several benchmarks of truss design and explore the performance of the Virtual Element Method (VEM) in driving the topology optimization procedure. In particular, we will show how the capability of VEM of efficiently approximating elasticity equations on very general polygonal meshes can contribute to overcome the aforementioned mesh-dependent instabilities exhibited by classical finite element based discretization techniques.
In the talk, the speaker shall first introduce the weak Galerkin (WG) finite element method for partial differential equations. Weak Galerkin is a finite element method for PDEs where the differential operators (e.g., gradient, divergence, curl, Laplacian etc.) in the weak forms are approximated by discrete generalized distributions. The WG discretization procedure often involves the solution of inexpensive problems defined locally on each element. The solution from the local problems can be regarded as a reconstruction of the corresponding differential operators. The fundamental difference between the weak Galerkin finite element method and other existing methods is the use of weak functions and weak derivatives (i.e., locally reconstructed differential operators) in the design of numerical schemes based on existing weak forms for the underlying PDEs. Weak Galerkin is a natural extension of the classical Galerkin finite element method with advantages in many aspects. Due to its great structural flexibility, the weak Galerkin finite element method is well suited to most partial differential equations by providing the needed stability and accuracy in approximation.

The talk will start with the second order elliptic equation, for which WG shall be applied and explained in detail. In particular, the concept of weak gradient will be introduced and discussed for its role in the design of weak Galerkin finite element schemes. The speaker will then introduce a general notion of weak differential operators, such as weak Hessian, weak divergence, and weak curl etc. These weak differential operators shall serve as building blocks for WG finite element methods for other class of partial differential equations, such as the Stokes equation, the biharmonic equation, the Maxwell equations in electron magnetics theory, div-curl systems, and PDEs in non-divergence form (such as the Fokker-Planck equation). In particular, the speaker will introduce a primal-dual formulation for second order elliptic PDEs in non-divergence form. Numerical results and error estimates shall be discussed. The talk should be accessible to graduate students with adequate training in computational methods.
WEAK GALERKIN METHODS AND APPLICATIONS

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The Weak Galerkin method is an extension of the standard Galerkin finite element method where classical derivatives were substituted by weakly defined derivatives on functions with discontinuity. The WG methods have the flexibility in handling complex geometry and the simplicity in analyzing real-world physical problems. Recent development of weak Galerkin methods will be discussed in the presentation.