MA3951/MA5352:
Numerical and Variational Methods for PDEs
Exercise sheet 1 and Answers

1. Let Ω be a domain, let L denote a linear differential operator and let D_L be an inner product space of functions defined on Ω for which L is defined.

   Define what it means for L to be symmetric on D_L and define what it means for L to be positive definite on D_L.

ANSWER

L is symmetric on D_L if $(Lu, v) = (u, Lv)$ for all $u, v \in D_L$.

L is positive definite on D_L if $(Lv, v) > 0$ for all $v \in D_L$ with $v \neq 0$.

In each of the following cases, determine whether or not the given differential operator L is symmetric on the function space D_L. For the differential operators which are symmetric determine whether or not they are positive definite.

(a) $\Omega = (0,1)$, $D_L = C^2[0,1]$, $Lu = -u''$.

(b) $\Omega = (0,1)$, $D_L = \{v \in C^2[0,1] : \quad v(0) = 0\}$, $Lu = -u''$.

(c) $\Omega = (0,1)$, $D_L = \{v \in C^2[0,1] : \quad v(0) = 0,\ v(1) = 0\}$, $Lu = -u''$.

(d) $\Omega = (0,1)$,

$$D_L = \{v \in C^2[0,1] : \quad v(0) = 0,\ v(1) = 0\}, \quad Lu = -u'' + q(x)u,$$

where $q \in C[0,1]$ and $q(x) \geq 0$ on $[0,1]$.

(e) $\Omega = (0,1)$,

$$D_L = \{v \in C^2[0,1] : \quad v(0) = 0,\ v'(1) + \beta v(1) = 0\}, \quad L = -u'' + q(x)u,$$

where $\beta \geq 0$ and where $q \in C[0,1]$ and $q(x) \geq 0$.

(f) $\Omega = (0,1)$,

$$D_L = \{v \in C^2[0,1] : \quad v'(0) = 0,\ v'(1) = 0\}, \quad L = -u''.$$

(g) $\Omega = (0,1)$,

$$D_L = \{v \in C^2[0,1] : \quad v'(0) - \beta_1 v(0) = 0,\ v'(1) + \beta_2 v(1) = 0\}, \quad L = -u'' + q(x)u,$$

where $\beta_1 > 0$ and $\beta_2 > 0$ and where $q \in C[0,1]$ and $q(x) \geq 0$.

ANSWER

All the above are second order ODEs and the main identity being used throughout is

$$
\begin{aligned}
(-u'', v) &= -\int_0^1 u''v \; \mathrm{d}x = -[u'v]_0^1 + \int_0^1 u'v' \; \mathrm{d}x \\
&= (-u'(1)v(1) + u'(0)v(0)) + \int_0^1 u'v' \; \mathrm{d}x.
\end{aligned}
$$

Whether or not we have symmetry depends on the boundary term and this in turn depends on the boundary conditions, if any, which are part of the definition of the space D_L.

(1a), **(1b)**, **(1c)**. These all correspond to $Lu = -u''$.
When $D_L = C^2[0,1]$ or when $D_L = \{C^2[0,1] : \; v(0) = 0\}$ the boundary term is not symmetric for all $u, v \in D_L$. As an example, take $u = x$ and $v = x^2$ and observe that $-u'(1)v(1) = -1$ and $-v'(1)u(1) = -2$.
When $D_L = \{C^2[0,1] : \; v(0) = v(1) = 0\}$ the boundary term is 0 and we have symmetry,

$$
(-u'', v) = \int_0^1 u'v' \; \mathrm{d}x \quad \text{for all } u, v \in D_L.
$$

L is also positive definite on D_L and this is shown below.

(1d). When $Lu = -u'' + qu$ and $D_L = \{C^2[0,1] : \; v(0) = v(1) = 0\}$ we similarly get

$$
(Lu, v) = \int_0^1 u'v' + quv \; \mathrm{d}x
$$

and we have symmetry. L is also positive definite on D_L and this is shown below.

(1e). When $v(0) = 0$ and $u'(1) = -\beta u(1)$ we have

$$
-u'(1)v(1) + u'(0)v(0) = -u'(1)v(1) = +\beta u(1)v(1).
$$

When $u, v \in D_L = \{v \in C^2[0,1] : \; v(0) = 0, \; v'(1) + \beta v(1) = 0\}$ and $Lu = -u'' + qu$ we hence have

$$
(Lu, v) = \int_0^1 u'v' + quv \; \mathrm{d}x + \beta u(1)v(1)
$$

and thus L is symmetric on D_L. L is also positive definite on D_L and this is shown below.

(1f). When $u'(0) = u'(1) = 0$ and $Lu = -u''$ the boundary term is 0 and

$$
(Lu, v) = \int_0^1 u'v' \; \mathrm{d}x
$$

for all $u, v \in D_L = \{v \in C^2[0,1] : \; v'(0) = v'(1) = 0\}$, and we have symmetry. L is NOT positive definite on D_L. This is because if $v(x) = 1$ then $v'(x) = 0$, $v \in D_L$, and $(Lv, v) = 0$. L is positive semi-definite on D_L but as we have a non-zero function giving $(Lv, v) = 0$ it follows that L is not positive definite on D_L.

(1g). When $u'(0) = \beta_1 u(0)$ and $u'(1) = -\beta_2 u(1)$ we have

$$-u'(1)v(1) + u'(0)v(0) = \beta_1 u(0)v(0) + \beta_2 u(1)v(1).$$

Thus if $D_L = \{v \in C^2[0,1] : \ v'(0) - \beta_1 v(0) = 0,\ v'(1) + \beta_2 v(1) = 0\}$, and $L = -u'' + q(x)u$ then

$$(Lu, v) = \int_0^1 u'v' + quv \ \mathrm{d}x + (\beta_1 u(0)v(0) + \beta_2 u(1)v(1))$$

and we have symmetry. L is also positive definite on D_L and this is considered next with all the positive definite cases.

Verifying the positive definite property.
We consider the expression

$$(Lv, v) = \int_0^1 v'^2 + qv^2 \ \mathrm{d}x + (\beta_1 v(0)^2 + \beta_2 v(1)^2)$$

which covers all the cases if we allow $q = 0$ or $\beta_1 = 0$ or $\beta_2 = 0$. As $\beta_1 \ge 0$ and $\beta_2 \ge 0$ and $q(x) \ge 0$ we have a sum of non-negative terms and the only way $(Lv, v) = 0$ is if each term individually is 0. With the spaces involved the integrand is continuous and thus we must have $v'(x) = 0$, $0 < x < 1$ and hence $v(x) = c$ where c is a constant. In (1f) there was no condition to force $c = 0$ but (1c), (1d) and (1e) all contain the condition $v(0) = 0$ to give $c = 0$. For (1g) the condition $(Lv, v) = 0$ also implies that

$$\beta_1 v(0)^2 + \beta_2 v(1)^2 = 0$$

and the condition here that $\beta_1 > 0$ and $\beta_2 > 0$ is sufficient to imply that $v(0) = v(1) = 0$ and we get $c = 0$ as required.

(h) $\Omega = (0, 1)$,

$$D_L = \{v \in C^4[0,1] : \quad v(0) = v'(0) = v(1) = v'(1) = 0\}, \quad L = u'''' - (pu')' + qu,$$

where $p \in C^1[0,1]$, $q \in C[0,1]$ and $p(x) \ge p_0 > 0$ and $q(x) \ge 0$ for all $x \in (0,1)$.

ANSWER

For the second order term and the property that $v(0) = v(1) = 0$ we get

$$-\int_0^1 (pu')'v \ \mathrm{d}x = \int_0^1 pu'v' \ \mathrm{d}x.$$

For the fourth order term we integrate by parts twice and we use all the boundary conditions.

$$\begin{aligned}\int_0^1 u''''v\ \mathrm{d}x &= [u'''v]_0^1 - \int_0^1 u'''v'\ \mathrm{d}x = -\int_0^1 u'''v'\ \mathrm{d}x \quad \text{because } v(0)=v(1)=0,\\ &= -[u''v']_0^1 + \int_0^1 u''v''\ \mathrm{d}x = \int_0^1 u''v''\ \mathrm{d}x, \quad \text{because } v'(0)=v'(1)=0.\end{aligned}$$

Putting everything together gives

$$(Lu,v) = \int_0^1 u''v'' + pu'v' + quv\ \mathrm{d}x$$

for all $u, v \in D_L$ and we have symmetry. We also have the positive definite property as

$$(Lv,v) = \int_0^1 v''^2 + pv'^2 + qv^2\ \mathrm{d}x \geq 0$$

and for $(Lv,v)=0$ we must have $v''(x)=0$ in $(0,1)$. $v''=0$ implies that v is linear and because $v(0)=v(1)=0$ we must have $v(x)=0$ as required.

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2. The following relates to problems in $\mathbb{R}^2$. This will be only briefly covered at the start of the module but if you already know results such as the divergence theorem then you could try this now.

   As in the previous question in each of the following cases, determine whether or not the given differential operator L is symmetric on the function space D_L. For the differential operators which are symmetric determine whether or not they are positive definite.

   (a)
   $$\Omega = \{(x,y):\quad 0<x<2,\ 0<y<1\},$$
   $$D_L = C^2(\bar{\Omega}),\quad Lu = -\Delta u.$$

   (b)
   $$\Omega = \{(x,y):\quad 0<x<2,\ 0<y<1\},$$
   $$D_L = \{v \in C^2(\bar{\Omega}):\quad v=0 \text{ on } \partial\Omega\},\quad Lu = -\Delta u.$$

   (c) Let
   $$\Omega = \{(x,y):\quad 0<x<2,\ 0<y<1\},$$
   let $\partial\Omega_1$ denote the 3 sides on $\partial\Omega$ corresponding to $x=0$, $x=2$ and $y=1$ and let $\partial\Omega_2$ denote the other side corresponding to $y=0$. Also let
   $$D_L = \{v \in C^2(\bar{\Omega}):\quad v=0 \text{ on } \partial\Omega_1 \text{ and } -\frac{\partial v}{\partial y}(x,0) + \beta v(x,0) = 0,\ 0<x<2\},$$
   where $\beta \geq 0$, and let $L = -\Delta$.

ANSWER

(2a), **(2b)** and **(2c)**.

$$\nabla \cdot (v \nabla u) = v\Delta v + \nabla u \cdot \nabla v \quad \text{gives} \quad -v\Delta v = \nabla u \cdot \nabla v - \nabla \cdot (v\nabla u).$$

The divergence theorem then gives

$$-\iint_\Omega v\Delta v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y - \int_{\partial\Omega} v\frac{\partial u}{\partial n} \, \mathrm{d}s.$$

The double integral on the right hand side is symmetric in u and v and thus symmetry depends on whether the boundary integral term is symmetric.

In (2a) there is no condition on u, v on $\partial\Omega$ and L is NOT symmetric on $D_L = C^2(\bar{\Omega})$. This can be confirmed by taking $u = 1$ and $v = x^2$. As $Lu = -\Delta u = 0$ and $Lv = -\Delta v = -2$ we have $(Lu, v) = 0$ and $(u, Lv) = -2 \times (\text{area of } \Omega) \neq 0$. Hence $(Lu, v) \neq (u, Lv)$.

In (2b) we have $v = 0$ on $\partial\Omega$ and

$$(Lu, v) = \iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y$$

for all $u, v \in D_L$ and hence L is symmetric on D_L. It is also positive definite on D_L since

$$(Lv, v) = \iint_\Omega \nabla v \cdot \nabla v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega (\nabla v)^2 \, \mathrm{d}x\mathrm{d}y \geq 0.$$

If $(Lv, v) = 0$ then the continuity of the integrand implies that $\nabla v = \underline{0}$ which in turn implies that v is constant. The condition that $v = 0$ on $\partial\Omega$ implies that the constant is 0.

In (2c) we have $v = 0$ on $\partial\Omega_1$ and

$$\frac{\partial u}{\partial y} = \beta u \quad \text{on } \partial\Omega_2.$$

Now on the side $y = 0$ the outward normal direction is $(0, -1)^T$ and thus

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial y} = -\beta u.$$

Hence

$$-\int_{\partial\Omega_2} v\frac{\partial u}{\partial n} \, \mathrm{d}s = +\int_{\partial\Omega_2} \beta uv \, \mathrm{d}s.$$

Thus

$$-\iint_\Omega v\Delta v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y + \int_{\partial\Omega_2} \beta uv \, \mathrm{d}s$$

and we have symmetry. It is also positive definite on D_L by a similar argument to the above where we now need to use the condition $v = 0$ on $\partial\Omega_1$.

(d) $\Omega \subset \mathbb{R}^2$,

$$D_L = \{C^4(\bar{\Omega}) : \quad v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\}, \quad Lu = \Delta^2 u = \Delta(\Delta u).$$

ANSWER

Let $w = \Delta u$ and note the vector identities

$$\begin{aligned} -v\Delta w &= \nabla v \cdot \nabla w - \nabla \cdot (v\nabla w), \\ -w\Delta v &= \nabla v \cdot \nabla w - \nabla \cdot (w\nabla v). \end{aligned}$$

Integrating over Ω for a function v satisfying $v = \partial v/\partial n = 0$ on $\partial\Omega$ we have

$$-\iint_\Omega v\Delta w \, \mathrm{d}x\mathrm{d}y = \iint_\Omega \nabla v \cdot \nabla w \, \mathrm{d}x\mathrm{d}y = -\iint_\Omega w\Delta v \, \mathrm{d}x\mathrm{d}y.$$

Thus

$$(Lu, v) = \iint_\Omega v\Delta^2 u \, \mathrm{d}x\mathrm{d}y = \iint_\Omega \Delta u \Delta v \, \mathrm{d}x\mathrm{d}y.$$

for all $u, v \in D_L$. Hence L is symmetric on D_L. It is positive definite on D_L because

$$(Lv, v) = \iint_\Omega (\Delta v)^2 \, \mathrm{d}x\mathrm{d}y \geq 0.$$

If $(Lv, v) = 0$ then the continuity of the integrand gives $\Delta v = 0$ in Ω. Then by using the boundary condition $v = 0$ on $\partial\Omega$ we similarly get

$$-\iint_\Omega v\Delta v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega (\nabla v)^2 \, \mathrm{d}x\mathrm{d}y = 0$$

and as before this gives $\nabla v = \underline{0}$ in Ω which in turn implies that $v = c$ where c is a constant. The boundary condition $v = 0$ on $\partial\Omega$ implies that $v = c = 0$ in Ω.

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3. Derive the expressions involved in the weak forms for the following problems stating in each case the appropriate function space involved and classify each boundary condition as an essential boundary condition or as a natural boundary condition.

(a)

$$-(pu')' + qu = f, \quad 0 < x < 1, \quad u(0) = u(1) = 0,$$

where $p \in C^1[0, 1]$, $q, f \in C[0, 1]$ and $p(x) > 0$ and $q(x) \geq 0$.

