MA3951/MA5352: Numerical and Variational Methods for PDEs Exercise sheet 1

1. Let Ω be a domain, let L denote a linear differential operator and let D_L be an inner product space of functions defined on Ω for which L is defined.

   Define what it means for L to be symmetric on D_L and define what it means for L to be positive definite on D_L.

   In each of the following cases, determine whether or not the given differential operator L is symmetric on the function space D_L. For the differential operators which are symmetric determine whether or not they are positive definite.

   (a) $\Omega = (0,1)$, $D_L = C^2[0,1]$, $Lu = -u''$.

   (b) $\Omega = (0,1)$, $D_L = \{v \in C^2[0,1] : \quad v(0) = 0\}$, $Lu = -u''$.

   (c) $\Omega = (0,1)$, $D_L = \{v \in C^2[0,1] : \quad v(0) = 0,\ v(1) = 0\}$, $Lu = -u''$.

   (d) $\Omega = (0,1)$,
   $$D_L = \{v \in C^2[0,1] : \quad v(0) = 0,\ v(1) = 0\}, \quad Lu = -u'' + q(x)u,$$
   where $q \in C[0,1]$ and $q(x) \geq 0$ on $[0,1]$.

   (e) $\Omega = (0,1)$,
   $$D_L = \{v \in C^2[0,1] : \quad v(0) = 0,\ v'(1) + \beta v(1) = 0\}, \quad L = -u'' + q(x)u,$$
   where $\beta \geq 0$ and where $q \in C[0,1]$ and $q(x) \geq 0$.

   (f) $\Omega = (0,1)$,
   $$D_L = \{v \in C^2[0,1] : \quad v'(0) = 0,\ v'(1) = 0\}, \quad L = -u''.$$

   (g) $\Omega = (0,1)$,
   $$D_L = \{v \in C^2[0,1] : \quad v'(0) - \beta_1 v(0) = 0,\ v'(1) + \beta_2 v(1) = 0\}, \quad L = -u'' + q(x)u,$$
   where $\beta_1 > 0$ and $\beta_2 > 0$ and where $q \in C[0,1]$ and $q(x) \geq 0$.

   (h) $\Omega = (0,1)$,
   $$D_L = \{v \in C^4[0,1] : \quad v(0) = v'(0) = v(1) = v'(1) = 0\}, \quad L = u'''' - (pu')' + qu,$$
   where $p \in C^1[0,1]$, $q \in C[0,1]$ and $p(x) \geq p_0 > 0$ and $q(x) \geq 0$ for all $x \in (0,1)$.

---

2. The following relates to problems in $\mathbb{R}^2$. This will be only briefly covered at the start of the module but if you already know results such as the divergence theorem then you could try this now.

   As in the previous question in each of the following cases, determine whether or not the given differential operator L is symmetric on the function space D_L. For the differential operators which are symmetric determine whether or not they are positive definite.

(a)
$$\Omega = \{(x, y) : \quad 0 < x < 2,\ 0 < y < 1\},$$
$$D_L = C^2(\bar{\Omega}), \quad Lu = -\Delta u.$$

(b)
$$\Omega = \{(x, y) : \quad 0 < x < 2,\ 0 < y < 1\},$$
$$D_L = \{v \in C^2(\bar{\Omega}) : \quad v = 0 \text{ on } \partial\Omega\}, \quad Lu = -\Delta u.$$

(c) Let
$$\Omega = \{(x, y) : \quad 0 < x < 2,\ 0 < y < 1\},$$
let $\partial\Omega_1$ denote the 3 sides on $\partial\Omega$ corresponding to $x = 0$, $x = 2$ and $y = 1$ and let $\partial\Omega_2$ denote the other side corresponding to $y = 0$. Also let
$$D_L = \{v \in C^2(\bar{\Omega}) : \quad v = 0 \text{ on } \partial\Omega_1 \text{ and } -\frac{\partial v}{\partial y}(x, 0) + \beta v(x, 0) = 0,\ 0 < x < 2\},$$
where $\beta \geq 0$, and let $L = -\Delta$.

(d) $\Omega \subset \mathbb{R}^2$,
$$D_L = \{C^4(\bar{\Omega}) : \quad v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\}, \quad Lu = \Delta^2 u = \Delta(\Delta u).$$

---

3. Derive the expressions involved in weak forms for the following problems stating in each case the appropriate function space involved and classify each boundary condition as an essential boundary condition or as a natural boundary condition. In the weak forms that you construct, the maximum order of derivatives in the the expressions should not be more than half that which appears in the given differential equations.

(a)
$$-(pu')' + qu = f, \quad 0 < x < 1, \quad u(0) = u(1) = 0,$$
where $p \in C^1[0, 1]$, $q, f \in C[0, 1]$ and $p(x) > 0$ and $q(x) \geq 0$.

(b)
$$-(pu')' + qu = f, \quad 0 < x < 1, \quad u(0) = 0, \quad u'(1) + u(1) = 0,$$
where $p \in C^1[0, 1]$, $q, f \in C[0, 1]$ and $p(x) > 0$ and $q(x) \geq 0$.

(c)
$$u'''' = f, \quad 0 < x < 1, \quad u(0) = u'(0) = u(1) = u'(1) = 0,$$
where $f \in C[0, 1]$.

(d)
$$u'''' = f, \quad 0 < x < 1, \quad u(0) = u''(0) = u(1) = u''(1) = 0,$$
where $f \in C[0, 1]$.

---

4. This was question 1 of the June 2004 MA3951 paper.

(a) Obtain the weak form expressions corresponding to each of the following boundary value problems. In your answer you should give in each case the space of functions involved and you should indicate if any of the boundary conditions are essential boundary conditions.

(i)
$$-((1+x)u')' = 1, \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

[2 marks]

(ii)
$$-u'' + 2u = x^2, \quad 0 < x < 1, \quad u(0) = u'(1) + 2u(1) = 0.$$

[2 marks]

(iii)
$$u'''' - u'' = 1, \quad 0 < x < 1, \quad u(0) = u'(0) = u'(1) = u'''(1) = 0.$$

[3 marks]

(b) In the following the ordinary differential equation

$$-u'' + 2u = x, \quad 0 < x < 1, \quad u(0) = 0, \quad u'(1) = 0,$$

is to be approximately solved using Galerkin's method.

(i) Derive the weak form for this problem in the form $a(u, v) = (f, v)$.

[1 mark]

(ii) Let $\psi_i(x) = x^i$, $i = 1, 2, \cdots$ and let $V_n = \text{span}\{\psi_1, \cdots, \psi_n\}$. Determine the Galerkin approximation from V_1 and determine a linear system for c_1 and c_2 in the Galerkin approximation $c_1 x + c_2 x^2$ from the space V_2. You do not need to solve these equations.

[5 marks]

(iii) Let $M \geq 1$ be a natural number and let $h = 1/M$, and $x_i = ih$, $i = 0, 1, \cdots, M$, denote equally spaced mesh points in $[0, 1]$. Also, on the ith element $[x_{i-1}, x_i]$, let

$$\tilde{\phi}_1(x) = \frac{x_i - x}{h}, \quad \tilde{\phi}_2(x) = \frac{x - x_{i-1}}{h},$$

denote the standard linear basis functions defined on $[x_{i-1}, x_i]$. Show that the element matrix K_i and the element vector $\underline{b}_i$ are given by

$$K_i = \frac{1}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{h}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \underline{b}_i = \frac{h}{2}\left(x_{i-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{h}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \right).$$

(In your answer you may assume that

$$\int_{x_{i-1}}^{x_i} \tilde{\phi}_1^2 \, \mathrm{d}x = \int_{x_{i-1}}^{x_i} \tilde{\phi}_2^2 \, \mathrm{d}x = \frac{h}{3} \quad \text{and} \quad \int_{x_{i-1}}^{x_i} \tilde{\phi}_1 \tilde{\phi}_2 \, \mathrm{d}x = \frac{h}{6}.)$$

[4 marks]

(iv) Let U denote the Galerkin finite element approximation to this problem using piecewise linears defined on the mesh in part (iii) in the case $M = 2$ and let $U_1 = U(x_1)$ and $U_2 = U(x_2)$. By using the results of part (iii) determine the linear equations that U_1 and U_2 satisfy. You do not need to solve these equations.

[3 marks]

---

5. Question 1 of the June 2004 `MA5352` paper had some parts in common with question 1 of the `MA3951` paper. These are some of the parts which were different. `MA3951` students had to answer 3 questions (from a choice of 4) in 3 hours. `MA5352` students had to answer 4 questions (from a choice of 5) in 3 hours. The extra question on the `MA5352` paper was from the finite element material taught by Simon Shaw.

