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Abstracts in alphabetical order

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DISCRETIZATION ERROR ESTIMATES FOR DIRICHLET CONTROL PROBLEMS IN POLYGONAL DOMAINS

Thomas Apel[1], Mariano Mateos[2], Johannes Pfefferer[3] and Arnd Rösch[4]

[1]Institut für Mathematik und Bauinformatik,
Universität der Bundeswehr München, Germany
thomas.apel@unibw.de

[2]Departamento de Matemáticas, Universidad de Oviedo, Spain
mmateos@uniovi.es

[3]Lehrstuhl für Optimalsteuerung,
Technische Universität München, Germany
pfefferer@ma.tum.de

[4]Fakultät für Mathematik, Universtät Duisburg-Essen, Germany
Arnd.Roesch@uni-due.de

In this talk we discuss convergence results for finite element discretized Dirichlet control problems in polygonal domains. We investigate unconstrained as well as control constrained problems. In both cases we discretize the state and the control by piecewise linear and continuous functions. The error estimates, which we obtain, mainly depend on the size of the interior angles but also on the presence of control constraints and the structure of the underlying mesh. For instance, considering non-convex domains, the convergence rates of the discrete optimal controls in the unconstrained case can even be worse than in the control constrained case.

A PRIORI ERROR ESTIMATES FOR AN OPTIMAL CONTROL PROBLEM RELATED TO QUASI-LINEAR PARABOLIC PDES

Konstantinos Chrysafinos[1] and Eduardo Casas[2]

[1]Department of Mathematics,
National Technical University (NTUA), Athens, Greece,
chrysafinos@math.ntua.gr

[2]Departmento de Matemática Aplicada y Ciencias de la Computación,
E.T.S.I. Industriales y de Telecomunicación,
Universidad de Cantabria, Spain,
eduardo.casas@unican.es

We consider an optimal control problem related to quasi-linear parabolic pdes. The standard tracking type of functional is minimized and the controls are of distrubuted type satisfying point-wise constraints. After presenting some results regarding existence and regularity of solutions, first and second order conditions, we focus on the discretization of the control to state mapping. A-priori error estimates for a fully-discrete scheme are presented. The scheme is based on the lowest order discontinuous Galerkin time stepping scheme combined with standard conforming finite elements (in space). We present estimates at the natural energy norm, as well as improved error estimates in $L^2(0,T;L^2(\Omega))$ norm. Using these estimates, as well as similar estimates for the discrete adjoint mapping, we discuss error estimates for the optimal control problem.

FINITE ELEMENT ANALYSIS OF FREE MATERIAL OPTIMIZATION PROBLEMS

Michael Hinze[a] and Tobias Jordan[b]

Department of Mathematics, University of Hamburg, Germany
[a]michael.hinze@uni-hamburg.de, [b]tobias.jordan@uni-hamburg.de

In Free Material Optimization, the design variable is the full material tensor of an elastic body. Written in matrix notation one obtains a control-in-the-coefficients problem for the material tensor.

In this talk we discuss recent results in the finite element analysis in Free Material Optimization. We employ the variational discretization approach, where the control, i.e., the material tensor, is only implicitly discretized. Using techniques from the identification of matrix-valued diffusion coefficients, we derive error estimates depending on the coupling of the discretization and Tikhonov regularization parameters. Furthermore, this approach allows to also take into account a noise level on the measured data. Numerical examples supplement our analytical findings.

OPTIMAL ERROR ESTIMATES OF PARABOLIC OPTIMAL CONTROL PROBLEMS WITH A MOVING POINT SOURCE

Dmitriy Leykekhman[1] and Boris Vexler[2]

[1]Department of Mathematics, University of Connecticut, USA
dmitriy.leykekhman@uconn.edu

[2]Faculty for Mathematics, Technical University of Munich, Germany
vexler@ma.tum.de

In this talk we discuss the following optimal control problem

$$\min_{q,u} J(q,u) := \frac{1}{2}\int_0^T \|u(t)-\widehat{u}(t)\|^2_{L^2(\Omega)}dt + \frac{\alpha}{2}\int_0^T |q(t)|^2 dt \tag{1}$$

subject to the second order parabolic equation

$$\begin{aligned} \partial_t u(t,x) - \Delta u(t,x) &= q(t)\delta_{\gamma(t)}, \quad (t,x) \in I\times\Omega, \\ u(t,x) &= 0, \quad (t,x)\in I\times\partial\Omega, \\ u(0,x) &= 0, \quad x\in\Omega \end{aligned}$$

Here $I = [0,T]$, $\Omega \subset R^2$ is a convex polygonal domain, and $\delta_{\gamma(t)}$ is the Dirac delta function along the curve $\gamma(t)\subset\Omega$. We assume that $\gamma(t)$ satisfies the following:

- $\gamma(t)\in C^1(0,T)$ and $max_t|\gamma'(t)|\le C_\gamma$.
- $\gamma(t)\subset\Omega_0\subset\subset\Omega$, for any $t\in I$.