(b)

$$-(pu')' + qu = f, \quad 0 < x < 1, \quad u(0) = 0, \quad u'(1) + u(1) = 0,$$

where $p \in C^1[0, 1]$, $q, f \in C[0, 1]$ and $p(x) > 0$ and $q(x) \geq 0$.

ANSWER

(3a), **(3b)**. When we multiply the ODE by a test function v and integrate from 0 to 1, one of the terms is

$$-\int_0^1 (pu')'v \, \mathrm{d}x = -[pu'v]_0^1 + \int_0^1 pu'v' \, \mathrm{d}x.$$

In (3a) we require $u(0) = u(1) = 0$ and if we also insist that $v(0) = v(1) = 0$ then we eliminate the boundary term. Hence we take the function space

$$V = \{v \in C^2[0,1] : \quad v(0) = v(1) = 0\}.$$

One version of the weak form is as follows. Find $u \in V$ such that

$$\int_0^1 pu'v' + quv \, \mathrm{d}x = \int_0^1 fv \, \mathrm{d}x \quad \text{for all } v \in V.$$

The boundary conditions at $x = 0$ and at $x = 1$ are both essential boundary conditions and need to be part of the specification of the function space V.

This derivation gets the correct expressions. To weaken the continuity requirements on the space we can replace the space V given by $V = H_0^1(0,1)$.

In (3b) we require that $u(0) = 0$ and thus if $v(0) = 0$ then we have that the solution u satifies

$$-[pu'v]_0^1 = -p(1)u'(1)v(1) = p(1)u(1)v(1).$$

This leads us to now define

$$V = \{v \in C^2[0,1] : \quad v(0) = 0\}.$$

One version of the weak form is as follows. Find $u \in V$ such that

$$\int_0^1 pu'v' + quv \, \mathrm{d}x + p(1)u(1)v(1) = \int_0^1 fv \, \mathrm{d}x \quad \text{for all } v \in V.$$

The boundary condition at $x = 0$ is an essential boundary conditions and needs to be part of the specification of the function space V. The boundary condition $u'(1) + u(1) = 0$ is a natural boundary condition for this particular weak form.

Again, to weaken the continuity requirements on the space we can replace the space V just given by

$$V = \{v \in H^1(0,1) : \; v(0) = 0\}.$$

(c)

$$u'''' = f, \quad 0 < x < 1, \quad u(0) = u'(0) = u(1) = u'(1) = 0,$$

where $f \in C[0,1]$.

(d)

$$u'''' = f, \quad 0 < x < 1, \quad u(0) = u''(0) = u(1) = u''(1) = 0,$$

where $f \in C[0,1]$.

ANSWER

(3c) and **(3d)**. As in (1h) we have

$$\begin{aligned}\int_0^1 u''''v \, \mathrm{d}x &= [u'''v]_0^1 - \int_0^1 u'''v' \, \mathrm{d}x = -\int_0^1 u'''v' \, \mathrm{d}x \quad \text{if } v(0)=v(1)=0, \\ &= -[u''v']_0^1 + \int_0^1 u''v'' \, \mathrm{d}x.\end{aligned}$$

In (3c) we need $u'(0) = u'(1) = 0$ and if we restrict to $v'(0) = v'(1) = 0$ then the boundary term in the last expression is 0. However in (3d) we are given $u''(0) = u''(1) = 0$ and we do not need to place any such condition on v' at the end points. This leads to the following weak problems.

For (3c) we take

$$V = \{v \in C^4[0,1] : \quad v(0) = v'(0) = v(1) = v'(1) = 0\}.$$

The problem is find $u \in V$ such that

$$\int_0^1 u''v'' \, \mathrm{d}x = \int_0^1 fv \, \mathrm{d}x \quad \text{for all } v \in V.$$

All the boundary conditions $u(0) = u'(0) = u(1) = u'(1) = 0$ are essential boundary conditions.
We can weaken the continuity conditions on the space V by taking instead

$$V = \{v \in H^2(0,1) : \quad v(0) = v'(0) = v(1) = v'(1) = 0\}.$$

For (3d) we take

$$V = \{v \in C^4[0,1] \quad v(0) = v(1) = 0\}.$$

The problem is find $u \in V$ such that

$$\int_0^1 u''v'' \, \mathrm{d}x = \int_0^1 fv \, \mathrm{d}x \quad \text{for all } v \in V.$$

All the boundary conditions $u(0) = u(1) = 0$ are essential boundary conditions but the conditions $u''(0) = u''(1) = 0$ are natural boundary conditions for this weak form.
We can weaken the continuity conditions on the space V by taking instead

$$V = \{v \in H^2(0,1) : \quad v(0) = v(1) = 0\}.$$

4. This was question 1 of the June 2004 MA3951 paper.

   (a) Obtain the weak form expressions corresponding to each of the following boundary value problems. In your answer you should give in each case the space of functions involved and you should indicate if any of the boundary conditions are essential boundary conditions.

   (i)
$$-((1+x)u')' = 1, \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

[2 MARKS]

   (ii)
$$-u'' + 2u = x^2, \quad 0 < x < 1, \quad u(0) = u'(1) + 2u(1) = 0.$$

[2 MARKS]

   (iii)
$$u'''' - u'' = 1, \quad 0 < x < 1, \quad u(0) = u'(0) = u'(1) = u'''(1) = 0.$$

[3 MARKS]

   (b) In the following the ordinary differential equation

$$-u'' + 2u = x, \quad 0 < x < 1, \quad u(0) = 0, \quad u'(1) = 0,$$

   is to be approximately solved using Galerkin's method.

   (i) Derive the weak form for this problem in the form $a(u, v) = (f, v)$.

[1 MARK]

   (ii) Let $\psi_i(x) = x^i$, $i = 1, 2, \cdots$ and let $V_n = \mathrm{span}\{\psi_1, \cdots, \psi_n\}$. Determine the Galerkin approximation from V_1 and determine a linear system for c_1 and c_2 in the Galerkin approximation $c_1 x + c_2 x^2$ from the space V_2. You do not need to solve these equations.

[5 MARKS]

   (iii) Let $M \ge 1$ be a natural number and let $h = 1/M$, and $x_i = ih$, $i = 0, 1, \cdots, M$, denote equally spaced mesh points in $[0, 1]$. Also, on the ith element $[x_{i-1}, x_i]$, let

$$\tilde{\phi}_1(x) = \frac{x_i - x}{h}, \quad \tilde{\phi}_2(x) = \frac{x - x_{i-1}}{h},$$

   denote the standard linear basis functions defined on $[x_{i-1}, x_i]$. Show that the element matrix K_i and the element vector $\underline{b}_i$ are given by

$$K_i = \frac{1}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{h}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \underline{b}_i = \frac{h}{2}\left(x_{i-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{h}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right).$$

   (In your answer you may assume that

$$\int_{x_{i-1}}^{x_i} \tilde{\phi}_1^2 \, \mathrm{d}x = \int_{x_{i-1}}^{x_i} \tilde{\phi}_2^2 \, \mathrm{d}x = \frac{h}{3} \quad \text{and} \quad \int_{x_{i-1}}^{x_i} \tilde{\phi}_1 \tilde{\phi}_2 \, \mathrm{d}x = \frac{h}{6}.)$$

[4 MARKS]

(iv) Let U denote the Galerkin finite element approximation to this problem using piecewise linears defined on the mesh in part (iii) in the case $M = 2$ and let $U_1 = U(x_1)$ and $U_2 = U(x_2)$. By using the results of part (iii) determine the linear equations that U_1 and U_2 satisfy. You do not need to solve these equations.

[3 MARKS]

ANSWER

(a) (i)

$$-\int_0^1 ((1+x)u')'v \, \mathrm{d}x = -\left[(1+x)u'v\right]_0^1 + \int_0^1 (1+x)u'v' \, \mathrm{d}x = \int_0^1 (1+x)u'v' \, \mathrm{d}x$$

if $v(0) = v(1) = 0$. Let $V = \{v \in H^1(0,1) : \ v(0) = v(1) = 0\}$. For the weak problem we have the following. Find $u \in V$ such that

$$\int_0^1 (1+x)u'v' \, \mathrm{d}x = \int_0^1 v \, \mathrm{d}x \quad \text{for all } v \in V.$$

Both boundary conditions which are part of the specification of the space V are essential boundary conditions.

2 MARKS

(ii) For the function u satisfying the ODE

$$-\int_0^1 u''v \, \mathrm{d}x = -\left[u'v\right]_0^1 + \int_0^1 u'v' \, \mathrm{d}x = \int_0^1 u'v' \, \mathrm{d}x - u'(1)v(1) = \int_0^1 u'v' \, \mathrm{d}x + 2u(1)v(1)$$

provided $v(0) = 0$. Let $V = \{v \in H^1(0,1) : \ v(0) = 0\}$. For the weak problem we have the following. Find $u \in V$ such that

$$\int_0^1 (u'v' + 2uv) \, \mathrm{d}x + 2u(1)v(1) = \int_0^1 x^2 v \, \mathrm{d}x \quad \text{for all } v \in V.$$

The boundary condition $u(0) = 0$ is an essential boundary condition.

3 MARKS

(iii)

$$\int_0^1 (u'''' - u'')v \, \mathrm{d}x = \left[(u''' - u')v\right]_0^1 - \int_0^1 (u''' - u')v' \, \mathrm{d}x = \int_0^1 (-u'''v' + u'v') \, \mathrm{d}x = -\left[u''v'\right]_0^1 + \int_0^1 (u''v'' + u'v') \, \mathrm{d}x$$

provided $v(0) = v'(0) = v'(1) = 0$ and by using the boundary conditions $u'(1) = u'''(1) = 0$ that the exact solution u satisfies at $x = 1$. Let

$$V = \{v \in H^2(0,1) : \; v(0) = v'(0) = v'(1) = 0\}.$$

For the weak problem we have the following. Find $u \in V$ such that

$$\int_0^1 (u''v'' + u'v') \, \mathrm{d}x = \int_0^1 v \, \mathrm{d}x \quad \text{for all } v \in V.$$

The 3 boundary conditions in the specification of V are essential boundary conditions.

3 MARKS

(b) (i) We let $V = \{v \in H^1(0,1) : \; v(0) = 0\}$. The weak form involves finding $u \in V$ such that for all $v \in V$ we have

$$a(u,v) := \int_0^1 (u'v' + 2uv) \, \mathrm{d}x = \int_0^1 xv \, \mathrm{d}x =: (f,v).$$

1 MARK

(ii) For any function

$$v = \sum_1^n \alpha_i \psi_i \in V_n$$

the bilinearity gives

$$a(v,v) = \underline{\alpha}^T K \underline{\alpha}.$$

As $a(.,.)$ is a positive definite bilinear form

$$\underline{\alpha}^T K \underline{\alpha} = a(v,v) \ge 0.$$

We only get 0 when $v = 0$ and as the basis functions are linearly independent this only occurs when $\underline{\alpha} = \underline{0}$. Thus the matrix K is positive definite.

3 MARKS

When $n = 1$ we have

$$a(\psi_1, \psi_1) c_1 = (f, \psi_1).$$

In the case $n = 2$ we have instead that c_1 and c_2 satisfy

$$\begin{pmatrix} a(\psi_1,\psi_1) & a(\psi_1,\psi_2) \\ a(\psi_2,\psi_1) & a(\psi_2,\psi_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (f,\psi_1) \\ (f,\psi_2) \end{pmatrix}.$$

Now $\psi_1' = 1$ and $\psi_2' = 2x$. For the integrals

$$\begin{aligned}
a(\psi_1,\psi_1) &= \int_0^1 (1 + 2x^2) \, \mathrm{d}x = 1 + \frac{2}{3} = \frac{5}{3}, \\
a(\psi_2,\psi_1) &= \int_0^1 (2x + 2x^3) \, \mathrm{d}x = 1 + \frac{2}{4} = \frac{3}{2}, \\
a(\psi_2,\psi_2) &= \int_0^1 (4x^2 + 2x^4) \, \mathrm{d}x = \frac{4}{3} + \frac{2}{5} = \frac{26}{15}, \\
(x,\psi_1) &= \int_0^1 x^2 \, \mathrm{d}x = \frac{1}{3}, \\
(x,\psi_2) &= \int_0^1 x^3 \, \mathrm{d}x = \frac{1}{4}.
\end{aligned}$$

Thus when $n = 1$ we have $c_1 = 1/5$. When $n = 2$ we have

$$\begin{pmatrix} 5/3 & 3/2 \\ 3/2 & 26/15 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/4 \end{pmatrix}.$$

6 MARKS

(iii) Let $\phi_1(s) = 1 - s$ and $\phi_2(s) = s$ and let $x = x_{i-1} + hs$ so that

$$\tilde{\phi}_1(x) = \phi_1(s) \quad \text{and} \quad \tilde{\phi}_2(x) = \phi_2(s).$$

We have $dx/ds = h$. Also let

$$a(u, v)_i = \int_{x_{i-1}}^{x_i} (u'v' + 2uv)\ \mathrm{d}x \quad \text{and} \quad (x, v)_i = \int_{x_{i-1}}^{x_i} xv\ \mathrm{d}x.$$

$$K_i = \begin{pmatrix} a(\tilde{\phi}_1, \tilde{\phi}_1)_i & a(\tilde{\phi}_1, \tilde{\phi}_2)_i \\ a(\tilde{\phi}_2, \tilde{\phi}_1)_i & a(\tilde{\phi}_2, \tilde{\phi}_2)_i \end{pmatrix} \quad \text{and} \quad \underline{b}_i = \begin{pmatrix} (f, \tilde{\phi}_1)_i \\ (f, \tilde{\phi}_2)_i \end{pmatrix}.$$

Now $\tilde{\phi}_1' = -1/h$ and $\tilde{\phi}_2' = 1/h$.

$$\int_{x_{i-1}}^{x_i} \begin{pmatrix} \tilde{\phi}_1'^2 & \tilde{\phi}_1'\tilde{\phi}_2' \\ \tilde{\phi}_2'\tilde{\phi}_1' & \tilde{\phi}_2'^2 \end{pmatrix}\ \mathrm{d}x = \frac{1}{h} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The result for K_i follows using the integrals given in the question.
On $[x_{i-1}, x_i]$ we have $x(s) = x_{i-1} + hs = x_{i-1}\phi_1(s) + x_i\phi_2(s)$, $0 \le s \le 1$ and thus

$$\begin{aligned} \underline{b}_i &= x_{i-1} h \int_0^1 \begin{pmatrix} 1 - s \\ s \end{pmatrix}\ \mathrm{d}s + h^2 \int_0^1 \begin{pmatrix} s(1 - s) \\ s^2 \end{pmatrix}\ \mathrm{d}s \\ &= x_{i-1} \frac{h}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{h^2}{6} \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \end{aligned}$$

4 MARKS

(iv)

$$K\underline{U} = \underline{b}$$

where using K_1 and K_2 with $h = 1/2$ we have

$$K = 2 \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}.$$

$$\underline{b}_1 = \frac{1}{24} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \underline{b}_2 = \frac{1}{8} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \underline{b}_1 = \frac{1}{24} \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Thus

$$\underline{b} = \frac{1}{24} \begin{pmatrix} 6 \\ 5 \end{pmatrix}.$$

3 MARKS

5. Question 1 of the June 2004 MA5352 paper had some parts in common with question 1 of the MA3951 paper. These are some of the parts which were different. MA3951 students had to answer 3 questions (from a choice of 4) in 3 hours. MA5352 students had to answer 4 questions (from a choice of 5) in 3 hours. The extra question on the MA5352 paper was from the finite element material taught by Simon Shaw.