(a) Obtain the weak form expressions corresponding to each of the following boundary value problems. In your answer you should give in each case the space of functions involved and you should indicate if any of the boundary conditions are essential boundary conditions.

(i)
$$-u'' + 2u = x^2, \quad 0 < x < 1, \quad u(0) = u'(1) + 2u(1) = 0.$$

[3 marks]

(ii)
$$u'''' - u'' = 1, \quad 0 < x < 1, \quad u(0) = u'(0) = u'(1) = u'''(1) = 0.$$

[3 marks]

(b) In the following the ordinary differential equation

$$-u'' + 2u = x, \quad 0 < x < 1, \quad u(0) = 0, \quad u'(1) = 0,$$

is to be approximately solved using Galerkin's method.

(i) Derive the weak form for this problem in the form $a(u, v) = (f, v)$ where $a(.,.)$ is a symmetric and positive bilinear form defined on an appropriate space of functions V.

[1 mark]

(ii) Let $\psi_i(x) = x^i$, $i = 1, 2, \cdots$ and let $V_n = \text{span}\{\psi_1, \cdots, \psi_n\}$. The coefficients $c_1, c_2, \cdots, c_n$ in the Galerkin approximation $U_n = \sum_1^n c_i\psi_i$ satisfy the linear system

$$K\underline{c} = \underline{b}$$

where $\underline{c} = (c_i)$, $K = (a(\psi_i, \psi_j))$ and $\underline{b} = ((f, \psi_i))$. Explain why the matrix K is positive definite.

[3 marks]

Determine the Galerkin approximation from V_1 and determine a linear system for c_1 and c_2 in the Galerkin approximation $c_1 x + c_2 x^2$ from the space V_2. You do not need to solve these equations.

[6 marks]

6. The following was question 1 of the June 2003 `MA3056S` paper.

   Obtain the weak form expressions corresponding to each of the following ODEs. In your answer you should give in each case the space of functions involved and you should indicate if any of the boundary conditions are natural boundary conditions.

   (a) (i)
   $$-u''(x) = x^2, \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$
   [2 marks]

   (ii)
   $$-u''(x) + u(x) = 1, \quad 0 < x < 1, \quad u(0) = u'(1) = 0.$$
   [2 marks]

   (iii)
   $$-u''(x) + 6u(x) = x, \quad 0 < x < 1, \quad u'(0) = 0, \quad u'(1) + u(1) = 0.$$
   [2 marks]

   (iv)
   $$u''''(x) = \cos(x), \quad 0 < x < 1, \quad u(0) = u'(0) = u(1) = u'(1) = 0.$$
   [3 marks]

   (b) Suppose that a weak problem is of the form:
   $$\text{find } u \in V \quad \text{such that} \quad a(u, v) = F(v) \quad \text{for all } v \in V$$
   where V is an appropriate space of functions, $a(.,.)$ is a symmetric and positive definite bilinear form on $V \times V$ and $F(.)$ is a linear functional on V. Let $\phi_1, \cdots, \phi_n$ denote linearly independent functions in V. Describe the Galerkin method for constructing an approximation $U_n \in \operatorname{span}\{\phi_1, \cdots, \phi_n\} \subset V$ and explain why the linear system which is obtained involves a matrix which is symmetric and positive definite.

   [5 marks]

   (c) Let $\phi_i(x) = x^i$, $i = 1, 2, \cdots$, let $V_n = \operatorname{span}\{\phi_1, \cdots, \phi_n\}$ and consider the problem
   $$-u''(x) + 6u(x) = 1, \quad 0 < x < 1, \quad u(0) = u'(1) = 0.$$
   Obtain the Galerkin approximation $U_1 \in V_1$.
   Also obtain the linear system $K\underline{c} = \underline{b}$ for the coefficients $\underline{c} = (c_1, c_2)^T$ of the approximation $U_2(x) = c_1 x + c_2 x^2$. You do not need to solve this system.

   [6 marks]

---

7. The following were parts of question 1 of the June 2003 `MA5156S` paper.

(a) Obtain the weak form expressions corresponding to each of the following ODEs. In your answer you should give in each case the space of functions involved and you should indicate if any of the boundary conditions are natural boundary conditions.

(i)

$$-u''(x) + 12u(x) = 1, \quad 0 < x < 1, \quad u(0) = u'(1) = 0.$$

[2 marks]

(ii)

$$u''''(x) - u''(x) = \cos(x), \quad 0 < x < 1, \quad u(0) = u''(0) = u(1) = u''(1) = 0.$$

[3 marks]

(b) In the following the problem,

$$-u'' + 12u = 1, \quad 0 < x < 1, \quad u(0) = u'(1) = 0,$$

given in part (7a)(i), is to be approximately solved using the Galerkin finite element method using a piecewise linear approximating function defined on a mesh $0 = x_0 < x_1 < \cdots < x_M = 1$. Do the following.

(i) When the finite element method is implemented in an element-by-element way an actual element $[x_{i-1}, x_i]$ is mapped to a standard element such as $[0, 1]$. State the linear basis functions $\phi_1(s)$, $\phi_2(s)$, $0 \le s \le 1$ defined on the standard element $[0, 1]$ and express both the mapping and the form of the approximation in terms of these functions.

[2 marks]

(ii) For the ith element $[x_{i-1}, x_i]$ describe what is meant by the element matrix K_i and the element vector $\underline{b}_i$ for this problem and determine K_i and $\underline{b}_i$.

[4 marks]

(iii) Let $\hat{\phi}_i$, $i = 0, 1, \cdots, M$ denote the piecewise linear hat functions associated with the points $0 = x_0 < x_1 < \cdots < x_M = 1$ with the property $\hat{\phi}_i(x_i) = 1$, $\hat{\phi}_i(x_j) = 0$, $j \ne i$. In the case $M = 2$ and uniformly spaced points $x_i = i/2$, $i = 0, 1, 2$ determine the 3×3 global matrix $\hat{K} = (a(\hat{\phi}_i, \hat{\phi}_j))$, $0 \le i, j \le 2$ and the 3×1 global vector $\hat{\underline{b}} = ((f, \hat{\phi}_i))$, $0 \le i \le 2$ where $f(x) = 1$.

[3 marks]

(iv) Determine the finite element approximation at $x = 1/2$ and at $x = 1$ using this mesh.

[2 marks]

---

8. The following was question 2 of the June 2003 `MA3056S` paper.

In the following the ordinary differential equation

$$-u'' + 12u = 1, \quad 0 < x < 1, \quad u(0) = u(1) = 0$$

is to be approximately solved using the Galerkin finite element method using a continuous piecewise linear approximating function defined on a mesh $0 = x_0 < x_1 < \cdots < x_M = 1$.

(a) Derive the weak form for this problem in the form $a(u,v) = (f,v)$.

[2 marks]

(b) When the finite element method is implemented in an element-by-element way an actual element $[x_{i-1}, x_i]$ is mapped to a standard element such as $[0,1]$. State the linear basis functions $\phi_1(s)$, $\phi_2(s)$, $0 \le s \le 1$ defined on the standard element $[0,1]$ and express both the mapping and the form of the approximation in terms of these functions.

[2 marks]

(c) Let $\hat{\phi}_i(x)$, $i = 0, 1, \cdots, M$ denote the piecewise linear hat functions with the property $\hat{\phi}_i(x_i) = 1$ and $\hat{\phi}_i(x_j) = 0$ for $j \ne i$. On the element $[x_{i-1}, x_i]$ indicate which of these functions are non-zero and indicate how the non-zero functions are related to ϕ_1 and ϕ_2 given in (b).

[2 marks]

(d) For the ith element $[x_{i-1}, x_i]$ describe what is meant by the element matrix K_i and the element vector $\underline{b}_i$ for this problem and determine K_i and $\underline{b}_i$.

[5 marks]

(e) In the case $M = 3$ and uniformly spaced points $x_i = i/3$, $i = 0, 1, 2, 3$ do the following.

(i) Give the 3 element matrices K_1, K_2, and K_3 and the 3 element vectors $\underline{b}_1$, $\underline{b}_2$ and $\underline{b}_3$.

[2 marks]

(ii) Determine the 4×4 global matrix $\hat{K} = (a(\hat{\phi}_i, \hat{\phi}_j))$, $0 \le i, j \le 3$ and the 4×1 global vector $\underline{\hat{b}} = ((f, \hat{\phi}_i))$, $0 \le i \le 3$ where $f(x) = 1$.

[4 marks]

(iii) Determine the finite element approximation at $x = 1/3$ and at $x = 2/3$ obtained using this mesh of 3 elements.