The parameter α is assumed to be positive and the desired state $\widehat{u}$ fulfills $\widehat{u}\in L^2(I;L^\infty(\Omega))$.

We discretize the problem with continuous Lagrange elements in space and discontinuous piecewise constant functions in time. Despite low regularity of the state equation we establish optimal (first order in time and the second order in space, modulo logarithmic terms) convergence rates for the fully discrete control variable. We will also discuss a new type of error estimates along the curve which are essential for our analysis.

ALGORITHMIC APPROACHES IN OPTIMAL SHAPE CONTROL OF INCOMPRESSIBLE FLOWS USING FINITE ELEMENTS

Thomas Apel and Edwin Mai[a]

Institute of Mathematics and Civil Engineering Computer Science,
Universität der Bundeswehr Munich, Germany
[a] edwin.mai@unibw.de

This presentation considers a specific aspect of optimal control for partial differential equations, where the control is given by the shape of the domain of interest. The key point in shape optimisation is the definition of the shape derivative, which is needed for the standard optimisation procedure. Several approaches exist whereas we follow the ideas of Sokolowski and Zolesio who provide a method to derive an analytical gradient. The shape gradient can be deduced from the state equation by applying shape calculus and solving an auxiliary adjoint equation. This approach leads to two analytically equivalent representations of the shape gradient, i.e. the boundary and the domain form. However this equivalence property does not transfer to the discrete case. The pros and cons will be discussed and how each representation influences the optimisation procedure.

The discussion is part of a research project for turbulence reduction in water pipes by modifying their shape. Hence it is embedded in the framework of incompressible flow equations, i.e. the Navier-Stokes equations and their simplifications. These equations were solved within a finite element approach, which is implemented by the finite element software package FEniCS.

A PRIORI AND A POSTERIORI ERROR ANALYSIS FOR OPTIMAL CONTROL OF THE OBSTACLE PROBLEM

Christian Meyer[1a], Andreas Rademacher[1b] and Winnifried Wollner[2]

[1]Fakultät für Mathematik, TU Dortmund, Germany
[a]constantin.christof@tu-dortmund.de, [b]arademac@math.tu-dortmund.de

[2]Fachbereich Mathematik, TU Darmstadt, Germany
wollner@mathematik.tu-darmstadt.de

We consider an optimal control problem governed by a variational inequality of obstacle type. Problems of this type are known to be challenging due to the non-differentiable control-to-state mapping, which permits the use of standard techniques for the derivation of optimality conditions. Nevertheless it is possible to rigorously derive a priori error estimate for the finite element (FE) discretization of such problems which turn out to be optimal w.r.t. to the generic regularity of the optimal control problem. In addition we present a more heuristic a posteriori approach based on the dual weighted residual method. While a rigorous analysis of the error estimator is still lacking, the numerical tests show promising results.

HIGHER ORDER FINITE ELEMENTS IN OPTIMAL CONTROL

Arnd Rösch[1] and Gerd Wachsmuth[2]

[1]Faculty of Mathematics, University of Duisburg-Essen, Germany
arnd.roesch@uni-due.de

[2]Faculty of Mathematics, Technische Universität Chemnitz, Germany
gerd.wachsmuth@mathematik.tu-chemnitz.de

In this talk we propose a new method for solving control constrained optimal control problems. We use a non-conform discretization with higher order finite elements. A mass lumping approach is proposed to obtain a simple and very accurate numerical scheme. Under certain assumptions we are able to show convergence order up to h^4.