   (a) Obtain the weak form expressions corresponding to each of the following boundary value problems. In your answer you should give in each case the space of functions involved and you should indicate if any of the boundary conditions are essential boundary conditions.

   (i)
$$-u'' + 2u = x^2, \quad 0 < x < 1, \quad u(0) = u'(1) + 2u(1) = 0.$$
[3 MARKS]

   (ii)
$$u'''' - u'' = 1, \quad 0 < x < 1, \quad u(0) = u'(0) = u'(1) = u'''(1) = 0.$$
[3 MARKS]

   (b) In the following the ordinary differential equation
$$-u'' + 2u = x, \quad 0 < x < 1, \quad u(0) = 0, \quad u'(1) = 0,$$
is to be approximately solved using Galerkin's method.

   (i) Derive the weak form for this problem in the form $a(u, v) = (f, v)$ where $a(.,.)$ is a symmetric and positive definite bilinear form defined on an appropriate space of functions V.

[1 MARK]

   (ii) Let $\psi_i(x) = x^i$, $i = 1, 2, \cdots$ and let $V_n = \text{span}\{\psi_1, \cdots, \psi_n\}$. The coefficients $c_1, c_2, \cdots, c_n$ in the Galerkin approximation $U_n = \sum_1^n c_i \psi_i$ satisfy the linear system
$$K\underline{c} = \underline{b}$$
where $\underline{c} = (c_i)$, $K = (a(\psi_i, \psi_j))$ and $\underline{b} = ((f, \psi_i))$. Explain why the matrix K is positive definite.

[3 MARKS]

Determine the Galerkin approximation from V_1 and determine a linear system for c_1 and c_2 in the Galerkin approximation $c_1 x + c_2 x^2$ from the space V_2. You do not need to solve these equations.

[6 MARKS]

ANSWER

(a) (i) For the function u satisfying the ODE

$$-\int_0^1 u''v\ \mathrm{d}x = -\left[u'v\right]_0^1 + \int_0^1 u'v'\ \mathrm{d}x = \int_0^1 u'v'\ \mathrm{d}x - u'(1)v(1)$$
$$= \int_0^1 u'v'\ \mathrm{d}x + 2u(1)v(1)$$

provided $v(0) = 0$. Let $V = \{v \in H^1(0,1) :\ v(0) = 0\}$. For the weak problem we have the following. Find $u \in V$ such that

$$\int_0^1 (u'v' + 2uv)\ \mathrm{d}x + 2u(1)v(1) = \int_0^1 x^2 v\ \mathrm{d}x \quad \text{for all } v \in V.$$

The boundary condition $u(0) = 0$ is an essential boundary condition.

3 MARKS

(ii)

$$\int_0^1 (u'''' - u'')v\ \mathrm{d}x = \left[(u''' - u')v\right]_0^1 - \int_0^1 (u''' - u')v'\ \mathrm{d}x$$
$$= \int_0^1 (-u'''v' + u'v')\ \mathrm{d}x$$
$$= -\left[u''v'\right]_0^1 + \int_0^1 (u''v'' + u'v')\ \mathrm{d}x$$

provided $v(0) = v'(0) = v'(1) = 0$ and by using the boundary conditions $u'(1) = u'''(1) = 0$ that the exact solution u satisfies at $x = 1$. Let

$$V = \{v \in H^2(0,1) :\ v(0) = v'(0) = v'(1) = 0\}.$$

For the weak problem we have the following. Find $u \in V$ such that

$$\int_0^1 (u''v'' + u'v')\ \mathrm{d}x = \int_0^1 v\ \mathrm{d}x \quad \text{for all } v \in V.$$

The 3 boundary conditions in the specification of V are essential boundary conditions.

3 MARKS

(b) (i) We let $V = \{v \in H^1(0,1) :\ v(0) = 0\}$. The weak form involves finding $u \in V$ such that for all $v \in V$ we have

$$a(u,v) := \int_0^1 (u'v' + 2uv)\ \mathrm{d}x = \int_0^1 xv\ \mathrm{d}x =: (f,v).$$

1 MARK

(ii) For any function

$$v = \sum_{1}^{n} \alpha_i \psi_i \in V_n$$

the bilinearity gives

$$a(v, v) = \underline{\alpha}^T K \underline{\alpha}.$$

As $a(.,.)$ is a positive definite bilinear form

$$\underline{\alpha}^T K \underline{\alpha} = a(v, v) \geq 0.$$

We only get 0 when $v = 0$ and as the basis functions are linearly independent this only occurs when $\underline{\alpha} = \underline{0}$. Thus the matrix K is positive definite.

3 MARKS

When $n = 1$ we have

$$a(\psi_1, \psi_1)c_1 = (f, \psi_1).$$

In the case $n = 2$ we have instead that c_1 and c_2 satisfy

$$\begin{pmatrix} a(\psi_1, \psi_1) & a(\psi_1, \psi_2) \\ a(\psi_2, \psi_1) & a(\psi_2, \psi_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (f, \psi_1) \\ (f, \psi_2) \end{pmatrix}.$$

Now $\psi_1' = 1$ and $\psi_2' = 2x$. For the integrals

$$\begin{aligned} a(\psi_1, \psi_1) &= \int_0^1 (1 + 2x^2)\ \mathrm{d}x = 1 + \frac{2}{3} = \frac{5}{3}, \\ a(\psi_2, \psi_1) &= \int_0^1 (2x + 2x^3)\ \mathrm{d}x = 1 + \frac{2}{4} = \frac{3}{2}, \\ a(\psi_2, \psi_2) &= \int_0^1 (4x^2 + 2x^4)\ \mathrm{d}x = \frac{4}{3} + \frac{2}{5} = \frac{26}{15}, \\ (x, \psi_1) &= \int_0^1 x^2\ \mathrm{d}x = \frac{1}{3}, \\ (x, \psi_2) &= \int_0^1 x^3\ \mathrm{d}x = \frac{1}{4}. \end{aligned}$$

Thus when $n = 1$ we have $c_1 = 1/5$. When $n = 2$ we have

$$\begin{pmatrix} 5/3 & 3/2 \\ 3/2 & 26/15 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/4 \end{pmatrix}.$$

6 MARKS

6. The following was question 1 of the June 2003 MA3056S paper.

Obtain the weak form expressions corresponding to each of the following ODEs. In your answer you should give in each case the space of functions involved and you should indicate if any of the boundary conditions are natural boundary conditions.

(a) (i)

$$-u''(x) = x^2, \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

[2 MARKS]

(ii)

$$-u''(x) + u(x) = 1, \quad 0 < x < 1, \quad u(0) = u'(1) = 0.$$

[2 MARKS]

(iii)

$$-u''(x) + 6u(x) = x, \quad 0 < x < 1, \quad u'(0) = 0, \quad u'(1) + u(1) = 0.$$

[2 MARKS]

(iv)

$$u''''(x) = \cos(x), \quad 0 < x < 1, \quad u(0) = u'(0) = u(1) = u'(1) = 0.$$

[3 MARKS]

(b) Suppose that a weak problem is of the form:

$$\text{find } u \in V \quad \text{such that} \quad a(u, v) = F(v) \quad \text{for all } v \in V$$

where V is an appropriate space of functions, $a(.,.)$ is a symmetric and positive definite bilinear form on $V \times V$ and $F(.)$ is a linear functional on V. Let $\phi_1, \cdots, \phi_n$ denote linearly independent functions in V. Describe the Galerkin method for constructing an approximation $U_n \in \text{span}\{\phi_1, \cdots, \phi_n\} \subset V$ and explain why the linear system which is obtained involves a matrix which is symmetric and positive definite.

[5 MARKS]

(c) Let $\phi_i(x) = x^i$, $i = 1, 2, \cdots$, let $V_n = \text{span}\{\phi_1, \cdots, \phi_n\}$ and consider the problem

$$-u''(x) + 6u(x) = 1, \quad 0 < x < 1, \quad u(0) = u'(1) = 0.$$

Obtain the Galerkin approximation $U_1 \in V_1$.

Also obtain the linear system $K\underline{c} = \underline{b}$ for the coefficients $\underline{c} = (c_1, c_2)^T$ of the approximation $U_2(x) = c_1 x + c_2 x^2$. You do not need to solve this system.

[6 MARKS]

ANSWER

(a) (i)

$$-\int_0^1 u''v \, \mathrm{d}x = -[u'v]_0^1 + \int_0^1 u'v' \, \mathrm{d}x = \int_0^1 u'v' \, \mathrm{d}x$$

if $v(0) = v(1) = 0$. Let $V = \{v \in C^2[0,1] : \ v(0) = v(1) = 0\}$. The exact solution $u \in V$ satisfies

$$\int_0^1 u'v' \, \mathrm{d}x = \int_0^1 x^2 v \, \mathrm{d}x \quad \text{for all } v \in V.$$

2 MARKS

(ii)

$$-\int_0^1 u''v \, \mathrm{d}x = -[u'v]_0^1 + \int_0^1 u'v' \, \mathrm{d}x = \int_0^1 u'v' \, \mathrm{d}x$$

if $v(0) = 0$ and using $u'(1) = 0$. Let $V = \{v \in C^2[0,1] : \ v(0) = 0\}$. The exact solution $u \in V$ satisfies

$$\int_0^1 u'v' + uv \, \mathrm{d}x = \int_0^1 v \, \mathrm{d}x \quad \text{for all } v \in V.$$

The boundary condition $u'(1) = 0$ is a natural boundary condition for this weak form.

2 MARKS

(iii)

$$\begin{aligned} -\int_0^1 u''v \, \mathrm{d}x &= -[u'v]_0^1 + \int_0^1 u'v' \, \mathrm{d}x \\ &= \int_0^1 u'v' \, \mathrm{d}x - u'(1)v(1) \\ &= \int_0^1 u'v' \, \mathrm{d}x + u(1)v(1) \end{aligned}$$

using $u'(0) = 0$ and $-u'(1) = u(1)$. Let $V = C^2[0,1]$. The exact solution $u \in V$ satisfies

$$\int_0^1 u'v' + 6uv \, \mathrm{d}x + u(1)v(1) = \int_0^1 xv \, \mathrm{d}x \quad \text{for all } v \in V.$$

The boundary conditions $u'(0) = 0$ and $u'(1) + u(1) = 0$ are natural boundary conditions for this weak form.

2 MARKS

(iv)

$$\begin{aligned} \int_0^1 u''''v \, \mathrm{d}x &= [u'''v]_0^1 - \int_0^1 u'''v' \, \mathrm{d}x = -\int_0^1 u'''v' \, \mathrm{d}x \\ &= -[u''v']_0^1 + \int_0^1 u''v'' \, \mathrm{d}x = \int_0^1 u''v'' \, \mathrm{d}x \end{aligned}$$

if $v(0) = v'(0) = v(1) = v'(1) = 0$. Let $V = \{v \in C^4[0,1] : v(0) = v'(0) = v(1) = v'(1) = 0\}$. The exact solution $u \in V$ satisfies

$$\int_0^1 u''v''\ \mathrm{d}x = \int_0^1 v\cos(x)\ \mathrm{d}x \quad \text{for all } v \in V.$$

3 MARKS

(b) With the Galerkin method we obtain $U_n \in V_n = \mathrm{span}\{\phi_1, \cdots, \phi_n\}$ such that

$$a(U_n, v) = F(v) \quad \text{for all } v \in V_n.$$

As $U_n \in V_n$ it is of the form $U_n = \sum_{j=1}^n c_j\phi_j$ and by taking $v = \phi_i$, $i = 1, \cdots, n$ we get the n equations

$$a(U_n, \phi_i) = a(\sum_{j=1}^n c_j\phi_j, \phi_i) = \sum_{j=1}^n a(\phi_j, \phi_i)c_j = F(\phi_i), \quad i = 1, \cdots, n.$$

In matrix-vector form this is

$$K\underline{c} = \underline{b} \quad \text{where } K = (a(\phi_j, \phi_i)), \quad \underline{c} = (c_i) \quad \text{and} \quad \underline{b} = (F(\phi_i)).$$

The matrix K is symmetric because the symmetry of $a(.,.)$ gives $a(\phi_j, \phi_i) = a(\phi_i, \phi_j)$. The matrix is positive definite because if $v = \sum_{j=1}^n d_j\phi_j$ then

$$\underline{d}^T K\underline{d} = a(v, v) \geq 0$$

and we get 0 only if $v = 0$ by the positive definite property of $a(.,.)$. Since the functions $\phi_1, \cdots, \phi_n$ are linearly independent this is only possible if $\underline{d} = \underline{0}$.

5 MARKS

(c) From (a)(ii) the weak form expression is

$$a(u, v) = \int_0^1 u'v' + 6uv\ \mathrm{d}x = \int_0^1 xv\ \mathrm{d}x = F(v).$$

For the linear system $K\underline{c} = \underline{b}$ we have

$$\begin{aligned}
k_{11} &= a(\phi_1, \phi_1) = \int_0^1 (1 + 6x^2)\ \mathrm{d}x = 1 + 2 = 3, \\
k_{22} &= a(\phi_2, \phi_2) = \int_0^1 (4x^2 + 6x^4)\ \mathrm{d}x = \frac{4}{3} + \frac{6}{5} = \frac{38}{15}, \\
k_{21} &= a(\phi_2, \phi_1) = \int_0^1 (2x + 6x^3)\ \mathrm{d}x = 1 + \frac{6}{4} = \frac{5}{2}, \\
b_1 &= F(\phi_1) = \int_0^1 x\ \mathrm{d}x = \frac{1}{2}, \\
b_2 &= F(\phi_2) = \int_0^1 x^2\ \mathrm{d}x = \frac{1}{3}.
\end{aligned}$$

Now $U_1 = c_1\phi_1 = c_1 x$ where $c_1 = b_1/k_{11} = 1/6$.
The approximation $U_2 = c_1\phi_1 + c_2\phi_2 = c_1 x + c_2 x^2$ where

$$\begin{pmatrix} 3 & \frac{5}{2} \\ \frac{5}{2} & \frac{38}{15} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}.$$

6 MARKS

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7. The following were parts of question 1 of the June 2003 `MA5156S` paper.