[3 marks]

---

9. Apart from a few minor changes and re-typing this was question 3 of the 1999 `MA3056S` paper.

The two point boundary value, with solution $u(x)$,

$$-u''(x) + 6u(x) = x, \quad 0 < x < 1, \quad u(0) = u(1) = 0,$$

is to be solved via its weak formulation using a Galerkin finite element based on piecewise linear functions. For test functions v where

$$v \in H_0^1(0,1) := \{v : \; v, v' \in L_2(0,1), \; v(0) = v(1) = 0\}, \tag{1}$$

derive the weak form of problem (1). In the case of the partition $0 = x_0 < x_1 < x_2 < \cdots < x_{ne} = 1$, show that the weak form can be expressed in the form

$$\sum_{i=1}^{ne} a(u,v)_i = \sum_{i=1}^{ne} (x,v)_i \quad \text{for all } v \in H_0^1(0,1), \tag{2}$$

where

$$a(u,v)_i = \int_{x_{i-1}}^{x_i} u'v' + 6uv \,\mathrm{d}x \quad \text{and} \quad (x,v)_i = \int_{x_{i-1}}^{x_i} xv \,\mathrm{d}x.$$

Explain what is meant by an isoparametric finite element method.

A linear isoparametric finite element, based on the equally spaced points $x_i = i/3$, $i = 0, 1, 2, 3$ is used to approximate the solution of (2). Let $U(x)$ denote the approximation and let $U_i = U(x_i)$. With $s = s(x)$ denoting the linear mapping of $[x_{i-1}, x_i]$ onto $[0, 1]$ and with $\tilde{u}(s) = u(x(s))$ and $\tilde{v}(s) = v(x(s))$ for $x_{i-1} < x < x_i$ show that

$$\begin{aligned} a(u,v)_i &= \int_0^1 \left(\frac{d\tilde{u}}{ds}\frac{d\tilde{v}}{ds}\frac{ds}{dx} + 6\tilde{u}\tilde{v}\frac{dx}{ds} \right) \mathrm{d}s, \\ (x,v)_i &= \int_0^1 x(s)\tilde{v}(s)\frac{dx}{ds}\,\mathrm{d}s. \end{aligned}$$

Then show that the element matrix for the element $[x_{i-1}, x_i]$ is,

$$\frac{1}{3}\begin{pmatrix} 11 & -8 \\ -8 & 11 \end{pmatrix}.$$

Assemble the 3 element matrices to form a 4×4 global matrix. Also compute the 3 element constant vectors and assemble these to create a 4×1 global vector. Then, by taking account of the homogeneous boundary conditions obtain the 2-by-2 system satisfied by U_1 and U_2. You do not have to solve the system.

---

10. Let V be an inner product space, let $a(.,.)$ be a symmetric and positive definite bilinear form on $V \times V$ and let $f \in V$. Show that a function $u \in V$ satisfying

$$a(u,v) = (f,v) \quad \text{for all } v \in V$$

uniquely minimises the functional

$$I(v) = \frac{1}{2}a(v,v) - (f,v).$$

---

11. (a) After first reformulating the two-point boundary value problem

$$-u'' + 6u = 1, \quad u(0) = u(1) = 0$$

into weak form, calculate the Galerkin method approximation $U(x) = c_1\phi_1(x)$ where $\phi_1(x) = x(1-x)$.

(b) After first reformulating the two-point boundary value problem

$$-u'' + 6u = x, \quad u'(0) = 0, \quad u'(1) + u(1) = 0$$

into weak form, calculate the Galerkin method approximation $U_1 \in \operatorname{span}\{\phi_1\}$ and $U_2 \in \operatorname{span}\{\phi_1, \phi_2\}$ where $\phi_1(x) = 1$ and $\phi_2(x) = x$.

---

12. Let $0 = x_0 < x_1 < \cdots < x_{ne} = 1$ and let $\hat{\phi}_i(x)$ denote the piecewise linear 'hat' function which takes the value 1 at $x = x_i$. Describe mathematically $\hat{\phi}_i(x)$ and $\hat{\phi}_i'(x)$, $0 \le x \le 1$.

If

$$a(u, v) = \int_0^1 u'v' + quv \, \mathrm{d}x,$$

where $q(x) \ge 0$, then explain why the matrix $K = (a(\hat{\phi}_i, \hat{\phi}_j))$, $1 \le i, j \le ne-1$ is banded and state the band width.

Let

$$U = \sum_{j=1}^{ne-1} c_j \hat{\phi}_j.$$

and consider the case of equally spaced points with spacing $h = 1/ne$.

(a) In the case $q = 0$ show that

$$a(U, \hat{\phi}_i) = \int_0^1 \hat{\phi}_i(x) \, \mathrm{d}x$$

gives

$$-c_{i-1} + 2c_i - c_{i+1} = h^2.$$

(b) In the case $q = \alpha > 0$ is constant on $[0, 1]$ show that the equation

$$a(U, \hat{\phi}_i) = \int_0^1 \hat{\phi}_i \, \mathrm{d}x$$

leads to a relation of the form

$$k_{i,i-1}c_{i-1} + k_{ii}c_i + k_{i,i+1}c_{i+1} = h.$$

In your answer you should give $k_{i,i-1}$, k_{ii} and $k_{i,i+1}$ in terms of h and α.

---

13. Apart from a few minor changes and re-typing this was part of question 1 of the 2001 `MA3056S` paper.

The weak form of the two point boundary value problem

$$L(u(x)) = f(x), \quad 0 < x < 1, \quad u(0) = A, \quad u(1) = B, \tag{1}$$

where L is a linear differential operator, $f(x)$ is a given function and A and B are given values, is set up by multiplying (1) by a test function $v \in H_0^1(0,1) := \{v : v \in$

$H^1(0,1),\ v(0) = v(1) = 0\}$, and integrating the product over $[0,1]$. After integrating the weak form of problem (1) is of the form:

$$\text{find } u \in H \text{ such that} \quad a(u,v) = \int_0^1 fv \,\mathrm{d}x \quad \forall v \in H_0^1(0,1), \tag{2}$$

where $H := \{w :\ w \in H^1(0,1),\ w(0) = A,\ w(1) = B\}$ and where $a(.,.)$ is a symmetric bilinear form.

(a) Explain how you would apply the Galerkin technique using polynomial basis functions over $[0,1]$ to derive an approximation U to $u \in H$, the solution of (2).

[6 marks]

(b) Derive the weak form of the problem

$$-u''(x) = x^4, \quad 0 < x < 1, \quad u(0) = 1, \quad u(1) = 0. \tag{3}$$

[4 marks]

(c) Using the Galerkin technique with trial function

$$U(x) = (1-x) + c_1 x(1-x) + c_2 x^2(1-x)$$

containing the unknown parameters c_1 and c_2, and the respective test functions

$$v_1(x) = x(1-x) \quad \text{and} \quad v_2(x) = x^2(1-x),$$

set up the 2×2 system of linear equations for c_1 and c_2.
[Do not attempt to evaluate the integrals involved and do not attempt to solve the system.]

[5 marks]

---

14. The following was question 3 of the June 2003 `MA3056S` paper.

Let V denote a Hilbert space with inner product $(.,.)$, let $a(.,.)$ denote a symmetric and positive definite bilinear form defined on $V \times V$ and let $F(.)$ denote a linear functional defined on V. Also let $V_h \subset V$ and suppose that there exists $u \in V$ and $U_h \in V_h$ satisfying

$$\begin{aligned} a(u,v) &= F(v), \quad \text{for all } v \in V, \\ a(U_h,v) &= F(v), \quad \text{for all } v \in V_h. \end{aligned}$$

(a) Define what it means for $a(.,.)$ to be positive definite on $V \times V$.

[1 mark]

(b) If $e = u - U_h$ then show that $a(e,v) = 0$ for all $v \in V_h$.

[1 mark]

(c) Let $\|v\|_E = a(v,v)^{1/2}$ denote the energy norm on V.
State the Cauchy Schwarz inequality as applied to $a(v,w)$ and show that

$$\|u - U_h\|_E \le \|u - v\|_E \quad \text{for all } v \in V_h.$$

Comment on this property of U_h.

[4 marks]

(d) Show that $u \in V$ and $U_h \in V_h$ uniquely minimise the functional

$$I(v) = \frac{1}{2}a(v,v) - F(v) \qquad (*)$$

over the spaces V and V_h respectively.

[5 marks]

(e) For the functional $I(.)$ defined in $(*)$ in part (d) show that

$$I(U_h) = I(u) + \frac{1}{2}a(u - U_h, u - U_h).$$

[2 marks]

(f) Let $\|.\|$ denote the usual norm of V given by $\|v\| = (v,v)^{1/2}$. Suppose that the bilinear form $a(.,.)$ and the linear functional $F(.)$ satisfy the coercive and bounded properties that

$$\kappa_1\|v\|^2 \le a(v,v), \quad |a(v,w)| \le \kappa_2\|v\|\|w\|, \quad |F(v)| \le \kappa_3\|v\|$$

for all $v, w \in V$ where $0 < \kappa_1 \le \kappa_2$ and $\kappa_3 > 0$ are constants. Show that under these conditions we have the following.