ERROR ESTIMATES FOR A DISCONTINUOUS FINITE VOLUME DISCRETIZATION OF THE BRINKMAN OPTIMAL CONTROL PROBLEM

Ruchi Sandilya[1a], Sarvesh Kumar[1b] and Ricardo Ruiz-Baier[2]

[1]Department of Mathematics, Indian Institute of Space Science and Technology, Thiruvananthapuram 695 547, Kerala, India
[a]ruchisandilya.12@iist.ac.in, [b]sarvesh@iist.ac.in

[2]Mathematical Institute, Oxford University, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, UK
ruizbaier@maths.ox.ac.uk

In this paper we discuss a discontinuous finite volume method for the approximation of distributed optimal control problem governed by the Brinkman equations written in terms of velocity and pressure. An additional force field is sought such that it produces a velocity matching a desired, known value. The discretization of state and co-state velocity and pressure fields follows a lowest order discontinuous finite volume scheme, whereas three different approaches are used for the control approximation: variational discretization, element-wise constant, and element-wise linear functions. We employ the *optimize-then-discretize* approach to approximate the control problem, and the resulting discretized formulation is non-symmetric. We derive *a priori* error estimates for velocity, pressure, and control in natural norms. A set of numerical examples is finally presented to illustrate the performance of the method and to confirm the predicted accuracy of the state, co-state and control approximations under various scenarios including 2D and 3D cases.

FINITE ELEMENT METHODS FOR PARABOLIC OPTIMAL CONTROL PROBLEMS WITH CONTROLS FROM MEASURE SPACES

Boris Vexler[1] and Dmitriy Leykekhman[2]

[1]Faculty for Mathematics, Technical University of Munich, Germany
vexler@ma.tum.de

[2]Department of Mathematics, University of Connecticut, USA
leykekhman@math.uconn.edu

In this talk we discuss optimal control problems subject to parabolic equations, where the support of the control is potentially of measure zero. This includes sparse optimal control problems [1] and problems with pointwise controls [4, 5]. For this type of problems we consider finite element discretizations in space and time and derive a priori error estimates. The main technical tools are recently established discrete maximal parabolic regularity [2] and pointwise best approximation results [3].

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EXPONENTIAL CONVERGENCE OF *hp*-FINITE ELEMENT DISCRETIZATION OF OPTIMAL BOUNDARY CONTROL PROBLEMS WITH ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Daniel Wachsmuth and Jan-Eric Wurst

Institut für Mathematik, Universität Würzburg, Würzburg, Germany
daniel.wachsmuth@mathematik.uni-wuerzburg.de

We investigate the numerical solution of a boundary control problem with elliptic partial differential equation by the *hp*-finite element method. We prove exponential convergence with respect to the number of unknowns for an a-priori chosen discretization. Here, we have to prove that derivatives of arbitrary order of the solution belong to suitably chosen weighted Sobolev spaces. This result relies on the assumption that the number of switching points of the optimal control is finite. Numerical experiments confirm the theoretical findings.

OPTIMAL CONVERGENCE ORDER FOR CONTROL CONSTRAINED OPTIMAL CONTROL PROBLEMS

René Schneider[a] and Gerd Wachsmuth[b]

Faculty of Mathematics, TU Chemnitz, Germany
[a]rene.schneider@feins.org, [b]gerd.wachsmuth@mathematik.tu-chemnitz.de

In this talk we consider the numerical solution of control constrained optimal control problems. We are interested in obtaining the optimal convergence rate for the $L^2(\Omega)$-error w.r.t. the number of degrees of freedom. Due to the control constraint, the optimal control possesses a kink at the interface between the active and inactive set w.r.t. the control constraint. This kink limits the convergence order of a uniform discretization to $h^{3/2}$.

We compare some approaches from the literature. Moreover, we provide a new, efficient and robust error estimator which is used for an adaptive refinement of the mesh.

We also present a new method for solving control constrained problems. In this method, we move the nodes of the mesh at the interface between the active and inactive set. This yields optimal order of convergence.

DISCRETIZATION OF PARABOLIC OPTIMIZATION PROBLEMS WITH CONSTRAINTS ON THE SPATIAL GRADIENT OF THE STATE

Francesco Ludovici[1] Ira Neitzel[2] and Winnifried Wollner[3]

[1]Department of Mathematics, Technische Universität Darmstadt, Germany
ludovici@mathematik.tu-darmstadt.de

[2]Institut für Numerische Simulation,
Rheinische Friedrich-Wilhelms-Universität Bonn, Germany
neitzel@ins.uni-bonn.de

[3]Department of Mathematics, Technische Universität Darmstadt, Germany
wollner@mathematik.tu-darmstadt.de

In this talk, optimization problems subject to a possibly semilinear parabolic partial differential equation (PDE) are considered. Moreover, additional pointwise constraints are imposed on the gradient of the state, i.e., the solution to the PDE. The optimization problems are discretized using a Galerkin-type approach and the convergence rates for the discretization error are discussed.