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Organisers:

Markus Bause and Florin Radu

Abstracts in alphabetical order

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SPACE-TIME FINITE ELEMENT APPROXIMATION OF FLOW IN DEFORMABLE POROUS MEDIA

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The modelling of coupled mechanical deformation and flow in porous media has become of increasing importance in several branches of natural sciences and technology including environmental, mechanical, petroleum and reservoir engineering, biomechanics and medicine. The numerical simulation of coupled mechanical deformation and flow is complex due to the structure of the model equations and continues to remain a challenging task. Recently, iterative coupling techniques have attracted researchers' interest and schemes were proposed [1, 5]. The appreciable advantage of these approaches is that by coupling the model components iteratively already highly developed simulation techniques for each component of the overall system can be used fully.

In this contribution we consider the quasi-static Biot system of poroelasticity,

$$-\nabla \cdot (\boldsymbol{\sigma}_0 + \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}) - b(p - p_0)\boldsymbol{I}) = \rho_b \boldsymbol{g}\,, \tag{1}$$

$$\partial_t \left(\frac{1}{M} p + \nabla \cdot (b\boldsymbol{u}) \right) + \nabla \cdot \boldsymbol{q} = f\,, \quad \boldsymbol{q} = -\frac{\boldsymbol{K}}{\eta} \left(\nabla p - \rho_f \boldsymbol{g} \right)\,. \tag{2}$$

We present a higher order space-time finite element approximation of the system (1), (2) that is based on an iterative coupling of properly defined subproblems of mechanical deformation and fluid flow; cf. [1]. For the discretization in time a discontinuous Galerkin method is used. Mixed finite element methods are applied for the spatial discretization of the flow subproblem. Error estimates for the discretization and efficient solution techniques for the arising algebraic systems of equations are addressed; cf. [2, 3, 4]. The stability and performance properties of the techniques are illustrated by applications of practical interest.

References

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DISCONTINUOUS GALERKIN METHOD FOR THE SOLUTION OF ELASTO-DYNAMIC AND FLUID-STRUCTURE INTERACTION PROBLEMS

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This contribution will be concerned with the numerical solution of dynamic elasticity by the discontinuous Galerkin (DG) method. We consider the linear case as well as the nonlinear St. Venant-Kirchhoff model. The space discretization is carried out by the DG method. For the time discretization several techniques are applied and tested. As the best method the DG discretization both in space and time appears. The applicability of the developed technique is demonstrated by several numerical experiments. Then the developed method is combined with the space-time DG method for the solution of compressible flow in a time dependent domain and used for the numerical simulation of fluid-structure interaction.

The results were obtained in cooperation with M. Balázsová, M. Hadrava, A. Kosík and J. Horáček.

The contribution will be presented in the minisymposium "Higher order space-time finite element methods".

HIGHER ORDER VARIATIONAL TIME DISCRETISATIONS FOR THE OSEEN EQUATIONS

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We discuss different time discretisations of variational type applied to time-dependent Oseen problems. As spatial discretisation, both inf-sup stable and equal-order pairs of finite element spaces for approximating velocity and pressure are considered.

Since Oseen problems are generally convection-dominated, a spatial stabilisation is applied. We will concentrate on local projection stabilisation methods which allow to stabilise the streamline derivative, the divergence constraint and, if needed, the pressure gradient separately.

To discretize in time, continuous Galerkin-Petrov methods (cGP) and discontinuous Galerkin methods (dG) as higher order variational time discretisation schemes are applied. These methods are known to be A-stable (cGP) or even strongly A-stable (dG). An adaption of the time postprocessing proposed by Matthies and Schieweck leads to numerical solutions which show for both velocity and pressure at the discrete time points a convergence rate of $2k+1$ for dG(k) and $2k$ for cGP(k), respectively.

HIGHER ORDER SPACE-TIME FINITE ELEMENTS FOR THE DIFFUSION EQUATION

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This work is devoted to a higher order scheme for the non-stationary diffusion equation. The scheme is based on continuous Galerkin in time and mixed finite element method (MFEM) in space. Precisely, Raviart-Thomas elements of arbitrary order are involved. Continuous, semi-discrete and fully-discrete variational formulations are set up. Existence and uniqueness of solutions for the all formulations is rigorously proved. A priori error estimates are derived to show the convergence of the scheme. This is done for arbitrary orders in time and space. To obtain optimal order estimates a duality argument is involved. Numerical experiments are shown to confirm the theoretical results. We refer to [1] for the details of the analysis.