(i)
$$\|u - U_h\|^2 \le \frac{\kappa_2}{\kappa_1}\|u - v\|^2 \quad \text{for all } v \in V_h.$$

[2 marks]

(ii)
$$\|u\| \le \frac{\kappa_3}{\kappa_1} \qquad \text{and} \qquad \|U_h\| \le \frac{\kappa_3}{\kappa_1}.$$

[2 marks]

(iii)
$$I(v) \ge -\frac{\kappa_3^2}{2\kappa_1}$$

for all $v \in V$, where $I(v)$ is defined in $(*)$ in part (d).

[3 marks]

---

15. Apart from a few minor changes and re-typing this was question 5 of the 2001 MA3056S paper.

(i) Let V denote a Hilbert space with inner product $(.,.)$ and let $a(.,.)$ denote a bilinear form which is such that $a(v,v) > 0$ for all $v \in V$ with $v \neq 0$. Also let $V_h \subset V$ denote another space of functions such that $u \in V$ and $U_h \in V_h$ minimise the functional

$$I(v) = \frac{1}{2}a(v,v) - (f,v)$$

over the spaces V and V_h respectively where $f \in V$. Show the following:

(a) $a(u,v) = (f,v)$ for all $v \in V$ and $a(U_h,v) = (f,v)$ for all $v \in V_h$.
(b) If $e = u - U_h$ then $a(e,v) = 0$ for all $v \in V_h$.
(c) $I(U_h) = I(u) + \frac{1}{2}a(e,e)$.
(d) $a(e,e) \le a(u-v,u-v)$ for all $v \in V_h$. Comment on this result.

[10 marks]

(ii) Show that the weak form of

$$u''''(x) = f(x), \quad 0 < x < 1 \quad \text{with } u(0) = u'(0) = u(1) = u'(1) = 0,$$

involves the bilinear form

$$a(u,v) = \int_0^1 u''v'' \, \mathrm{d}x.$$

If the finite element method is used to obtain an approximation U_h to u using piecewise cubic Hermite elements on a mesh $0 = x_0 < x_1 < \cdots < x_{ne} = 1$ then it can be shown that the approximate solution and its first derivative are exact at the point x_i. In the case $f(x) = 1$ give the exact solution. Also in the case $ne = 1$ give the finite element solution. Under these conditions of $f(x) = 1$ and $ne = 1$ and with $e = u - U_h$ show the following:

(a) $e'(1/2) = 0$.
(b) $e''(\alpha) = 0$ where $\alpha = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$.
(c) $e'''(1/2) = 0$.

[10 marks]

---

16. The following relates more to what is done in the MA5352 part of the module but anyone who has done Fourier series before can attempt it.

Let $v \in C^1[0,1]$ be a function satisfying $v(0) = v(1) = 0$ and let

$$v(x) = \sum_{k=1}^{\infty} c_k \sin k\pi x$$

be its Fourier series representation. Show that for all such functions

$$\int_0^1 v(x)^2 \, \mathrm{d}x \le \frac{1}{\pi^2} \int_0^1 v'(x)^2 \, \mathrm{d}x.$$

Use this result to determine the values of k for which $Lu = -u'' - k^2 u$ is positive definite on the space

$$D_L = \{v \in C^2[0,1] : \quad v(0) = v(1) = 0\}.$$

MA3951/MA5352: Numerical and Variational Methods for PDEs Exercise sheet 2

1. This was question 2 of the June 2004 MA5352B paper.

   (a) Let Ω denote a bounded simply connected domain with a piecewise smooth boundary $\partial\Omega$ and consider the Poisson problem

   $$-\Delta u = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega_1, \quad \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega_2$$

   where $\partial\Omega_1$ and $\partial\Omega_2$ give a partition of $\partial\Omega$, g is a constant and where $\partial/\partial n$ denotes partial differentiation in the direction of the outward normal to $\partial\Omega_2$.

   Show that if u satisfies this problem then $u \in V$ also satisfies the weak formulation

   $$\iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega v \, \mathrm{d}x\mathrm{d}y + g \int_{\partial\Omega_2} v \, \mathrm{d}s$$

   for all $v \in V$ where

   $$V = \{v \in H^1(\Omega) : \; v = 0 \;\text{ on } \partial\Omega_1\}.$$

   [6 marks]

   (b) Suppose that the domain Ω is partitioned into n triangular elements $\Omega_1, \cdots, \Omega_n$ such that

   $$\bar{\Omega} = \bigcup_{r=1}^{n} \bar{\Omega}_r, \quad \Omega_i \cap \Omega_j = \text{empty set for } i \neq j$$

   and let T denote the standard triangle with vertices $\underline{s}_1^T = (0,0)$, $\underline{s}_2^T = (1,0)$ and $\underline{s}_3^T = (0,1)$ in the (s,t) plane.

   (i) State the three standard linear Lagrange basis functions ϕ_1, ϕ_2 and ϕ_3 defined on T such that $\phi_i(\underline{s}_i) = 1$ and $\phi_i(\underline{s}_j) = 0$ if $j \neq i$.

   [1 mark]

   (ii) Let $\underline{x}_1 = (x_1, y_1)^T$, $\underline{x}_2 = (x_2, y_2)^T$ and $\underline{x}_3 = (x_3, y_3)^T$ denote the vertices of Ω_r, let $U(\underline{x})$ denote a linear function defined on Ω_r and let $U_i = U(\underline{x}_i)$, $i = 1, 2, 3$. Describe the mapping $\underline{x} : T \to \Omega_r$ which is such that $\underline{s}_i \to \underline{x}_i$, $i = 1, 2, 3$, give $U(\underline{x}(s,t))$ for $(s,t) \in T$ and show that the gradient vector ∇U can be written as

   $$\nabla U = \frac{1}{J} \begin{pmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{pmatrix} \begin{pmatrix} U_2 - U_1 \\ U_3 - U_1 \end{pmatrix}$$

   where $J = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$.

   [6 marks]

(iii) Explain what is meant by the element matrix K_r for the Poisson problem of part (a) and show that in the case $\underline{x}_1 = (0,0)^T$, $\underline{x}_2 = (2,0)^T$ and $\underline{x}_3 = (0,1)^T$ we obtain

$$K_r = \begin{pmatrix} 5/4 & -1/4 & -1 \\ -1/4 & 1/4 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

In the case $g = 0$ also give the 3×1 element vector $\underline{b}_r$.

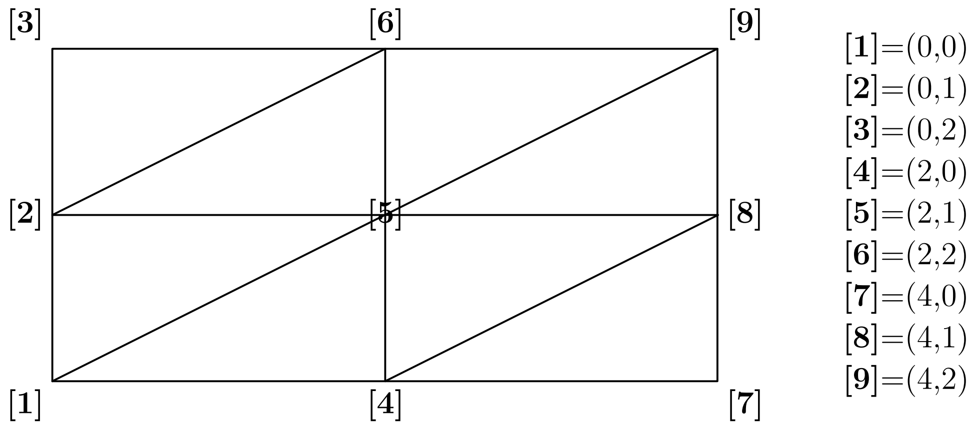
[6 marks]

(iv) Consider the uniform mesh of $\Omega := (0,4) \times (0,2)$ consisting of eight triangles and nine nodes as shown in the figure below and let $\hat{\phi}_i(\underline{x})$, $i = 1, 2, \cdots, 9$ denote the piecewise linear basis functions defined on $\bar{\Omega}$ which are such that

$$\hat{\phi}_i(\underline{x}_j) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases}$$

Also let $U_j = U(\underline{x}_j)$, $j = 1, 2, \cdots, 9$ denote the Galerkin approximation at the nodal points. Given the result of part (iii) and given that the element matrices have an invariance to translation, uniform scaling, reflection and rotations determine $a(\hat{\phi}_j, \hat{\phi}_5)$, $j = 1, 2, \cdots, 9$ and $(1, \hat{\phi}_5)$ in the equation

$$\sum_{j=1}^{9} a(\hat{\phi}_j, \hat{\phi}_5) U_j = (1, \hat{\phi}_5).$$



[6 marks]

---

2. Derive the weak forms for the following problems involving Poisson's equation

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^2, \quad f \in C(\bar{\Omega}),$$

subject to the boundary conditions given below, stating in each case the appropriate space of functions involved (in terms of Sobolev spaces) and classify each boundary condition as an essential boundary condition or as a natural boundary condition.