   (a) Obtain the weak form expressions corresponding to each of the following ODEs. In your answer you should give in each case the space of functions involved and you should indicate if any of the boundary conditions are natural boundary conditions.

   (i)

$$-u''(x) + 12u(x) = 1, \quad 0 < x < 1, \quad u(0) = u'(1) = 0.$$

[2 MARKS]

   (ii)

$$u''''(x) - u''(x) = \cos(x), \quad 0 < x < 1, \quad u(0) = u''(0) = u(1) = u''(1) = 0.$$

[3 MARKS]

ANSWER

(i)

$$-\int_0^1 u''v \, \mathrm{d}x = -[u'v]_0^1 + \int_0^1 u'v' \, \mathrm{d}x = \int_0^1 u'v' \, \mathrm{d}x$$

if $v(0) = 0$ and using $u'(1) = 0$. Let $V = \{v \in C^2[0,1] : \ v(0) = 0\}$. The exact solution $u \in V$ satisfies

$$\int_0^1 u'v' + 12uv \, \mathrm{d}x = \int_0^1 v \, \mathrm{d}x \quad \text{for all } v \in V.$$

The boundary condition $u'(1) = 0$ is a natural boundary condition for this weak form.

2 MARKS

(ii)

$$\begin{aligned} \int_0^1 u''''v \, \mathrm{d}x &= [u'''v]_0^1 - \int_0^1 u'''v' \, \mathrm{d}x = -\int_0^1 u'''v' \, \mathrm{d}x \\ &= -[u''v']_0^1 + \int_0^1 u''v'' \, \mathrm{d}x = \int_0^1 u''v'' \, \mathrm{d}x \end{aligned}$$

if $v(0) = v(1) = 0$ and using the boundary conditions $u''(0) = u''(1) = 0$. Let $V = \{v \in C^4[0,1] : \ v(0) = v(1) = 0\}$. The exact solution $u \in V$ satisfies

$$\int_0^1 u''v'' + u'v' \, \mathrm{d}x = \int_0^1 v\cos(x) \, \mathrm{d}x \quad \text{for all } v \in V.$$

The boundary conditions $u''(0) = u''(1) = 0$ are natural boundary conditions for this weak problem.

3 MARKS

(b) In the following the problem,

$$-u'' + 12u = 1, \quad 0 < x < 1, \quad u(0) = u'(1) = 0,$$

given in part (7a)(i), is to be approximately solved using the Galerkin finite element method using a piecewise linear approximating function defined on a mesh $0 = x_0 < x_1 < \cdots < x_M = 1$. Do the following.

(i) When the finite element method is implemented in an element-by-element way an actual element $[x_{i-1}, x_i]$ is mapped to a standard element such as $[0, 1]$. State the linear basis functions $\phi_1(s)$, $\phi_2(s)$, $0 \le s \le 1$ defined on the standard element $[0, 1]$ and express both the mapping and the form of the approximation in terms of these functions.

[2 MARKS]

(ii) For the ith element $[x_{i-1}, x_i]$ describe what is meant by the element matrix K_i and the element vector $\underline{b}_i$ for this problem and determine K_i and $\underline{b}_i$.

[4 MARKS]

(iii) Let $\hat{\phi}_i$, $i = 0, 1, \cdots, M$ denote the piecewise linear hat functions associated with the points $0 = x_0 < x_1 < \cdots < x_M = 1$ with the property $\hat{\phi}_i(x_i) = 1$, $\hat{\phi}_i(x_j) = 0$, $j \ne i$. In the case $M = 2$ and uniformly spaced points $x_i = i/2$, $i = 0, 1, 2$ determine the 3×3 global matrix $\hat{K} = (a(\hat{\phi}_i, \hat{\phi}_j))$, $0 \le i, j \le 2$ and the 3×1 global vector $\hat{\underline{b}} = ((f, \hat{\phi}_i))$, $0 \le i \le 2$ where $f(x) = 1$.

[3 MARKS]

(iv) Determine the finite element approximation at $x = 1/2$ and at $x = 1$ using this mesh.

[2 MARKS]

ANSWER

(i) The basis functions are $\phi_1(s) = 1 - s$, $\phi_2(s) = s$. Let $x : [0, 1] \to [x_{i-1}, x_i]$ denote the map, let U denote the approximation and let $U_j = U(x_j)$.

$$\begin{aligned} x(s) &= x_{i-1}\phi_1(s) + x_i\phi_2(s) = x_{i-1} + h_i s, \quad h_i = x_i - x_{i-1}, \\ U(x(s)) &= U_{i-1}\phi_1(s) + U_i\phi_2(s) = U_{i-1} + (U_i - U_{i-1})s. \end{aligned}$$

2 MARKS

(ii) Let

$$a(u, v)_i = \int_{x_{i-1}}^{x_i} u'v' + 12uv \,\mathrm{d}x \quad \text{and} \quad (f, v)_i = \int_{x_{i-1}}^{x_i} v \,\mathrm{d}x.$$

Also let $\tilde{\phi}_i$ be such that $\tilde{\phi}_i(x(s)) = \phi_i(s)$. The element matrix and element vector are

$$K_i = \begin{pmatrix} a(\tilde{\phi}_1, \tilde{\phi}_1)_i & a(\tilde{\phi}_1, \tilde{\phi}_2)_i \\ a(\tilde{\phi}_2, \tilde{\phi}_1)_i & a(\tilde{\phi}_2, \tilde{\phi}_2)_i \end{pmatrix} \quad \text{and} \quad \underline{b}_i = \begin{pmatrix} (f, \tilde{\phi}_1)_i \\ (f, \tilde{\phi}_2)_i \end{pmatrix}.$$

$$\frac{d\tilde{\phi}_i}{dx} = \frac{ds}{dx}\frac{d\phi_i}{ds} = \frac{1}{h_i}\frac{d\phi_i}{ds}.$$

As $\phi_1' = -1$ and $\phi_2' = 1$ we have

$$K_i = \frac{1}{h_i}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + 12h_i \int_0^1 \begin{pmatrix} (1-s)^2 & s(1-s) \\ s(1-s) & s^2 \end{pmatrix} \, \mathrm{d}s$$

$$\int_0^1 s^2 \, \mathrm{d}s = \int_0^1 (1-s)^2 \, \mathrm{d}s = \frac{1}{3}, \quad \int_0^1 s(1-s) \, \mathrm{d}s = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

The element matrix is

$$K_i = \frac{1}{h_i}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{12h_i}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The element vector is

$$\underline{b}_i = h_i \int_0^1 \begin{pmatrix} 1-s \\ s \end{pmatrix} \, \mathrm{d}s = \frac{h_i}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

4 MARKS

(iii) The element matrices and the element vectors are the same for all the elements. With $M = 2$, $h_i = 1/2$, $1/h_i = 2$ and

$$K_1 = K_2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix},$$

$$\underline{b}_1 = \underline{b}_2 = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\hat{K} = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -1 & 4 \end{pmatrix}.$$

$$\underline{\hat{b}} = \frac{1}{4}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

3 MARKS

(iv) At the node $x_0 = 0$ the approximation satisfies the essential boundary conditions and thus $U(x_0) = 0$.

$$U(x) = U(1/2)\hat{\phi}_1(x) + U(1)\hat{\phi}_2(x)$$

where

$$\begin{pmatrix} 8 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} U(1/2) \\ U(1) \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We get

$$\begin{pmatrix} U(1/2) \\ U(1) \end{pmatrix} = \frac{1}{31}\begin{pmatrix} 4 & 1 \\ 1 & 8 \end{pmatrix} \frac{1}{4}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{124}\begin{pmatrix} 9 \\ 10 \end{pmatrix}.$$

2 MARKS

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8. The following was question 2 of the June 2003 `MA3056S` paper.

In the following the ordinary differential equation

$$-u'' + 12u = 1, \quad 0 < x < 1, \quad u(0) = u(1) = 0$$

is to be approximately solved using the Galerkin finite element method using a continuous piecewise linear approximating function defined on a mesh $0 = x_0 < x_1 < \cdots < x_M = 1$.

(a) Derive the weak form for this problem in the form $a(u, v) = (f, v)$.

[2 MARKS]

(b) When the finite element method is implemented in an element-by-element way an actual element $[x_{i-1}, x_i]$ is mapped to a standard element such as $[0, 1]$. State the linear basis functions $\phi_1(s)$, $\phi_2(s)$, $0 \le s \le 1$ defined on the standard element $[0, 1]$ and express both the mapping and the form of the approximation in terms of these functions.

[2 MARKS]

(c) Let $\hat{\phi}_i(x)$, $i = 0, 1, \cdots, M$ denote the piecewise linear hat functions with the property $\hat{\phi}_i(x_i) = 1$ and $\hat{\phi}_i(x_j) = 0$ for $j \ne i$. On the element $[x_{i-1}, x_i]$ indicate which of these functions are non-zero and indicate how the non-zero functions are related to ϕ_1 and ϕ_2 given in (b).

[2 MARKS]

(d) For the ith element $[x_{i-1}, x_i]$ describe what is meant by the element matrix K_i and the element vector $\underline{b}_i$ for this problem and determine K_i and $\underline{b}_i$.

[5 MARKS]

(e) In the case $M = 3$ and uniformly spaced points $x_i = i/3$, $i = 0, 1, 2, 3$ do the following.

(i) Give the 3 element matrices K_1, K_2, and K_3 and the 3 element vectors $\underline{b}_1$, $\underline{b}_2$ and $\underline{b}_3$.

[2 MARKS]

(ii) Determine the 4×4 global matrix $\hat{K} = (a(\hat{\phi}_i, \hat{\phi}_j))$, $0 \le i, j \le 3$ and the 4×1 global vector $\hat{\underline{b}} = ((f, \hat{\phi}_i))$, $0 \le i \le 3$ where $f(x) = 1$.

[4 MARKS]

(iii) Determine the finite element approximation at $x = 1/3$ and at $x = 2/3$ obtained using this mesh of 3 elements.

[3 MARKS]

ANSWER

(a) If $v \in H_0^1(0,1)$ then
$$-\int_0^1 u''v \, \mathrm{d}x = \int_0^1 u'v' \, \mathrm{d}x.$$
The weak form involves finding $v \in V = H_0^1(0,1)$ such that
$$a(u,v) = \int_0^1 u'v' + 12uv \, \mathrm{d}x = \int_0^1 v \, \mathrm{d}x = (f,v) \quad \text{for all } v \in V$$
with $f(x) = 1$.

2 MARKS

(b) The basis functions are $\phi_1(s) = 1-s$, $\phi_2(s) = s$. Let $x : [0,1] \to [x_{i-1}, x_i]$ denote the map, let U denote the approximation and let $U_j = U(x_j)$.
$$\begin{aligned} x(s) &= x_{i-1}\phi_1(s) + x_i\phi_2(s) = x_{i-1} + h_i s, \quad h_i = x_i - x_{i-1}, \\ U(x(s)) &= U_{i-1}\phi_1(s) + U_i\phi_2(s) = U_{i-1} + (U_i - U_{i-1})s. \end{aligned}$$

2 MARKS

(c) On $[x_{i-1}, x_i]$ the functions $\hat{\phi}_{i-1}(x)$ and $\hat{\phi}_i(x)$ are the only non-zero basis functions.
$$\begin{aligned} \hat{\phi}_{i-1}(x(s)) &= \phi_1(s), \\ \hat{\phi}_i(x(s)) &= \phi_2(s). \end{aligned}$$

2 MARKS

(d) Let
$$a(u,v)_i = \int_{x_{i-1}}^{x_i} u'v' + 12uv \, \mathrm{d}x \quad \text{and} \quad (f,v)_i = \int_{x_{i-1}}^{x_i} v \, \mathrm{d}x.$$
Also let $\tilde{\phi}_i$ be such that $\tilde{\phi}_i(x(s)) = \phi_i(s)$. The element matrix and element vector are
$$K_i = \begin{pmatrix} a(\tilde{\phi}_1, \tilde{\phi}_1)_i & a(\tilde{\phi}_1, \tilde{\phi}_2)_i \\ a(\tilde{\phi}_2, \tilde{\phi}_1)_i & a(\tilde{\phi}_2, \tilde{\phi}_2)_i \end{pmatrix} \quad \text{and} \quad \underline{b}_i = \begin{pmatrix} (f, \tilde{\phi}_1)_i \\ (f, \tilde{\phi}_2)_i \end{pmatrix}.$$
$$\frac{d\tilde{\phi}_i}{dx} = \frac{ds}{dx}\frac{d\phi_i}{ds} = \frac{1}{h_i}\frac{d\phi_i}{ds}.$$
As $\phi_1' = -1$ and $\phi_2' = 1$ we have
$$K_i = \frac{1}{h_i}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + 12h_i \int_0^1 \begin{pmatrix} (1-s)^2 & s(1-s) \\ s(1-s) & s^2 \end{pmatrix} \mathrm{d}s,$$
$$\int_0^1 s^2 \, \mathrm{d}s = \int_0^1 (1-s)^2 \, \mathrm{d}s = \frac{1}{3}, \quad \int_0^1 s(1-s) \, \mathrm{d}s = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$
The element matrix is
$$K_i = \frac{1}{h_i}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{12h_i}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The element vector is

$$\underline{b}_i = h_i \int_0^1 \begin{pmatrix} 1-s \\ s \end{pmatrix} \, \mathrm{d}s = \frac{h_i}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

5 MARKS

(e) (i) The element matrices and the element vectors are the same for all the elements. With $M = 3$, $h_i = 1/3$, $1/h_i = 3$ and

$$K_1 = K_2 = K_3 = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{13}{3} & -\frac{-7}{3} \\ \frac{-7}{3} & \frac{13}{3} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 13 & -7 \\ -7 & 13 \end{pmatrix},$$

$$\underline{b}_1 = \underline{b}_2 = \underline{b}_3 = \frac{1}{6} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

2 MARKS

(ii)

$$\hat{K} = \frac{1}{3} \begin{pmatrix} 13 & -7 & 0 & \\ -7 & 26 & -7 & 0 \\ 0 & -7 & 26 & -7 \\ 0 & 0 & -7 & 13 \end{pmatrix}.$$

$$\underline{\hat{b}} = \frac{1}{6} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

4 MARKS

(iii) At the nodes $x_0 = 0$ and $x_3 = 1$ the approximation satisfies the essential boundary conditions and thus $U(x_0) = U(x_3) = 0$.

$$U(x) = U(1/3)\hat{\phi}_1(x) + U(2/3)\hat{\phi}_2(x)$$

where

$$\frac{1}{3} \begin{pmatrix} 26 & -7 \\ -7 & 26 \end{pmatrix} \begin{pmatrix} U(1/3) \\ U(2/3) \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

We get

$$\begin{pmatrix} U(1/3) \\ U(2/3) \end{pmatrix} = \frac{1}{627} \begin{pmatrix} 26 & 7 \\ 7 & 26 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{33}{627} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

3 MARKS

9. Apart from a few minor changes and re-typing this was question 3 of the 1999 MA3056S paper.

The two point boundary value, with solution $u(x)$,

$$-u''(x) + 6u(x) = x, \quad 0 < x < 1, \quad u(0) = u(1) = 0,$$

is to be solved via its weak formulation using a Galerkin finite element based on piecewise linear functions. For test functions v where

$$v \in H_0^1(0,1) := \{v : \ v, v' \in L_2(0,1), \ v(0) = v(1) = 0\}, \tag{1}$$

derive the weak form of problem (1).