References

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ANALYSIS OF A DG-METHOD IN TIME WITH POST-PROCESSING FOR THE TRANSIENT STOKES PROBLEM

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We study the discontinuous Galerkin time discretization (dG(k)-method) for the transient Stokes problem [3, 2] which is discretized in space by means of an inf-sup stable pair of finite element spaces (V_h, Q_h) for velocity and pressure, respectively. Here, the fully discrete solution $(u_h(t), p_h(t))$ on each time interval is a polynomial in time of order k with values in the finite element product space $V_h \times Q_h$. By means of a simple post-processing step we can compute in a very inexpensive way a lifted solution $(\tilde{u}_h(t), \tilde{p}_h(t))$ which is globally continuous in time and a polynomial of order $k+1$ on each time interval. For this approximation $(\tilde{u}_h(t), \tilde{p}_h(t))$, we prove an optimal estimate for the velocity error in $L^2(L^2)$ of the higher order in time $\tau^{k+2} + h^{r+1}$, where τ denotes the time step size, h the mesh size and r the polynomial degree for the velocity approximation in V_h. Moreover, we prove an optimal $L^2(L^2)$ estimate for the pressure error of the order $\tau^{k+2} + h^r$, where the polynomial degree for the pressure approximation in Q_h is $r-1$ due to the inf-sup condition. Key ingredients of the analysis are a special higher order interpolate in time of the exact solution and a special stability estimate for the lifted velocity error (for both see [1]) applied in the discretely divergence free subspace of V_h as well as the proof of superconvergence of the error in the time derivative for the velocity. We present some numerical results which confirm the theoretical error bounds.

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SPACE-TIME GALERKIN APPROXIMATION OF WAVE PROPAGATION IN DISPERSIVE MEDIA

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Viscoelastic media such as polymers and biotissue are dispersive and are usually described by a hereditary constitutive law. The physically reasonable assumption of fading memory in these problems makes it possible to derive stability and error bounds which are 'sharp' in so much as they can be derived without recourse to Gronwall's inequality. This means that they do not contain an exponential growth in time, and this provides some confidence in the quality of long-time simulations.

An example of this type of result will be given for a high order space-time Galerkin finite element method (continuous in space; discontinuous in time) for a dynamic linear solid viscoelasticity problem. This problem is of interest to us because, in a proof-of-concept project, we as a multidisciplinary group are aiming to model the passage of shear waves from the wall of a diseased coronary artery to the chest surface. Our long term aim is a relatively cheap and non-invasive screening or diagnostic device, based on solving the inverse problem, for coronary artery disease.

Within the context of that project we have followed the heat equation formulations in [Werder *et al.*, Comput. Methods Appl. Mech. Engrg., 190:6685—6708, 2001] and developed a time diagonalised space-time finite element solver for the viscodynamic wave equation. This approach allows for both coarse and fine grained parallelism, and high degree polynomial approximation in both space and time. This formulation will be illustrated for the simpler case of the acoustic wave equation in order to describe the main points.

Surprisingly, perhaps, Maxwell's equations for a Debye media have at a high enough level of abstraction essentially the same structure as those for viscodynamics. The same type of sharp estimates will be illustrated, for finite difference time discretization, for this application along with some further results for Lorentz media. Difficulties in extending the space-time Galerkin formulation (as above) for these materials, as well as for the Drude model for metamaterials, will be touched upon.

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Various aspects of this material are joint work with any or all of the following: *SE Greenwald* (QMUL); *MJ Birch, MP Brewin* (Barts and the London NHS Trust); *HT Banks, ZR Kenz, S Hu* (NC State); *J Li* (UNLV); *C Kruse and JR Whiteman* (Brunel).

ADAPTIVE WAVELET METHODS FOR SPACE-TIME VARIATIONAL FORMULATIONS OF EVOLUTIONARY PDES

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Space-time discretization methods require a well-posed space-time variational formulation. Such formulations are well-known for parabolic problems. The (Navier)-Stokes equations can be viewed as a parabolic problem for the divergence-free velocities. Yet to avoid the cumbersome construction of divergence-free trial spaces, we present well-posed variational formulations for the saddle-point problem involving the pair of velocities and pressure. We discuss adaptive wavelet methods for the optimal adaptive solution of simultaneous space-time variational formulations of evolutionary PDEs. Thanks to use of tensor products of temporal and spatial wavelets, the whole time evolution problem can be solved at a complexity of solving one instance of the corresponding stationary problem.

DISCRETE MAXIMAL PARABOLIC REGULARITY AND BEST APPROXIMATION RESULTS FOR GALERKIN FINITE ELEMENT SOLUTIONS OF PARABOLIC PROBLEMS

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In this talk we present discrete maximal parabolic regularity results [1] for linear parabolic equations discretized by discontinuous Galerkin methods in time and Lagrange finite elements in space. These results provide a novel flexible technique for establishing optimal error estimates in various non-Hilbertian norms without any coupling conditions between the spatial mesh size and time steps. Especially we present global and interior best approximation type estimates in the $L^\infty((0,T)\times\Omega)$ norm [2].

References

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