(a) $u = 0$ on $\partial\Omega$.

(b) $u = 0$ on $\partial\Omega_D$ and $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega_N$ where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ and assuming that $\partial\Omega_D$ is not empty.

(c) $u = 0$ on $\partial\Omega_D$ and $\dfrac{\partial u}{\partial n} + \beta u = g$ on $\partial\Omega_N$ where $\beta \ge 0$, $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ and assuming that $\partial\Omega_D$ is not empty.

---

3. Let T denote the standard triangle with vertices $\underline{s}_1^T = (0,0)$, $\underline{s}_2^T = (1,0)$ and $\underline{s}_3^T = (0,1)$ and let Ω_r denote an actual triangle with vertices $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$.

   (a) State the linear basis functions ϕ_1, ϕ_2 and ϕ_3 defined on T which have the property that $\phi_i(\underline{s}_i) = 1$ and $\phi_i(\underline{s}_j) = 0$ for $j \ne i$ and give the affine mapping of T onto Ω_r such that $\underline{x}(\underline{s}_i) = \underline{x}_i$, $i = 1,2,3$.
   State the Jacobian matrix $\tilde{J}$ of this mapping.

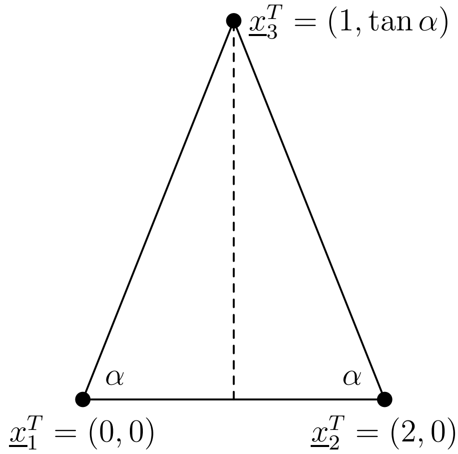
   (b) Let $U(\underline{x})$ denote a linear function defined on T and let $U_i = U(\underline{x}_i)$, $i = 1,2,3$. Show that the gradient vector ∇U can be written as
   $$\nabla U = \frac{1}{J}\begin{pmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{pmatrix}\begin{pmatrix} U_2 - U_1 \\ U_3 - U_1 \end{pmatrix}$$
   where $J = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$.

   (c) Given that we have reformulated Poisson's equation $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$ into weak form involving
   $$a(u,v) = \iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega f v \, \mathrm{d}x\mathrm{d}y = (f,v),$$
   explain what is meant by the 3×3 element stiffness matrix K_r for the triangle Ω_r and show that it can be written in the form
   $$K_r = \frac{|J|}{2} B^T B \quad \text{where} \quad B = \tilde{J}^{-T}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$
   where, as above, $\tilde{J}$ is Jacobian matrix and $J = \det(\tilde{J})$.

   (d) Explain why the matrix K_r given in part (c) is unchanged if we do any of the following.
   We shift the vertices to $\underline{x}_i + \underline{c}$, $i = 1,2,3$, i.e. we do a translation.
   We do a uniform scaling, i.e. $\underline{x}_i \to \alpha \underline{x}_i$, $i = 1,2,3$ where $\alpha > 0$.
   We rotate the points $\underline{x}_i$ about a point or we reflect the points $\underline{x}_i$ about a line which corresponds to replacing the Jacobian matrix $\tilde{J}$ by $Q\tilde{J}$ where Q is a 2×2 orthogonal matrix.

   (e) In the case of the isosceles triangle Ω_r with vertices $\underline{x}_1^T = (0,0)$, $\underline{x}_2^T = (2,0)$ and $\underline{x}_3^T = (1, \tan\alpha)$ show that the element stiffness matrix is
   $$K_r = \frac{1}{4\tan\alpha}\begin{pmatrix} \sec^2\alpha & 2 - \sec^2\alpha & -2 \\ 2 - \sec^2\alpha & \sec^2\alpha & -2 \\ -2 & -2 & 4 \end{pmatrix}.$$

[Note that we obtain from this the two specific cases covered in the notes by letting $\alpha = \pi/2$ (for the standard triangle) and letting $\alpha = \pi/3$ (for the equilateral triangle).]



---

4. The following was question 4 of the June 2003 `MA3056S` paper.

   Let Ω denote a bounded domain with a polygonal boundary $\partial\Omega$ and consider the Poisson equation
   $$-\Delta u = 1 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

   (a) Show that the weak form for this problem involves the expression
   $$\iint_\Omega \nabla u \cdot \nabla v \ \mathrm{d}x\mathrm{d}y = \iint_\Omega v \ \mathrm{d}x\mathrm{d}y$$
   and state an appropriate space of functions in which u and v should lie.

   [5 marks]

   (b) Suppose that the domain Ω is partitioned into n triangular elements $\Omega_1, \cdots, \Omega_n$ such that
   $$\bar{\Omega} = \bigcup_{i=1}^{n} \bar{\Omega}_i, \quad \Omega_i \cap \Omega_j = \text{empty set for } i \neq j$$
   and let T denote the standard triangle with vertices $\underline{s}_1^T = (0, 0)$, $\underline{s}_2^T = (1, 0)$ and $\underline{s}_3^T = (0, 1)$ in the (s, t) plane.

   (i) State the 3 standard Lagrange basis functions ϕ_1, ϕ_2 and ϕ_3 defined on T such that $\phi_i(\underline{s}_i) = 1$ and $\phi_i(\underline{s}_j) = 0$ if $j \neq i$.

   [1 mark]

   (ii) Suppose that the triangle Ω_i has vertices $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$ and let $U_i = U(\underline{x}_i)$, $i = 1, 2, 3$ denote the values of the function $U(\underline{x})$ which is linear for $\underline{x} \in \Omega_i$. Describe the mapping $\underline{x} : T \to \Omega_i$ and describe the function $U(\underline{x})$ at any point $\underline{x} \in \Omega_i$ using ϕ_1, ϕ_2 and ϕ_3 of part (i). Also give the Jacobian matrix $\tilde{J}$ of the mapping $\underline{x} : T \to \Omega_i$ in terms of $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$.

   [4 marks]

(iii) Explain what is meant by the element stiffness matrix K_i in this case of Poisson's equation and piecewise linear basis functions and show that K_i can be written in the form

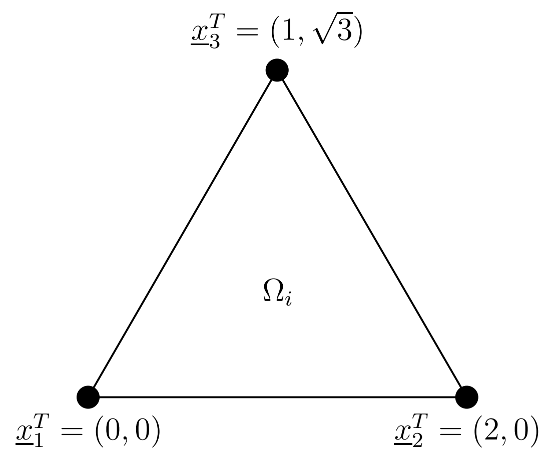
$$K_i = \frac{|\det \tilde{J}|}{2} \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{J}^{-1} \tilde{J}^{-T} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

[5 marks]

(iv) If Ω_i is the equilateral triangle shown below with vertices $\underline{x}_1^T = (0,0)$, $\underline{x}_2^T = (2,0)$ and $\underline{x}_3^T = (1,\sqrt{3})$ then give an affine mapping $\underline{x} : T \to \Omega_i$ and show that

$$K_i = \frac{1}{2\sqrt{3}} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

[3 marks]