ANSWER

$$-\int_0^1 u''v \ \mathrm{d}x = -[u'v]_0^1 + \int_0^1 u'v' \ \mathrm{d}x = \int_0^1 u'v' \ \mathrm{d}x \quad \text{for all } v \in H_0^1(0,1).$$

The weak form involves finding $u \in H_0^1(0,1)$ such that

$$a(u,v) = \int_0^1 u'v' + 6uv \ \mathrm{d}x = \int_0^1 xv \ \mathrm{d}x =: F(v) \quad \text{for all } v \in H_0^1(0,1).$$

In the case of the partition $0 = x_0 < x_1 < x_2 < \cdots < x_{ne} = 1$, show that the weak form can be expressed in the form

$$\sum_{i=1}^{ne} a(u,v)_i = \sum_{i=1}^{ne} (x,v)_i \quad \text{for all } v \in H_0^1(0,1), \tag{2}$$

where

$$a(u,v)_i = \int_{x_{i-1}}^{x_i} u'v' + 6uv \, \mathrm{d}x \quad \text{and} \quad (x,v)_i = \int_{x_{i-1}}^{x_i} xv \, \mathrm{d}x.$$

ANSWER

Because of the properties of the integral we have for any function $g \in L_2(0,1)$ that

$$\int_0^1 g(x) \ \mathrm{d}x = \sum_{i=1}^{ne} \int_{x_{i-1}}^{x_i} g(x) \, \mathrm{d}x.$$

Thus

$$a(u,v) = \sum_{i=1}^{ne} a(u,v)_i \quad \text{and} \quad (x,v) = \sum_{i=1}^{ne} (x,v)_i.$$

The result then follows.

Explain what is meant by an isoparametric finite element method.

ANSWER

Let S denote a standard element and let $I_i = (x_{i-1}, x_i)$ denote an actual element and let $x : S \to I_i$ be a mapping which is one-to-one and onto. Also let U denote the finite element approximation and let $\phi_1, \cdots, \phi_m$ denote the basis functions defined on S. An isoparametric finite element method is a method in which on each element the mapping and the approximation are both of the form

$$\begin{aligned} x(s) &= \sum_{j=1}^{m} \tilde{x}_j \phi_j(x), \\ U(x(s)) &= \sum_{j=1}^{m} \tilde{U}_j \phi_j(x). \end{aligned}$$

A linear isoparametric finite element, based on the equally spaced points $x_i = i/3$, $i = 0, 1, 2, 3$ is used to approximate the solution of (2). Let $U(x)$ denote the approximation and let $U_i = U(x_i)$. With $s = s(x)$ denoting the linear mapping of $[x_{i-1}, x_i]$ onto $[0, 1]$ and with $\tilde{u}(s) = u(x(s))$ and $\tilde{v}(s) = v(x(s))$ for $x_{i-1} < x < x_i$ show that

$$\begin{aligned} a(u, v)_i &= \int_0^1 \left(\frac{d\tilde{u}}{ds}\frac{d\tilde{v}}{ds}\frac{ds}{dx} + 6\tilde{u}\tilde{v}\frac{dx}{ds} \right) \mathrm{d}s, \\ (x, v)_i &= \int_0^1 x(s)\tilde{v}(s)\frac{dx}{ds}\, \mathrm{d}s. \end{aligned}$$

Then show that the element matrix for the element $[x_{i-1}, x_i]$ is,

$$\frac{1}{3}\begin{pmatrix} 11 & -8 \\ -8 & 11 \end{pmatrix}.$$

Assemble the 3 element matrices to form a 4×4 global matrix. Also compute the 3 element constant vectors and assemble these to create a 4×1 global vector. Then, by taking account of the homogeneous boundary conditions obtain the 2-by-2 system satisfied by U_1 and U_2. You do not have to solve the system.

ANSWER

We have

$$\tilde{u}(s) = u(x(s)), \quad \frac{d\tilde{u}}{ds} = \frac{du}{dx}\frac{dx}{ds}, \quad \frac{du}{dx} = \frac{d\tilde{u}}{ds}\frac{ds}{dx}.$$

Thus

$$\begin{aligned} a(u, v)_i &= \int_{x_{i-1}}^{x_i} (u'v' + 6uv)\, \mathrm{d}x = \int_0^1 \left(\left(\frac{ds}{dx}\right)^2 \frac{d\tilde{u}}{ds}\frac{d\tilde{v}}{ds} + 6\tilde{u}\tilde{v} \right) \frac{dx}{ds}\, \mathrm{d}s \\ &= \int_0^1 \left(\frac{ds}{dx}\frac{d\tilde{u}}{ds}\frac{d\tilde{v}}{ds} + 6\tilde{u}\tilde{v}\frac{dx}{ds} \right) \mathrm{d}s \\ (x, v)_i &= \int_0^1 x(s)\tilde{v}(s)\frac{dx}{ds}\, \mathrm{d}s. \end{aligned}$$

The linear basis functions on $[0,1]$ are $\tilde\phi_1(s) = 1-s$ and $\tilde\phi_2(s) = s$. For the derivatives $\tilde\phi_1' = -1$ and $\tilde\phi_2' = 1$. The mapping is

$$x(s) = x_{i-1} + hs, \quad h = x_i - x_{i-1}, \quad \frac{dx}{ds} = h, \quad \frac{ds}{dx} = \frac{1}{h}.$$

The element matrix K_i and the element vector $\underline{b}_i$ are

$$K_i = (a(\phi_k, \phi_l)_i), \quad 1 \le k,l \le 2 \quad \text{and} \quad \underline{b}_i = ((x,\phi_k)_i), \quad 1 \le k \le 2$$

where $\phi_k(x(s)) = \tilde\phi_k(s)$.

$$\begin{aligned}
a(\phi_1,\phi_1) &= \int_0^1 \left(\frac{1}{h} + 6(1-s)^2 h\right) \mathrm{d}s = \frac{1}{h} + 6h\left[\frac{(s-1)^3}{3}\right]_0^1 = \frac{1}{h} + 2h, \\
a(\phi_2,\phi_2) &= \int_0^1 \left(\frac{1}{h} + 6s^2 h\right) \mathrm{d}s = \frac{1}{h} + 2h, \\
a(\phi_1,\phi_2) &= \int_0^1 \left(\frac{-1}{h} + 6s(1-s)h\right) \mathrm{d}s = \frac{-1}{h} + 6h\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{-1}{h} + h.
\end{aligned}$$

As $h = 1/3$ we have

$$K_i = \frac{1}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + h\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 11 & -8 \\ -8 & 11 \end{pmatrix}.$$

If $\hat\phi_k$, $k = 0,1,2,3$ denote the piecewise linear hat functions defined for all $x \in [0,1]$ then the 4×4 global matrix is $\hat K = (a(\hat\phi_k, \hat\phi_l))$, $0 \le k,l \le 3$. The nodes $x_1 = 1/3$ and $x_2 = 2/3$ are both on two elements and thus the $2,2$ and $3,3$ entries of the global matrix get contributions from two elements. Assembling gives

$$\hat K = \frac{1}{3}\begin{pmatrix} 11 & -8 & & \\ -8 & 22 & -8 & \\ & -8 & 22 & -8 \\ & & -8 & 11 \end{pmatrix}.$$

For the element vector we have

$$\underline{b}_i = h\int_0^1 x(s)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mathrm{d}s = h\int_0^1 (x_{i-1} + hs)\begin{pmatrix} 1-s \\ s \end{pmatrix} \mathrm{d}s = hx_{i-1}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + h^2\begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \end{pmatrix}.$$

Thus

$$\begin{aligned}
\underline{b}_1 &= \frac{1}{9}\begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \end{pmatrix}, \\
\underline{b}_2 &= \frac{1}{9}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \frac{1}{9}\begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \end{pmatrix} = \frac{1}{9}\begin{pmatrix} \frac{2}{3} \\ \frac{5}{6} \end{pmatrix}, \\
\underline{b}_3 &= \frac{2}{9}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \frac{1}{9}\begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \end{pmatrix} = \frac{1}{9}\begin{pmatrix} \frac{7}{6} \\ \frac{4}{3} \end{pmatrix}.
\end{aligned}$$

Assembling these element vectors gives the global vector

$$\underline{\hat{b}} = \frac{1}{9}\begin{pmatrix} \frac{1}{6} \\ 1 \\ 2 \\ \frac{4}{3} \end{pmatrix}.$$

The approximation $U(x) = U_1\hat{\phi}_1(x) + U_2\hat{\phi}_2(x)$ where U_1 and U_2 satisfy

$$\begin{pmatrix} a(\hat{\phi}_1,\hat{\phi}_1) & a(\hat{\phi}_1,\hat{\phi}_2) \\ a(\hat{\phi}_2,\hat{\phi}_1) & a(\hat{\phi}_2,\hat{\phi}_2) \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 22 & -8 \\ -8 & 22 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} (x,\hat{\phi}_1) \\ (x,\hat{\phi}_2) \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

---

10. Let V be an inner product space, let $a(.,.)$ be a symmetric and positive definite bilinear form on $V \times V$ and let $f \in V$. Show that a function $u \in V$ satisfying

$$a(u,v) = (f,v) \quad \text{for all } v \in V$$

uniquely minimises the functional

$$I(v) = \frac{1}{2}a(v,v) - (f,v).$$

ANSWER

This is a standard book work question.

As we are attempting to show that u minimises $I(.)$ over V we compare $I(u+v)$ with $I(u)$ for any $v \in V$, $v \neq 0$. We have

$$\begin{aligned} I(u+v) - I(u) &= \frac{1}{2}(a(u+v,u+v) - a(u,u)) - ((f,u+v) - (f,u)), \\ &= \frac{1}{2}(2a(u,v) + a(v,v))) - (f,v), \\ &= \frac{1}{2}a(v,v) > 0 \end{aligned}$$

using the properties of $a(.,.)$ to expand the $a(u+v,u+v)$ term, using $a(u,v)-(f,v) = 0$ and using the positive definite property. This tells us that u uniquely minimises the functional.

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11. (a) After first reformulating the two-point boundary value problem

$$-u'' + 6u = 1, \quad u(0) = u(1) = 0$$

into weak form, calculate the Galerkin method approximation $U(x) = c_1\phi_1(x)$ where $\phi_1(x) = x(1-x)$.

ANSWER

We have essential boundary conditions at $x = 0$ and at $x = 1$. Let

$$V = \{v \in C^2[0,1] : \quad v(0) = v(1) = 0\}.$$

The exact solution $u \in V$ satisfies

$$a(u, v) = \int_0^1 u'v' + 6uv \, \mathrm{d}x = \int_0^1 v \, \mathrm{d}x = (1, v) \quad \text{for all } v \in V.$$

The Galerkin approximation $U = c_1\phi_1$ satisfies

$$a(U, \phi_1) = a(\phi_1, \phi_1)c_1 = (1, \phi_1) \quad \text{giving} \quad c_1 = \frac{(1, \phi_1)}{a(\phi_1, \phi_1)}.$$

$\phi_1 = x - x^2$, $\phi_1' = 1 - 2x$.

$$(1, \phi_1) = \int_0^1 x - x^2 \, \mathrm{d}x = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

$$a(\phi_1, \phi_1) = \int_0^1 {\phi_1'}^2 + \phi_1^2 \, \mathrm{d}x = \int_0^1 (1 - 2x)^2 + 6(x - x^2)^2 \, \mathrm{d}x.$$

The integrand is

$$(1 - 4x + 4x^2) + 6(x^2 - 2x^3 + x^4)$$

Thus

$$a(\phi_1, \phi_1) = (1 - \frac{4}{2} + \frac{4}{3}) + 6\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}.$$

Thus

$$c_1 = \frac{1/6}{8/15} = \frac{15}{48} = \frac{5}{16}.$$

(b) After first reformulating the two-point boundary value problem

$$-u'' + 6u = x, \quad u'(0) = 0, \quad u'(1) + u(1) = 0$$

into weak form, calculate the Galerkin method approximation $U_1 \in \mathrm{span}\{\phi_1\}$ and $U_2 \in \mathrm{span}\{\phi_1, \phi_2\}$ where $\phi_1(x) = 1$ and $\phi_2(x) = x$.

ANSWER

Let $Lu = -u'' + 6u$ and let $f(x) = x$. Then

$$(Lu, v) = -[u'v]_0^1 + \int_0^1 u'v' + 6uv \, \mathrm{d}x.$$

Using the boundary conditions we have

$$-[u'v]_0^1 = -u'(1)v(1) + u'(0)v(0) = u(1)v(1).$$

Let $V = C^2[0,1]$. The exact solution $u \in V$ satisfies

$$a(u,v) = \int_0^1 u'v' + 6uv \, \mathrm{d}x + u(1)v(1) = \int_0^1 fv \, \mathrm{d}x \quad \text{for all } v \in V.$$

Both the boundary conditions are natural boundary conditions for this weak form.

$$U_1 = c_1\phi_1, \quad \text{where} \quad a(\phi_1,\phi_1)c_1 = (f,\phi_1).$$

With $\phi_1 = 1$, $\phi_1' = 0$.

$$a(\phi_1,\phi_1) = \int_0^1 6\phi_1^2 \, \mathrm{d}x + \phi_1(1)^2 = 6 + 1 = 7, \quad (f,\phi_1) = \int_0^1 x \, \mathrm{d}x = \frac{1}{2}.$$

Thus

$$c_1 = \frac{1}{14} \quad \text{giving} \quad U_1 = \frac{1}{14}.$$

$$U_2 = c_1\phi_1 + c_2\phi_2 \quad \text{where now} \quad \begin{pmatrix} a(\phi_1,\phi_1) & a(\phi_1,\phi_2) \\ a(\phi_2,\phi_1) & a(\phi_2,\phi_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (f,\phi_1) \\ (f,\phi_2) \end{pmatrix}.$$

Now $\phi_2 = x$, $\phi_2' = 1$. Thus

$$\begin{aligned} (f,\phi_2) &= \int_0^1 x^2 \, \mathrm{d}x = \frac{1}{3}, \\ a(\phi_1,\phi_2) &= \int_0^1 6\phi_1\phi_2 \, \mathrm{d}x + \phi_1(1)\phi_2(1) = \int_0^1 6x \, \mathrm{d}x + 1 = 3 + 1 = 4, \\ a(\phi_2,\phi_2) &= \int_0^1 1 + 6x^2 \, \mathrm{d}x + 1 = 1 + 2 + 1 = 4. \end{aligned}$$

The linear system is

$$\begin{pmatrix} 7 & 4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}.$$

Solving

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 4 & -4 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} = \frac{1}{72} \begin{pmatrix} 4 & -4 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{1}{72} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{1}{36} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Hence $U_2 = (2+x)/36$.