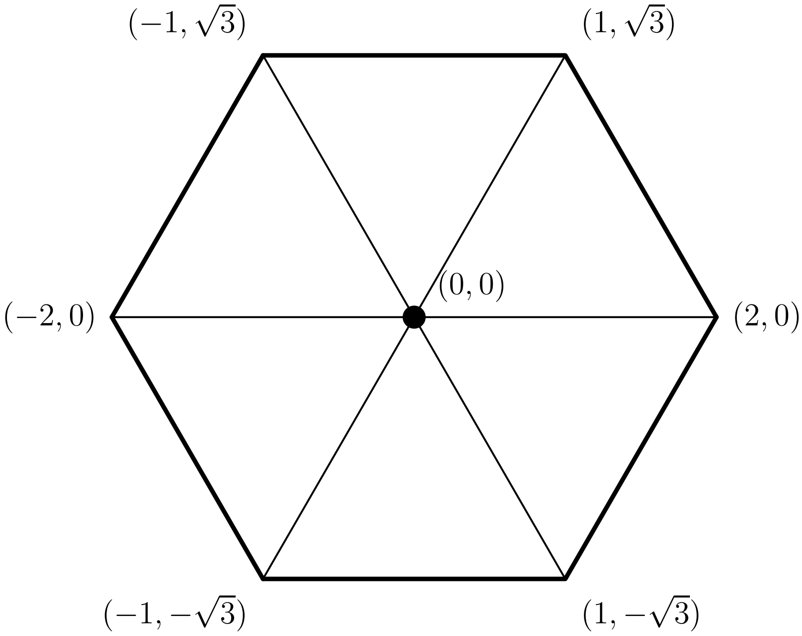


Suppose that Ω is the hexagonal domain which is the union of the 6 equilateral triangles shown below. Given that the element contribution to the right hand side vector for each triangle is

$$\iint_{\Omega_i} \begin{pmatrix} \tilde{\phi}_1(\underline{x}) \\ \tilde{\phi}_2(\underline{x}) \\ \tilde{\phi}_3(\underline{x}) \end{pmatrix} \mathrm{d}x\mathrm{d}y = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where $\tilde{\phi}_1$, $\tilde{\phi}_2$ and $\tilde{\phi}_3$ are the linear basis functions on Ω_i, and that all the element matrices are the same and are as given above, determine the finite element solution at the centre $(0,0)$ using piecewise linears on this mesh of 6 triangles.

[2 marks]



---

5. The following was question 5 of the June 2003 `MA3056S` paper.
Let $\Omega = \{(x, y) : \ 0 < x < 2, \ 0 < y < 1\}$, a 2×1 rectangle, and consider Poisson's equation with the mixed boundary conditions

$$\begin{aligned}
&-\Delta u = f \quad \text{in } \Omega, \\
&u(0, y) = u(2, y) = 0, \quad 0 < y < 1, \\
&\frac{\partial u}{\partial y}(x, 0) = \frac{\partial u}{\partial y}(x, 1) = 0, \quad 0 < x < 2,
\end{aligned}$$

where f is a continuous function on $\bar{\Omega}$.

(a) Show that the weak form for the problem can be written as follows.
Find $u \in V$ such that

$$a(u, v) := \iint_\Omega \nabla u \cdot \nabla v \ \mathrm{d}x\mathrm{d}y = \iint_\Omega fv \ \mathrm{d}x\mathrm{d}y =: (f, v)$$

for all $v \in V$. In your answer you should give the space of functions V taking into account any natural boundary conditions.

[5 marks]

(b) Suppose that the weak form is to be approximately solved by the finite element method using a piecewise bilinear function defined on a uniform mesh of square elements each of length h. In an implementation each square element is mapped to a standard element $S = \{(s, t) : \ -1 < s, t < 1\}$ with vertices $\underline{s}_1^T = (-1, -1)$, $\underline{s}_2^T = (1, -1)$, $\underline{s}_3^T = (1, 1)$ and $\underline{s}_4^T = (-1, 1)$ in the (s, t) plane.

(i) State the 4 standard basis functions ϕ_1, ϕ_2, ϕ_3 and ϕ_4 defined on S such that $\phi_i(\underline{s}_i) = 1$ and $\phi_i(\underline{s}_j) = 0$ for $j \neq i$.

[2 marks]

(ii) Let $\underline{x}_1^T = (c, d)$, $\underline{x}_2^T = (c+h, d)$, $\underline{x}_3^T = (c+h, d+h)$ and $\underline{x}_4^T = (c, d+h)$ be the vertices of an actual element Ω_i and let U_1, U_2, U_3 and U_4 denote respectively the value of the approximation at these points. Let $\underline{x} : S \to \Omega_i$ denote the mapping of the standard element to Ω_i. Explain why this can be written in the form

$$\underline{x} = \begin{pmatrix} c \\ d \end{pmatrix} + \frac{h}{2} \begin{pmatrix} 1+s \\ 1+t \end{pmatrix}.$$

[1 mark]

Further show that the gradient vector can be written in the form

$$\nabla U(\underline{x}(s,t)) = \frac{1}{2h} \begin{pmatrix} t-1 & 1-t & t+1 & -(t+1) \\ s-1 & -(s+1) & s+1 & 1-s \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}.$$

[3 marks]

(iii) It can be shown that the element stiffness matrix for this problem has the form

$$\begin{pmatrix} 2\beta & \gamma & -\beta & \gamma \\ \gamma & 2\beta & \gamma & -\beta \\ -\beta & \gamma & 2\beta & \gamma \\ \gamma & -\beta & \gamma & 2\beta \end{pmatrix}.$$

Show that $\beta = 1/3$ and $\gamma = -1/6$.
[Hint: First explain why $\beta + 2\gamma = 0$.]

[4 marks]

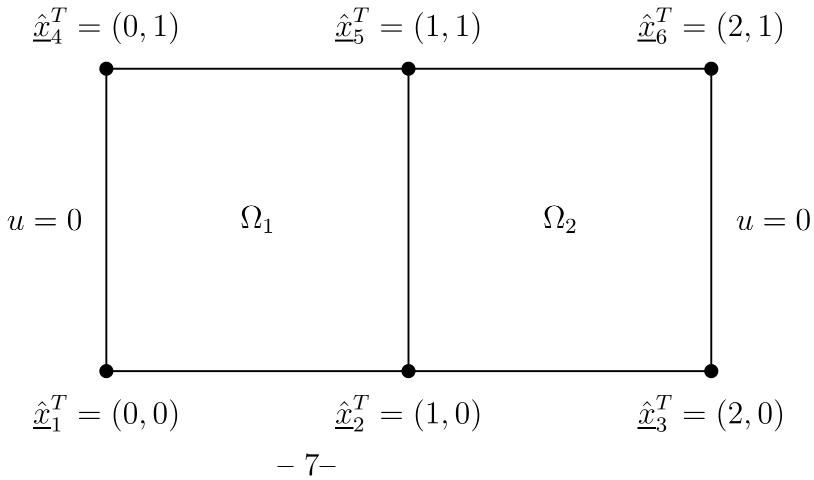
(iv) Consider the mesh involving the two squares shown in the figure. Let $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\phi}_3$, $\hat{\phi}_4$, $\hat{\phi}_5$ and $\hat{\phi}_6$ be the 6 piecewise bilinear functions defined on $\bar{\Omega}$ which satisfy $\hat{\phi}_i(\hat{\underline{x}}_i) = 1$ and $\hat{\phi}_i(\hat{\underline{x}}_j) = 0$ for $j \neq i$. Describe $\hat{\phi}_2(\underline{x})$ and $\hat{\phi}_5(\underline{x})$ for all $\underline{x} \in \Omega$.

[3 marks]

By using the element stiffness matrix given in part (iii), determine the entries $a(\hat{\phi}_2, \hat{\phi}_2)$, $a(\hat{\phi}_2, \hat{\phi}_5)$ and $a(\hat{\phi}_5, \hat{\phi}_5)$ in the linear system

$$\begin{pmatrix} a(\hat{\phi}_2, \hat{\phi}_2) & a(\hat{\phi}_2, \hat{\phi}_5) \\ a(\hat{\phi}_2, \hat{\phi}_5) & a(\hat{\phi}_5, \hat{\phi}_5) \end{pmatrix} \begin{pmatrix} U(\hat{\underline{x}}_2) \\ U(\hat{\underline{x}}_5) \end{pmatrix} = \begin{pmatrix} (f, \hat{\phi}_2) \\ (f, \hat{\phi}_5) \end{pmatrix}.$$

[2 marks]



6. Apart from a few minor changes and re-typing this was question 2 of the 2001 `MA5056S` paper. (`MA5156S` was previously labelled as `MA5056S`).

   For the boundary value problem in which $u(\underline{x}) = u(x,y)$ satisfies $-\Delta u(\underline{x}) = f(\underline{x})$, $\underline{x} \in \Omega$; $u(\underline{x}) = 0$, $\underline{x} \in \partial\Omega$, where $\Omega \subset \mathbb{R}^2$ has a polygonal boundary $\partial\Omega$ derive the weak formulation

$$a(u,v) = \iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \iint_\Omega fv \; \mathrm{d}x\mathrm{d}y \quad \text{for all } v \in S,$$

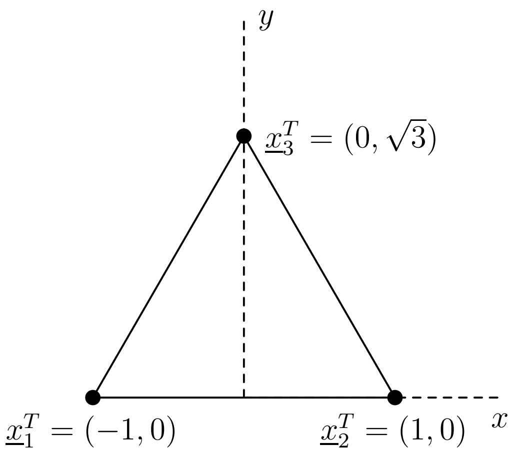
   giving the appropriate space of functions S.

[4 marks]

The weak problem is to be approximated using a finite element method based on piecewise linear functions defined over a triangular partition of Ω. The finite element method approximation is $U_h(\underline{x})$. What is meant by a finite element method being conforming and what conditions must $U_h(\underline{x})$ satisfy in order to be a conforming approximation?

[2 marks]

Suppose the standard triangular element is taken as the equilateral triangle T shown in the Figure with nodes $\underline{x}_1^T = (-1,0)$, $\underline{x}_2^T = (1,0)$, $\underline{x}_3^T = (0,\sqrt{3})$.



What are meant by the Lagrange basis functions on an element and what conditions must they satisfy?

[1 mark]

(a) Write down the three linear Lagrange basis functions $\phi_1(x,y)$, $\phi_2(x,y)$ and $\phi_3(x,y)$.

[4 marks]

(b) Hence write down the approximation $U_h(x,y)$ for (x,y) in T.

[1 mark]

(c) Derive the 3×3 element matrix arising from $a(u, v)|_T$.

[6 marks]

(d) Without evaluating the integrals give the 3×1 element vector arising from $(f, v)|_T$.

[1 mark]

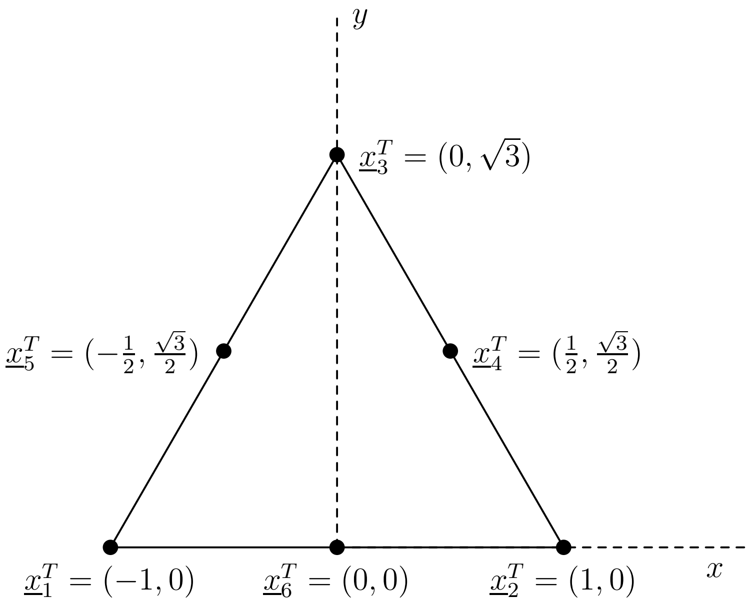
(e) Explain how the linear form for $U_h(x, y)$ as in (b) will produce a conforming finite element approximation.

[1 mark]

(f) For the equilateral triangle shown below with the six nodes

$$\underline{x}_1^T = (-1, 0), \quad \underline{x}_2^T = (1, 0), \qquad \underline{x}_3^T = (0, \sqrt{3}),$$
$$\underline{x}_4^T = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}), \quad \underline{x}_5^T = (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}), \quad \underline{x}_6^T = (0, 0)$$

write down the six quadratic Lagrange basis functions $N_j(x, y)$, $i = 1, 2, 3, 4, 5, 6$. [Hint: The functions N_j can be expressed in terms of the ϕ_i of part (a).]



[5 marks]

---

7. This question is taken from the last part of question 2 of the 1998 `MA3056S` paper.

The context of the question is that of Poisson's equation $-\Delta u = f$ which has been reformulated into the weak form

$$a(u, v) = (f, v), \ \forall v \in S$$

where

$$a(u, v) = \iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) \, \mathrm{d}x\mathrm{d}y \quad \text{and} \quad (f, v) = \iint_\Omega f v \, \mathrm{d}x\mathrm{d}y.$$

An earlier part of the question has also involved deriving the element matrix for the standard triangle T with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Using the basis functions $\phi_1(x,y) = 1 - x - y$, $\phi_2(x,y) = x$ and $\phi_3(x,y) = y$ the matrix is

$$K_i = \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

(i) Give the 3×1 element vector arising from $(f,v)|_T$.

(ii) Give the approximation to the element vector obtained when a one-point quadrature rule is used to evaluate the integrals.

Let Ω be the unit square $\{(x,y) : 0 < x, y < 1\}$ and $\partial\Omega_D \equiv \{(x,0) : 0 \le x \le 1\}$. Consider the triangular meshes shown in Fig. A and B, and in each case construct the 4×4 global stiffness matrix, by assembling the appropriate 3×3 element stiffness matrices, and the 4×1 global vector assuming that the one-point quadrature rule has been used. Then from the relevant linear equation systems satisfied by $(u_h(0,1), u_h(1,1))^T$, show that both meshes lead to the same solution only when

$$f(1/3,1/3) + f(2/3,2/3) = f(1/3,2/3) \quad \text{and} \quad f(2/3,1/3) + f(1/3,1/3) = f(2/3,2/3)\,.$$

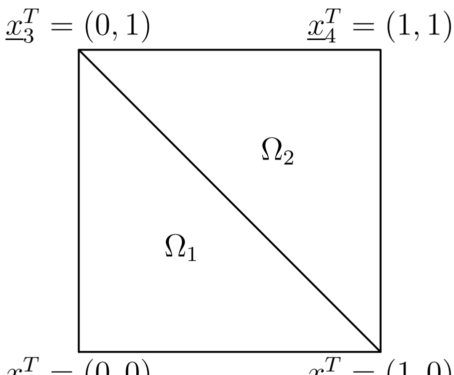


Fig. A

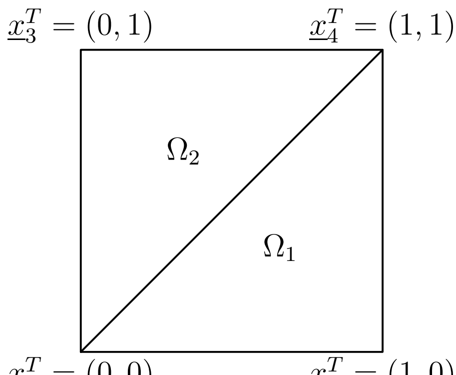


Fig. B

---

8. Apart from a few minor changes and re-typing this was question 2 of the 1999 `MA3056S` paper.

For the mixed boundary value problem in which $u(x,y)$ satisfies

$$\begin{aligned} -\Delta(u(x,y)) &= f(x,y), & (x,y) &\in \Omega, \\ u(x,y) &= 0, & (x,y) &\in \partial\Omega_D, \\ \frac{\partial u}{\partial n}(x,y) &= 0, & (x,y) &\in \partial\Omega_N \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is a rectangle with boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ and $\partial\Omega_D \cap \partial\Omega_N$ =empty set and $\dfrac{\partial}{\partial n}$ is the derivative in the outward normal direction to $\partial\Omega_N$, derive the weak formulation: find $u \in V$ such that

$$a(u,v) = \iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \right)\, \mathrm{d}x\mathrm{d}y = \iint_\Omega fv\, \mathrm{d}x\mathrm{d}y = (f,v) \quad \forall v \in V,$$

describing the appropriate space V.

The weak form is to be approximated using the finite element method. Explain what is meant by the conforming condition for the finite element approximation, giving sufficient conditions for this when piecewise polynomial functions are used.

The finite element approximation $U(x, y)$ is a piecewise bilinear function defined over a partition of Ω into square elements. In the case of the standard square S where $S = \{(x, y) : \ -1 \le x, y \le 1\}$ with nodes $(-1, -1)$, $(1, -1)$, $(1, 1)$ and $(-1, 1)$ write down the four Lagrange basis functions each of which is associated with one of the nodes. Then for this element do the following.