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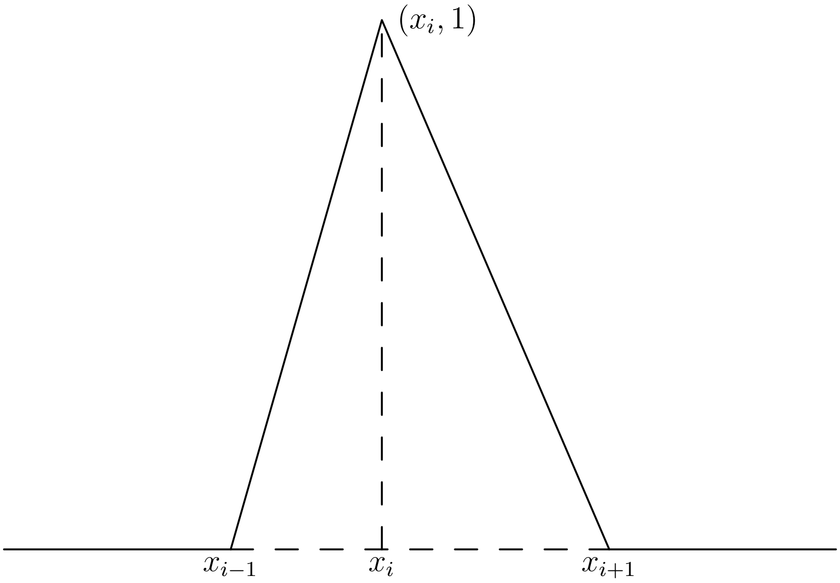
12. Let $0 = x_0 < x_1 < \cdots < x_{ne} = 1$ and let $\hat{\phi}_i(x)$ denote the piecewise linear 'hat' function which takes the value 1 at $x = x_i$. Describe mathematically $\hat{\phi}_i(x)$ and $\hat{\phi}_i'(x)$, $0 \le x \le 1$.

ANSWER

The graph of a hat function is



Let $h_i = x_i - x_{i-1}$, $h_{i+1} = x_{i+1} - x_i$.

$$\hat\phi_i(x) = \begin{cases} \dfrac{(x - x_{i-1})}{h_i}, & x_{i-1} \le x \le x_i, \\ \dfrac{(x_i - x)}{h_{i+1}}, & x_i \le x \le x_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \hat\phi_i'(x) = \begin{cases} \dfrac{1}{h_i}, & x_{i-1} < x < x_i, \\ \dfrac{-1}{h_{i+1}}, & x_i < x < x_{i+1}, \\ 0, & x < x_{i-1} \text{ or } x > x_i. \end{cases}$$

$\hat\phi_i'(x)$ is not defined at $x = x_{i-1}$, $x = x_i$ and $x = x_{i+1}$.

If

$$a(u, v) = \int_0^1 u'v' + quv \, \mathrm{d}x,$$

where $q(x) \ge 0$, then explain why the matrix $K = (a(\hat\phi_i, \hat\phi_j))$, $1 \le i, j \le ne-1$ is banded and state the band width.

ANSWER

$\hat\phi_i$ is only non-zero in (x_{i-1}, x_{i+1}) and if i and j are such that $(x_{i-1}, x_{i+1}) \cap (x_{j-1}, x_{j+1}) =$ empty set then $a(\hat\phi_i, \hat\phi_j) = 0$. If we keep i fixed then we get non-zero entries only if $j = i-1$, $j = i$ and $j = i+1$. Hence K is a tri-diagonal matrix, i.e. a matrix with band width=3.

Let

$$U = \sum_{j=1}^{ne-1} c_j \hat{\phi}_j$$

and consider the case of equally spaced points with spacing $h = 1/ne$.

(a) In the case $q = 0$ show that

$$a(U, \hat{\phi}_i) = \int_0^1 \hat{\phi}_i(x)\ \mathrm{d}x$$

gives

$$-c_{i-1} + 2c_i - c_{i+1} = h^2.$$

ANSWER

For the right hand side and using $h_i = h_{i+1} = h$ we have

$$\int_0^1 \hat{\phi}_i(x)\ \mathrm{d}x = \int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{h}\ \mathrm{d}x + \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - x}{h}\ \mathrm{d}x = \frac{h}{2} + \frac{h}{2} = h.$$

For the left hand side

$$a(U, \hat{\phi}_i) = a(\hat{\phi}_{i-1}, \hat{\phi}_i)c_{i-1} + a(\hat{\phi}_i, \hat{\phi}_i)c_i + a(\hat{\phi}_{i+1}, \hat{\phi}_i)c_{i+1}.$$

For the integrals note that the derivatives of the basis functions are constant on each element.

$$\begin{aligned}
a(\hat{\phi}_{i-1}, \hat{\phi}_i) &= \int_{x_{i-1}}^{x_i} \frac{-1}{h^2}\ \mathrm{d}x = \frac{-1}{h}, \\
a(\hat{\phi}_{i+1}, \hat{\phi}_i) &= \int_{x_i}^{x_{i+1}} \frac{-1}{h^2}\ \mathrm{d}x = \frac{-1}{h}, \\
a(\hat{\phi}_i, \hat{\phi}_i) &= \int_{x_{i-1}}^{x_i} \frac{1}{h^2}\ \mathrm{d}x + \int_{x_i}^{x_{i+1}} \frac{1}{h^2}\ \mathrm{d}x = \frac{2}{h}.
\end{aligned}$$

The equation is

$$\frac{1}{h}(-c_{i-1} + 2c_i - c_{i+1}) = h.$$

(b) In the case $q = \alpha > 0$ is constant on $[0, 1]$ show that the equation

$$a(U, \hat{\phi}_i) = \int_0^1 \hat{\phi}_i\ \mathrm{d}x$$

leads to a relation of the form

$$k_{i,i-1}c_{i-1} + k_{ii}c_i + k_{i,i+1}c_{i+1} = h.$$

In your answer you should give $k_{i,i-1}$, k_{ii} and $k_{i,i+1}$ in terms of h and α.

ANSWER

We again have

$$a(U,\hat\phi_i)=a(\hat\phi_{i-1},\hat\phi_i)c_{i-1}+a(\hat\phi_i,\hat\phi_i)c_i+a(\hat\phi_{i+1},\hat\phi_i)c_{i+1}=h$$

where now

$$a(u,v)=\int_0^1 u'v'+\alpha uv\,\mathrm{d}x.$$

We consider the integrals of terms such as $\hat\phi_i\hat\phi_j$.

$$\int_{x_{i-1}}^{x_i}\hat\phi_i(x)\hat\phi_{i-1}(x)\,\mathrm{d}x=\frac{1}{h^2}\int_{x_{i-1}}^{x_i}(x-x_{i-1})(x_i-x)\,\mathrm{d}x.$$

To evaluate the integral let $y=x-x_{i-1}$ so that $x_i-x=-(x-x_i)=-(y-h)=h-y$.

$$\int_{x_{i-1}}^{x_i}(x-x_{i-1})(x_i-x)\,\mathrm{d}x.=\int_0^h y(h-y)\,\mathrm{d}x=h\frac{h^2}{2}-\frac{h^3}{3}=\frac{h^3}{6}.$$

Similarly

$$\int_{x_i}^{x_{i+1}}\hat\phi_i(x)\hat\phi_{i+1}(x)\,\mathrm{d}x=\frac{1}{h^2}\int_{x_i}^{x_{i+1}}(x-x_i)(x_{i+1}-x)\,\mathrm{d}x=\frac{h^3}{6}.$$

Hence

$$\begin{aligned}k_{i,i-1}&=a(\hat\phi_{i-1},\hat\phi_i)=-\frac{1}{h}+\alpha\frac{h}{6},\\ k_{i,i+1}&=a(\hat\phi_{i+1},\hat\phi_i)=-\frac{1}{h}+\alpha\frac{h}{6}.\end{aligned}$$

For the $a(\hat\phi_i,\hat\phi_i)$ term we have to consider the following.

$$\begin{aligned}\int_{x_{i-1}}^{x_i}\hat\phi_i^2\,\mathrm{d}x+\int_{x_i}^{x_{i+1}}\hat\phi_i^2\,\mathrm{d}x&=\frac{1}{h^2}\left(\int_{x_{i-1}}^{x_i}(x-x_{i-1})^2\,\mathrm{d}x+\int_{x_i}^{x_{i+1}}(x-x_{i+1})^2\,\mathrm{d}x\right)\\&=\frac{1}{h^2}\left(\frac{h^3}{3}+\frac{h^3}{3}\right)=\frac{2h}{3}.\end{aligned}$$

Thus

$$k_{ii}=a(\hat\phi_i,\hat\phi_i)=\frac{2}{h}+\alpha\frac{2h}{3}.$$

An alternative way of organising the computations here is to get the terms $k_{i,i-1}$, $k_{i,i}$ and $k_{i,i+1}$ after first computing the element matrices which is part of the next question. The details of calculating the element matrices are given in the answer to the next question. To summarise the results we have for (x_{i-1},x_i) an element stiffness matrix corresponding to the derivative terms and an element mass matrix corresponding to integrating the basis functions of respectively

$$\frac{1}{h_i}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\quad\text{and}\quad\frac{h_i}{6}\begin{pmatrix}2&1\\1&2\end{pmatrix}.$$

Combining these appropriately we get the element matrix associated with the weak form of

$$K_i = \frac{1}{h_i}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \alpha \frac{h_i}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The terms $k_{i,i-1}$ and $k_{i,i+1}$ only have a contribution from one element in each case. As $h_i = h$ for all elements we get

$$k_{i,i-1} = k_{i,i+1} = -\frac{1}{h} + \alpha\frac{h}{6}.$$

The $k_{i,i}$ term involves combining the contributions from elements (x_{i-1}, x_i) and (x_i, x_{i+1}) which in turn involves adding the appropriate diagonal terms of K_i and K_{i+1}. We have

$$k_{i,i} = \frac{2}{h} + \alpha\frac{4h}{6}.$$

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13. Apart from a few minor changes and re-typing this was question 1 of the 2001 `MA3056S` paper.

The weak form of the two point boundary value problem

$$L(u(x)) = f(x), \quad 0 < x < 1, \quad u(0) = A, \quad u(1) = B, \tag{1}$$

where L is a linear differential operator, $f(x)$ is a given function and A and B are given values, is set up by multiplying (1) by a test function $v \in H_0^1(0,1) := \{v : \ v \in H^1(0,1),\ v(0) = v(1) = 0\}$, and integrating the product over $[0,1]$. After integrating the weak form of problem (1) is of the form:

$$\text{find } u \in H \text{ such that} \quad a(u,v) = \int_0^1 fv \ \mathrm{d}x \quad \forall v \in H_0^1(0,1), \tag{2}$$

where $H := \{w : \ w \in H^1(0,1),\ w(0) = A,\ w(1) = B\}$ and where $a(.,.)$ is a symmetric bilinear form.

(a) Explain how you would apply the Galerkin technique using polynomial basis functions over $[0,1]$ to derive an approximation U to $u \in H$, the solution of (2).

[6 MARKS]

<u>ANSWER</u>

The problem as given has non-homogeneous boundary conditions but can be converted to one with homogeneous conditions by defining

$$u_1(x) = A + (B - A)x \quad \text{which satisfies} \quad u_1(0) = A \quad \text{and} \quad u_1(1) = B.$$

The solution $u \in H$ is of the form $u = u_1 + \tilde{u}$ where $\tilde{u} \in H_0^1(0,1)$. We have

$$a(u,v) = a(u_1 + \tilde{u}, v) = (f,v) \quad \text{giving} \quad a(\tilde{u}, v) = (f,v) - a(u_1, v) =: F(v).$$

With polynomial basis functions $\phi_1, \phi_2, \cdots$ in $H_0^1(0,1)$ we define

$$V_n := \text{span}\{\phi_1, \cdots, \phi_n\}.$$

The Galerkin technique involves computing $\tilde{U}_n \in V_n$ such that

$$a(\tilde{U}_n, v) = F(v) \quad \text{for all } v \in V_n$$

from which we get $U_n = u_1 + \tilde{U}_n$. With $\tilde{U}_n = \sum_1^n c_j\phi_j$ the coefficients $\underline{c} = (c_i)$ satisfy $K\underline{c} = \underline{b}$ where

$$K = (a(\phi_i, \phi_j)), \quad \underline{b} = (F(\phi_i)).$$

(b) Derive the weak form of the problem

$$-u''(x) = x^4, \quad 0 < x < 1, \quad u(0) = 1, \quad u(1) = 0. \tag{3}$$

[4 MARKS]

ANSWER

$$-\int_0^1 u''v \, \mathrm{d}x = \int_0^1 u'v' \, \mathrm{d}x \quad \text{for } v \text{ satisfying} \quad v(0) = v(1) = 0.$$

The weak form can be written as find $u \in \{v \in H^1(0,1) : \ v(0) = 0, \ v(1) = 1\}$ such that

$$a(u, v) = \int_0^1 u'v' \, \mathrm{d}x = \int_0^1 x^4 v \, \mathrm{d}x \quad \text{for all } v \in H_0^1(0,1).$$

(c) Using the Galerkin technique with trial function

$$U(x) = (1-x) + c_1 x(1-x) + c_2 x^2(1-x)$$

containing the unknown parameters c_1 and c_2, and the respective test functions

$$v_1(x) = x(1-x) \quad \text{and} \quad v_2(x) = x^2(1-x),$$

set up the 2×2 system of linear equations for c_1 and c_2.
[Do not attempt to evaluate the integrals involved and do not attempt to solve the system.]