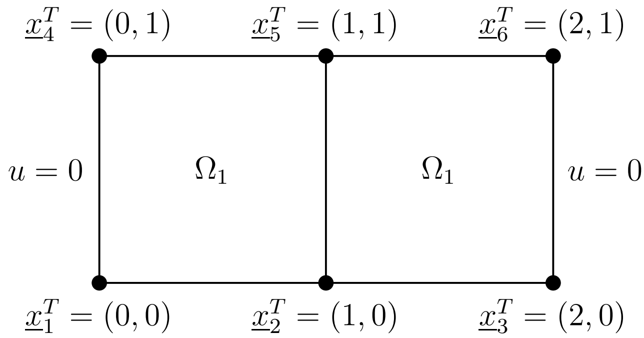
(i) It can be shown that the 4×4 element stiffness matrix arising from $a(u, v)|_S$ when bilinears is used has the form

$$\begin{pmatrix} \alpha & \beta & \beta & \gamma \\ \beta & \alpha & \gamma & \beta \\ \beta & \gamma & \alpha & \gamma \\ \gamma & \beta & \beta & \alpha \end{pmatrix}.$$

Evaluate α, β and γ.

(ii) Give the 4×1 element vector arising from S, leaving each component in integral form.

Let Ω be the rectangle $\{(x, y) : \ 0 < x < 2, \ 0 < y < 1\}$ and let $\partial\Omega_D = \{(0, y) : \ 0 < y < 1\} \cup \{2, y) : \ 0 < y < 1\}$ as in the Figure. Let $\underline{x}_i$, $i = 1, \cdots, 6$ be the nodes of the elements Ω_1 and Ω_2 of the mesh of Ω and let $U_i = U(\underline{x}_i)$.



By constructing the 4×4 element stiffness matrix for each of the elements Ω_1 and Ω_2 assemble the 6×6 global stiffness matrix associated with the given mesh of Ω. Apply the essential boundary conditions of the boundary value problem to obtain the 2×2 matrix K in the system

$$K \begin{pmatrix} U_2 \\ U_5 \end{pmatrix} = \begin{pmatrix} b_2 \\ b_5 \end{pmatrix}$$

that determine U_2 and U_5.

---

9. This question is taken from the last part of question 2 of the 1997 `MA5056S` paper.

The context of the question is that of Poisson's equation $-\Delta u = f$ in Ω with boundary conditions $u = 0$ on $\partial\Omega_D$ and $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega_N$ where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ with $\partial\Omega_D$ being non-empty. In an earlier part of the question this has been reformulated into the weak form

$$a(u,v) = (f,v), \ \forall v \in S$$

where

$$a(u,v) = \iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) \,\mathrm{d}x\mathrm{d}y \quad \text{and} \quad (f,v) = \iint_\Omega fv \,\mathrm{d}x\mathrm{d}y.$$

and where $S = \{v \in H^1(\Omega) : \ v = 0 \text{ on } \partial\Omega_D\}$.

An earlier part of the question has also been concerned with constructing the element matrix for the standard triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ and with the unit square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$. These matrices are

$$\frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{6}\begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix}.$$

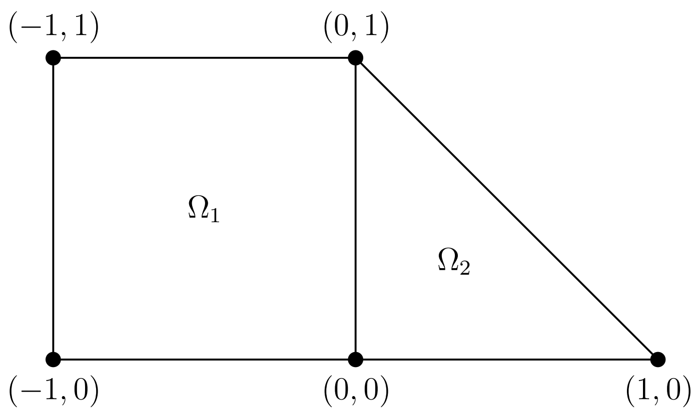
The last part of the question follows.

Let $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ denote the region shown in the Figure where Ω_1 is a unit square and Ω_2 is the standard triangle and suppose that $\partial\Omega_D = \{(x,y) : \ -1 \le x \le 1, \ y = 0\}$. If the finite element method is used to approximate the solution using a bilinear function on Ω_1 and a linear function on Ω_2 with

$$U(x,y) = U(-1,1)B_1(x,y) + U(0,1)B_2(x,y)$$

describing the approximation then give the functions B_1 and B_2 and derive the 2×2 matrix K in the linear system

$$K\begin{pmatrix} U(-1,1) \\ U(0,1) \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad b_i = \iint_\Omega f(x,y)B_i(x,y) \,\mathrm{d}x\mathrm{d}y, \quad i = 1,2.$$

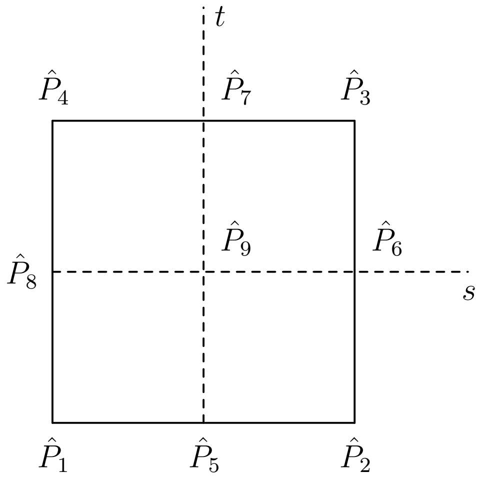


10. Apart from a few minor changes and re-typing this was question 1 of the 2002 MA5156S paper.

Let S be the square element in the (s, t) plane, as shown in the figure, with vertices $(-1, 1)$, $(1, -1)$, $(1, 1)$ and $(-1, 1)$, labelled respectively as $\hat{P}_1$, $\hat{P}_2$, $\hat{P}_3$ and $\hat{P}_4$. The points $(0, -1)$, $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, 0)$ are labelled respectively as $\hat{P}_5$, $\hat{P}_6$, $\hat{P}_7$, $\hat{P}_8$ and $\hat{P}_9$. For $i = 1, \cdots, 9$ write down the biquadratic basis functions $\phi_i(s, t)$, $(s, t) \in S$ which are such that

$$\phi_i(\hat{P}_j) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

[7 marks]



Consider the points (nodes) P_i of the element Ω_r in the (x, y) plane with coordinates (x_i, y_i), $i = 1, \cdots, 9$, where

$$\begin{aligned} &x_1 = x_8 = x_4 = -1, \quad y_1 = y_5 = y_2 = -1, \\ &x_5 = x_9 = 0, \quad y_6 = y_8 = y_9 = 0, \\ &x_2 = x_3 = x_6 = 1, \quad y_3 = y_4 = 1 \end{aligned}$$

and where (x_7, y_7) is such that $|x_7| < 1$ and $y_7 > 0$. With $\underline{x}_i^T = (x_i, y_i)$ and $\underline{x}^T = (x, y)$, the biquadratic mapping which takes S onto Ω_r is given by

$$\underline{x}(s, t) = \sum_{i=1}^{9} \underline{x}_i \phi_i(s, t).$$

What is the form of the mapping in the particular case $x_7 = 0$ and $y_7 = 1$.

[2 marks]

Hence explain why for a general point (x_7, y_7), $|x_7| < 1$, $y_7 > 0$ we have

$$\begin{aligned} x &= s + x_7 \phi_7(s, t) \\ y &= t + (y_7 - 1)\phi_7(s, t). \end{aligned}$$

[3 marks]

Hence sketch the form of the element Ω_r which is the image of S under the mapping.

[2 marks]

Give the partial derivatives $\dfrac{\partial \phi_7}{\partial s}$ and $\dfrac{\partial \phi_7}{\partial t}$ and show that the determinant of the Jacobian of the mapping is given by

$$J = 1 + x_7 \frac{\partial \phi_7}{\partial s} + (y_7 - 1)\frac{\partial \phi_7}{\partial t}.$$

[6 marks]

Show that for $(s, t) \in S$

$$-2 \le \frac{\partial \phi_7}{\partial s} \le 2; \quad -\frac{1}{2} \le \frac{\partial \phi_7}{\partial t} \le \frac{3}{2}.$$

Also show that if (x_7, y_7) satisfies

$$-\frac{1}{4} < x_7 < \frac{1}{4} \quad \text{and} \quad \frac{2}{3} < y_7 < 2,$$

then $J > 0$.

[5 marks]