[5 MARKS]

ANSWER

$$U(x) = u_1 + c_1\phi_1(x) + c_2\phi_2(x), \quad \text{with} \quad \begin{array}{l} u_1 = 1-x, \\ \phi_1(x) = x(1-x), \\ \phi_2(x) = x^2(1-x). \end{array}$$

The integrands in the $a(.,.)$ expression involves derivatives of the basis functions ϕ_1 and ϕ_2. We have

$$\phi_1'(x) = 1-2x \quad \text{and} \quad \phi_2'(x) = 2x-3x^2.$$

The equations are

$$K\underline{c} = \begin{pmatrix} (x^4,\phi_1) \\ (x^4,\phi_2) \end{pmatrix} - \begin{pmatrix} a(u_1,\phi_1) \\ a(u_1,\phi_2) \end{pmatrix}$$

where

$$K = \int_0^1 \begin{pmatrix} {\phi_1'}^2 & \phi_1'\phi_2' \\ \phi_2'\phi_1' & {\phi_2'}^2 \end{pmatrix} \,\mathrm{d}x = \int_0^1 \begin{pmatrix} (1-2x)^2 & (1-2x)(2x-3x^2) \\ (1-2x)(2x-3x^2) & (2x-3x^2)^2 \end{pmatrix} \,\mathrm{d}x.$$

14. The following was question 3 of the June 2003 `MA3056S` paper.

Let V denote a Hilbert space with inner product $(.,.)$, let $a(.,.)$ denote a symmetric and positive definite bilinear form defined on $V\times V$ and let $F(.)$ denote a linear functional defined on V. Also let $V_h \subset V$ and suppose that there exists $u\in V$ and $U_h \in V_h$ satisfying

$$\begin{aligned} a(u,v) &= F(v), \quad \text{for all } v\in V, \\ a(U_h,v) &= F(v), \quad \text{for all } v \in V_h. \end{aligned}$$

(a) Define what it means for $a(.,.)$ to be positive definite on $V\times V$.

[1 MARK]

(b) If $e = u - U_h$ then show that $a(e,v)=0$ for all $v\in V_h$.

[1 MARK]

(c) Let $\|v\|_E = a(v,v)^{1/2}$ denote the energy norm on V.
State the Cauchy Schwarz inequality as applied to $a(v,w)$ and show that

$$\|u-U_h\|_E \le \|u-v\|_E \quad \text{for all } v\in V_h.$$

Comment on this property of U_h.

[4 MARKS]

(d) Show that $u \in V$ and $U_h \in V_h$ uniquely minimise the functional

$$I(v) = \frac{1}{2}a(v,v) - F(v) \qquad (*)$$

over the spaces V and V_h respectively.

[5 MARKS]

(e) For the functional $I(.)$ defined in $(*)$ in part (d) show that

$$I(U_h) = I(u) + \frac{1}{2}a(u - U_h, u - U_h).$$

[2 MARKS]

(f) Let $\|.\|$ denote the usual norm of V given by $\|v\| = (v,v)^{1/2}$. Suppose that the bilinear form $a(.,.)$ and the linear functional $F(.)$ satisfy the coercive and bounded properties that

$$\kappa_1\|v\|^2 \le a(v,v), \quad |a(v,w)| \le \kappa_2\|v\|\|w\|, \quad |F(v)| \le \kappa_3\|v\|$$

for all $v, w \in V$ where $0 < \kappa_1 \le \kappa_2$ and $\kappa_3 > 0$ are constants. Show that under these conditions we have the following.

(i)

$$\|u - U_h\|^2 \le \frac{\kappa_2}{\kappa_1}\|u - v\|^2 \quad \text{for all } v \in V_h.$$

[2 MARKS]

(ii)

$$\|u\| \le \frac{\kappa_3}{\kappa_1} \qquad \text{and} \qquad \|U_h\| \le \frac{\kappa_3}{\kappa_1}.$$

[2 MARKS]

(iii)

$$I(v) \ge -\frac{\kappa_3^2}{2\kappa_1}$$

for all $v \in V$, where $I(v)$ is defined in $(*)$ in part (d).

[3 MARKS]

ANSWER

(a) $a(.,.)$ is positive definite on $V \times V$ if $a(v,v) \ge 0$ for all $v \in V$ with $a(v,v) = 0$ only when $v = 0$.

1 MARK

(b) Subtracting the two relations when $v \in V_h$ and using the properties of the bilinear form we have

$$0 = a(u,v) - a(U_h,v) = a(u - U_h, v).$$

1 MARK

(c) The Cauchy Schwarz inequality is $|a(v,w)| \le \|v\|_E \|w\|_E$.
Using the result in (b) we have for all $v \in V_h$

$$\begin{aligned}\|u - U_h\|^2 &= a(u-U_h, u-U_h) \\ &= a(u-U_h, u-U_h) + a(u-U_h, U_h - v) \\ &= a(u-U_h, u-v) \\ &\le \|u-U_h\|_E \|u-v\|_E.\end{aligned}$$

using the Cauchy Schwarz inequality. Dividing by $\|u-U_h\|_E \neq 0$ gives the result. This result tells us that U_h is the best approximation to u from the space V_h in the energy norm.

4 MARKS

(d) Let $v \in V$ and consider $I(u+v)$.

$$\begin{aligned}I(u+v) &= \frac{1}{2}a(u+v, u+v) - F(u+v) \\ &= \frac{1}{2}(a(u,u) + 2a(u,v) + a(v,v)) - (F(u) + F(v)) \\ &= I(u) + (a(u,v) - F(v)) + \frac{1}{2}a(v,v).\end{aligned}$$

From the property of u we have $a(u,v) - F(v) = 0$ and thus

$$I(u+v) = I(u) + \frac{1}{2}a(v,v) > I(u)$$

for all $v \neq 0$ using the positive definite property of $a(.,.)$. u hence uniquely minimises the functional over V.
Similarly if $v \in V_h$ and $v \neq 0$ then we have

$$I(U_h + v) = I(U_h) + \frac{1}{2}a(v,v) > I(U_h)$$

showing that U_h uniquely minimises the functional over V_h.

5 MARKS

(e) From the derivation used in (d) we take $v = U_h - u$ to immediately get the result.

2 MARKS

(f) (i) If we use the coercive and bounded properties on the terms in the best approximation result in part (c) we get for all $v \in V$ that

$$\kappa_1 \|u - U_h\|^2 \le \|u - U_h\|_E^2 \le \|u - v\|_E^2 \le \kappa_2 \|u - v\|^2.$$

The result then follows.

2 MARKS

(ii)

$$\kappa_1\|u\|^2 \le a(u,u) = F(u) \le \kappa_3\|u\|$$

giving $\|u\| \le \kappa_3/\kappa_1$.
Similarly

$$\kappa_1\|U_h\|^2 \le a(U_h,U_h) = F(U_h) \le \kappa_3\|U_h\|$$

giving $\|U_h\| \le \kappa_3/\kappa_1$.

2 MARKS

(iii) Using the coercive property of $a(v,v)$ and the bounded property of $F(v)$ we have

$$I(v) \ge \frac{\kappa_1}{2}\|v\|^2 - \kappa_3\|v\| = \frac{1}{2}(\kappa_1\|v\|^2 - 2\kappa_3\|v\|)$$

Now the quadratic

$$\begin{aligned} g(t) &= \kappa_1 t^2 - 2\kappa_3 t \\ &= \kappa_1\left(t^2 - 2\frac{\kappa_3}{\kappa_1}t\right) \\ &= \kappa_1\left(t - \frac{\kappa_3}{\kappa_1}t\right)^2 - \frac{\kappa_3^2}{\kappa_1} \\ &\ge -\frac{\kappa_3^2}{\kappa_1} \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

Hence

$$I(v) \ge -\frac{\kappa_3^2}{2\kappa_1}.$$

3 MARKS

15. Apart from a few minor changes and re-typing this was question 5 of the 2001 `MA3056S` paper.

(i) Let V denote a Hilbert space with inner product $(.,.)$ and let $a(.,.)$ denote a bilinear form which is such that $a(v,v) > 0$ for all $v \in V$ with $v \ne 0$. Also let $V_h \subset V$ denote another space of functions such that $u \in V$ and $U_h \in V_h$ minimise the functional

$$I(v) = \frac{1}{2}a(v,v) - (f,v)$$

over the spaces V and V_h respectively where $f \in V$. Show the following:

(a) $a(u,v) = (f,v)$ for all $v \in V$ and $a(U_h,v) = (f,v)$ for all $v \in V_h$.

ANSWER

As $u \in V$ minimises $I(v)$ over V it follows that for all $v \neq 0$ the quadratic $g(t) = I(u+tv) - I(u)$ has a minimum at $t = 0$, i.e. $g'(0) = 0$. Now

$$\begin{aligned} g(t) &= \frac{1}{2}(a(u+tv, u+tv) - a(u,u)) - ((f, u+tv) - (f,u)) \\ &= \frac{1}{2}(2a(u,v)t + a(v,v)t^2) - (f,v)t \\ &= (a(u,v) - (f,v))t + \frac{t^2}{2}a(v,v) \end{aligned}$$

and $g'(0) = a(u,v) - (f,v)$. Thus for all $v \in V$ we have $a(u,v) = (f,v)$.

Similarly if $v \in V_h$, $v \neq 0$ and we now let $g(t) = I(U_h + tv) - I(U_h)$ then g has a minimum at $t = 0$ with

$$0 = g'(0) = a(U_h, v) - (f, v)$$

and we have $a(U_h, v) = (f, v)$.

(b) If $e = u - U_h$ then $a(e, v) = 0$ for all $v \in V_h$.

ANSWER

This follows immediately from

$$(f,v) = a(u,v) = a(U_h, v) \quad \text{for all } v \in V_h$$

and the properties of the bilinear form, i.e. $a(u - U_h, v) = a(u,v) - a(u_h, v) = 0$.

(c) $I(U_h) = I(u) + \frac{1}{2}a(e,e)$.

ANSWER

From the answer to part (a) (with $t = 1$) we have

$$I(u+v) = I(u) + \frac{1}{2}a(v,v).$$

Taking $v = U_h - u = -e$ gives

$$I(U_h) = I(u) + \frac{1}{2}a(-e,-e) = I(u) + \frac{1}{2}a(e,e).$$

(d) $a(e,e) \le a(u-v, u-v)$ for all $v \in V_h$. Comment on this result.

ANSWER

Let $v \in V_h$. We have

$$\begin{aligned} a(e,e) &= a(e, u - U_h) = a(e, u - U_h + (U_h - v)) \\ &= a(e, u - v) \\ &\le a(e,e)^{1/2} a(u-v, u-v)^{1/2} \quad \text{by the Cauchy Schwarz inequality.} \end{aligned}$$

Hence

$$a(e,e) \le a(u-v, u-v).$$

Let $\|v\|_E := a(v,v)^{1/2}$ denote the energy norm. The result just given shows that the Galerkin approximation U_h is the best approximation to u from the space V_h in the energy norm.

[10 MARKS]

(ii) Show that the weak form of

$$u''''(x) = f(x), \quad 0 < x < 1 \quad \text{with } u(0) = u'(0) = u(1) = u'(1) = 0,$$

involves the bilinear form

$$a(u, v) = \int_0^1 u''v'' \, \mathrm{d}x.$$

ANSWER

Let v satisfy $v(0) = v'(0) = v(1) = v'(1) = 0$. We use integration by parts twice to get

$$\begin{aligned} \int_0^1 u''''v \, \mathrm{d}x &= -\int_0^1 u'''v' \, \mathrm{d}x, \quad \text{using } v(0) = v(1) = 0, \\ &= +\int_0^1 u''v'' \, \mathrm{d}x, \quad \text{using } v'(0) = v'(1) = 0. \end{aligned}$$

The weak form involves finding $u \in V = \{v \in H^2(0,1) : \ v(0) = v'(0) = v(1) = v'(1) = 0\}$ such that

$$\int_0^1 u''v'' \, \mathrm{d}x = \int_0^1 fv \, \mathrm{d}x \quad \text{for all } v \in V.$$

If the finite element method is used to obtain an approximation U_h to u using piecewise cubic Hermite elements on a mesh $0 = x_0 < x_1 < \cdots < x_{ne} = 1$ then it can be shown that the approximate solution and its first derivative are exact at the point x_i. In the case $f(x) = 1$ give the exact solution. Also in the case $ne = 1$ give the finite element solution. Under these conditions of $f(x) = 1$ and $ne = 1$ and with $e = u - U_h$ show the following:

(a) $e'(1/2) = 0$.

(b) $e''(\alpha) = 0$ where $\alpha = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$.

(c) $e'''(1/2) = 0$.

[10 MARKS]

ANSWER

I have not done much on the module on cubic Hermite elements but there is not much to know to answer this question.

Firstly, for the exact solution we just need to integrate $f(x)$ four times and apply the 4 boundary conditions. As $f(x) = 1$ the exact solution $u(x)$ is a polynomial of degree 4 with the coefficient of x^4 being $1/4! = 1/24$ and the boundary conditions imply that $u(x)$ has double roots at $x = 0$ and at $x = 1$. Hence

$$u(x) = \frac{1}{24}x^2(1-x)^2 = \frac{x^2}{24}(x^2 - 2x + 1).$$

With one cubic element the finite element solution is completely determined by the boundary conditions and we have $U_h = 0$ and thus the error $e = u - U_h = u$. Differentiating we have

$$\begin{aligned} e'(x) &= \frac{1}{24}(4x^3 - 6x^2 + 2x) = \frac{x}{12}(2x^2 - 3x + 1) = \frac{x}{12}(2x-1)(x-1), \\ e''(x) &= \frac{1}{24}(12x^2 - 12x + 2) = \frac{1}{12}(6x^2 - 6x + 1), \\ e'''(x) &= \frac{1}{12}(12x - 6) = \frac{1}{2}(2x - 1). \end{aligned}$$

From the factors we immediately have the result for parts (a) and (c). For part (b) the roots of $e''(x)$ are

$$x = 6 \pm \frac{\sqrt{(6^2 - 4(6)}}{(2)(6)} = \frac{6 \pm \sqrt{12}}{12} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}.$$

---

16. The following relates more to what is done in the MA5352 part of the module but anyone who has done Fourier series before can attempt it.

Let $v \in C^1[0,1]$ be a function satisfying $v(0) = v(1) = 0$ and let

$$v(x) = \sum_{k=1}^{\infty} c_k \sin k\pi x$$

be its Fourier series representation. Show that for all such functions

$$\int_0^1 v(x)^2 \, \mathrm{d}x \le \frac{1}{\pi^2} \int_0^1 v'(x)^2 \, \mathrm{d}x.$$

Use this result to determine the values of k for which $Lu = -u'' - k^2 u$ is positive definite on the space

$$D_L = \{v \in C^2[0,1] : \quad v(0) = v(1) = 0\}.$$

ANSWER

Let v_n be the sum of the first n terms. We have

$$\begin{aligned} v_n(x) &= \sum_{k=1}^{n} c_k \sin k\pi x, \\ v_n'(x) &= \sum_{k=1}^{n} k c_k \cos k\pi x. \end{aligned}$$

Then

$$\begin{aligned} v_n(x)^2 &= \sum_{k=1}^{n}\sum_{l=1}^{n} c_k c_l \sin k\pi x \sin l\pi x, \\ v_n'(x)^2 &= \sum_{k=1}^{n}\sum_{l=1}^{n} k l c_k c_l \cos k\pi x \cos l\pi x. \end{aligned}$$

Now from the identities

$$\begin{aligned}\cos(y+z) &= \cos y\cos z - \sin y\sin z,\\ \cos(y-z) &= \cos y\cos z + \sin y\sin z\end{aligned}$$

we get

$$\begin{aligned}\cos y\cos z &= \frac{1}{2}(\cos(y+z)+\cos(y-z)),\\ \sin y\sin z &= \frac{1}{2}(\cos(y-z)-\cos(y+z)).\end{aligned}$$

We need to consider integrals of the form

$$\begin{aligned}\int_0^1 \cos(k+l)\pi x\ \mathrm{d}x &= \frac{1}{(k+l)\pi}[\sin(k+l)\pi x]_0^1 = 0,\\ \int_0^1 \cos(k-l)\pi x\ \mathrm{d}x &= \frac{1}{(k-l)\pi}[\sin(k-l)\pi x]_0^1 = 0, \quad \text{provided } k \ne l,\\ \int_0^1 \mathrm{d}x &= 1, \quad \text{corresponding to } k = l.\end{aligned}$$

Thus

$$\int_0^1 \sin k\pi x \sin l\pi x\ \mathrm{d}x = \int_0^1 \cos k\pi x\cos l\pi x\ \mathrm{d}x = \begin{cases}\frac{1}{2} & \text{if } k = l,\\ 0 & \text{if } k \ne l.\end{cases}$$

From this it follows that

$$\begin{aligned}\int_0^1 v_n(x)^2\ \mathrm{d}x &= \frac{1}{2}\sum_{k=1} c_k^2,\\ \int_0^1 v_n'(x)^2\ \mathrm{d}x &= \frac{1}{2}\pi^2\sum_{k=1} k^2c_k^2.\end{aligned}$$

As $k^2c_k^2 \ge c_k^2$ for $k = 1, 2, \cdots$ it follows that

$$\int_0^1 v_n'(x)^2\ \mathrm{d}x \ge \pi^2 \int_0^1 v_n(x)^2\ \mathrm{d}x.$$

(In fact we can only have equality here if $c_k = 0$ for $k \ge 2$ and this corresponds to $v(x) = c_1 \sin \pi x$.) Letting $n \to \infty$ gives the required inequality.

Using the boundary condition $v(0) = v(1) = 0$ we have

$$(Lv, v) = \int_0^1 (-v'' - k^2 v)v\ \mathrm{d}x = \int_0^1 v'^2 - k^2v^2\ \mathrm{d}x \ge \int_0^1 (\pi^2 - k^2)v^2\ \mathrm{d}x.$$

We have the positive definite property if $\pi^2 - k^2 > 0$, i.e. $|k| < \pi$. L fails to be positive definite if $k = \pi$. In this case the function $v(x) = \sin \pi x$ is non-zero and gives $(Lv, v) = 0$.

As a comment here, if you have taken courses on ODEs which has included sections on eigenvalues then the above is connected with the eigenvalues and eigenfunctions of $Lu = -u''$ from a space of functions which includes the conditions $u(0) = u(1) = 0$. An eigenvalue λ with corresponding eigenfunction $u \neq 0$ of L satisfy $Lu = \lambda u$, i.e. $-u'' - \lambda u = 0$. There are no eigenvalues for $\lambda < \pi^2$. $\lambda = \pi^2$ is the smallest positive eigenvalue. The complete set of eigenvalues are $\lambda_k = k^2\pi^2$, $k = 1, 2, \cdots$ corresponding respectively to the eigenfunctions $\sin k\pi x$. $k = 1, 2, \cdots$.

MA3951/MA5352: Numerical and Variational Methods for PDEs Exercise sheet 2 and Answers

1. This was question 2 of the June 2004 MA5352B paper.

   (a) Let Ω denote a bounded simply connected domain with a piecewise smooth boundary $\partial\Omega$ and consider the Poisson problem

   $$-\Delta u = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega_1, \quad \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega_2$$

   where $\partial\Omega_1$ and $\partial\Omega_2$ give a partition of $\partial\Omega$, g is a constant and where $\partial/\partial n$ denotes partial differentiation in the direction of the outward normal to $\partial\Omega_2$.
   Show that if u satisfies this problem then $u \in V$ also satisfies the weak formulation

   $$\iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega v \, \mathrm{d}x\mathrm{d}y + g \int_{\partial\Omega_2} v \, \mathrm{d}s$$

   for all $v \in V$ where

   $$V = \{v \in H^1(\Omega) : \ v = 0 \ \text{ on } \partial\Omega_1\}.$$

   [6 MARKS]

   (b) Suppose that the domain Ω is partitioned into n triangular elements $\Omega_1, \cdots, \Omega_n$ such that

   $$\bar{\Omega} = \bigcup_{r=1}^{n} \bar{\Omega}_r, \quad \Omega_i \cap \Omega_j = \text{empty set for } i \neq j$$

   and let T denote the standard triangle with vertices $\underline{s}_1^T = (0,0)$, $\underline{s}_2^T = (1,0)$ and $\underline{s}_3^T = (0,1)$ in the (s,t) plane.

   (i) State the three standard linear Lagrange basis functions ϕ_1, ϕ_2 and ϕ_3 defined on T such that $\phi_i(\underline{s}_i) = 1$ and $\phi_i(\underline{s}_j) = 0$ if $j \neq i$.

   [1 MARK]

   (ii) Let $\underline{x}_1 = (x_1, y_1)^T$, $\underline{x}_2 = (x_2, y_2)^T$ and $\underline{x}_3 = (x_3, y_3)^T$ denote the vertices of Ω_r, let $U(\underline{x})$ denote a linear function defined on Ω_r and let $U_i = U(\underline{x}_i)$, $i = 1, 2, 3$. Describe the mapping $\underline{x} : T \to \Omega_r$ which is such that $\underline{s}_i \to \underline{x}_i$, $i = 1, 2, 3$, give $U(\underline{x}(s,t))$ for $(s,t) \in T$ and show that the gradient vector ∇U can be written as

   $$\nabla U = \frac{1}{J} \begin{pmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{pmatrix} \begin{pmatrix} U_2 - U_1 \\ U_3 - U_1 \end{pmatrix}$$

   where $J = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$.

   [6 MARKS]

(iii) Explain what is meant by the element matrix K_r for the Poisson problem of part (a) and show that in the case $\underline{x}_1 = (0,0)^T$, $\underline{x}_2 = (2,0)^T$ and $\underline{x}_3 = (0,1)^T$ we obtain

$$K_r = \begin{pmatrix} 5/4 & -1/4 & -1 \\ -1/4 & 1/4 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

In the case $g = 0$ also give the 3×1 element vector $\underline{b}_r$.

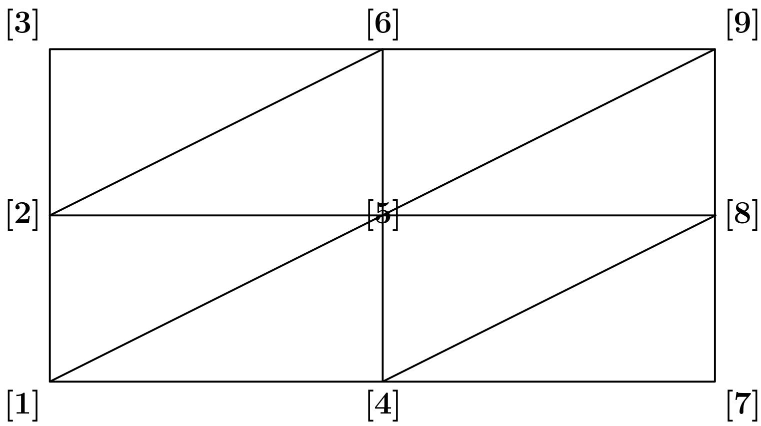
[6 MARKS]

(iv) Consider the uniform mesh of $\Omega := (0,4) \times (0,2)$ consisting of eight triangles and nine nodes as shown in the figure below and let $\hat{\phi}_i(\underline{x})$, $i = 1, 2, \cdots, 9$ denote the piecewise linear basis functions defined on $\bar{\Omega}$ which are such that

$$\hat{\phi}_i(\underline{x}_j) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases}$$

Also let $U_j = U(\underline{x}_j)$, $j = 1, 2, \cdots, 9$ denote the Galerkin approximation at the nodal points. Given the result of part (iii) and given that the element matrices have an invariance to translation, uniform scaling, reflection and rotations determine $a(\hat{\phi}_j, \hat{\phi}_5)$, $j = 1, 2, \cdots, 9$ and $(1, \hat{\phi}_5)$ in the equation

$$\sum_{j=1}^{9} a(\hat{\phi}_j, \hat{\phi}_5) U_j = (1, \hat{\phi}_5).$$



[1]=(0,0)
[2]=(0,1)
[3]=(0,2)
[4]=(2,0)
[5]=(2,1)
[6]=(2,2)
[7]=(4,0)
[8]=(4,1)
[9]=(4,2)

[6 MARKS]

ANSWER

(a) From the vector identity

$$\nabla \cdot (v \nabla u) = v \Delta u + \nabla v \cdot \nabla u$$

we have using the divergence theorem that

$$\begin{aligned} -\iint_\Omega v\Delta u \,\mathrm{d}x\mathrm{d}y &= \iint_\Omega \nabla u\cdot\nabla v \,\mathrm{d}x\mathrm{d}y - \int_{\partial\Omega} v\nabla u\cdot \underline{n}\,\mathrm{d}s \\ &= \iint_\Omega \nabla u\cdot\nabla v \,\mathrm{d}x\mathrm{d}y - g\int_{\partial\Omega_2} v\,\mathrm{d}s \end{aligned}$$

if $v=0$ on $\partial\Omega_1$ and by using the given boundary condition on $\partial\Omega_2$. Thus if $v\in V$ then the exact solution $u\in V$ satisfies

$$\iint_\Omega \nabla u\cdot\nabla v \,\mathrm{d}x\mathrm{d}y = \iint_\Omega v \,\mathrm{d}x\mathrm{d}y + g\int_{\partial\Omega_2} v\,\mathrm{d}s.$$

6 MARKS

(b) (i) The basis functions are

$$\begin{aligned} \phi_1(s,t) &= 1-s-t, \\ \phi_2(s,t) &= s, \\ \phi_3(s,t) &= t. \end{aligned}$$

1 MARK

(ii)

$$\begin{aligned} \underline{x}(s,t) &= \underline{x}_1\phi_1(s,t)+\underline{x}_2\phi_2(s,t)+\underline{x}_3\phi_3(s,t) \\ &= \underline{x}_1 + (\underline{x}_2-\underline{x}_1)s + (\underline{x}_3-\underline{x}_1)t, \\ U(\underline{x}(s,t)) &= U_1\phi_1(s,t)+U_2\phi_2(s,t)+U_3\phi_3(s,t) \\ &= U_1+(U_2-U_1)s+(U_3-U_1)t. \end{aligned}$$

By the chain rule of partial differentiation

$$\begin{pmatrix} \dfrac{\partial U(\underline{x}(s,t))}{\partial s} \\ \dfrac{\partial U(\underline{x}(s,t))}{\partial t} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{pmatrix} \begin{pmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{\partial U}{\partial y} \end{pmatrix} = \tilde{J}^T \begin{pmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{\partial U}{\partial y} \end{pmatrix}$$

where $\tilde{J}$ is the Jacobian matrix. Thus

$$\nabla U = \begin{pmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{\partial U}{\partial y} \end{pmatrix} = \tilde{J}^{-T} \begin{pmatrix} \dfrac{\partial U(\underline{x}(s,t))}{\partial s} \\ \dfrac{\partial U(\underline{x}(s,t))}{\partial t} \end{pmatrix} = \tilde{J}^{-T}\begin{pmatrix} U_2-U_1 \\ U_3-U_1 \end{pmatrix}.$$

The Jacobian matrix $\tilde{J}$ is given by

$$\tilde{J} = \begin{pmatrix} \underline{x}_2-\underline{x}_1, & \underline{x}_3-\underline{x}_1 \end{pmatrix} = \begin{pmatrix} x_2-x_1 & x_3-x_1 \\ y_2-y_1 & y_3-y_1 \end{pmatrix}$$

and hence the determinant is

$$J = (x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)$$

and the inverse is

$$\tilde{J}^{-1} = \frac{1}{J}\begin{pmatrix} y_3-y_1 & x_1-x_3 \\ y_1-y_2 & x_2-x_1 \end{pmatrix}$$

and the result follows.

6 MARKS

(iii) Let $\tilde{\phi}_1$, $\tilde{\phi}_2$ and $\tilde{\phi}_3$ be the basis functions defined on Ω_r given by

$$\tilde{\phi}_i(\underline{x}(s,t)) = \phi_i(s,t), \quad i = 1,2,3$$

and let

$$a(\tilde{\phi}_i, \tilde{\phi}_j)_r = \iint_{\Omega_r} \nabla\tilde{\phi}_i \cdot \nabla\tilde{\phi}_j \, \mathrm{d}x\mathrm{d}y$$

The element matrix is the 3×3 matrix $(a(\tilde{\phi}_i, \tilde{\phi}_j)_r)$.
For the specific element given we have $x = 2s$ and $y = t$ so that

$$\tilde{J} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = 2, \quad \tilde{J}^{-1} = \tilde{J}^{-T} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now let

$$B = \begin{pmatrix} \frac{\partial\tilde{\phi}_1}{\partial x} & \frac{\partial\tilde{\phi}_2}{\partial x} & \frac{\partial\tilde{\phi}_3}{\partial x} \\ \frac{\partial\tilde{\phi}_1}{\partial y} & \frac{\partial\tilde{\phi}_2}{\partial y} & \frac{\partial\tilde{\phi}_3}{\partial y} \end{pmatrix} = \tilde{J}^{-T} \begin{pmatrix} \frac{\partial\phi_1}{\partial s} & \frac{\partial\phi_2}{\partial s} & \frac{\partial\phi_3}{\partial s} \\ \frac{\partial\phi_1}{\partial t} & \frac{\partial\phi_2}{\partial t} & \frac{\partial\phi_3}{\partial t} \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

As B is constant on Ω_r and as Ω_r has area 1 we have

$$K_r = B^T B = \begin{pmatrix} 5/4 & -1/4 & -1 \\ -1/4 & 1/4 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

The element vector in this case is is the 3×1 vector $((1, \tilde{\phi}_j)_r)$. As the integrand is linear, this can be computed exactly using the one-point quadrature rule. At the centroid of the element each basis function has value 1/3 and the area of the triangle is 1 and thus

$$\underline{b}_r = \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

6 MARKS

(iv) All 8 triangles in the mesh have the same angles as the triangle in part (iii) and hence all 8 element matrices are the same provided the 3 vertices of each triangle are taken in the appropriate order. Now

$$a(\hat{\phi}_j, \hat{\phi}_5) = \sum_{r=1}^{8} a(\hat{\phi}_j, \hat{\phi}_5)_r.$$

As points 3 and 5 and points 7 and 5 are not on the same element we get $a(\hat{\phi}_3, \hat{\phi}_5) = a(\hat{\phi}_7, \hat{\phi}_5) = 0$.
As $(K_r)_{23} = 0$ we get $a(\hat{\phi}_1, \hat{\phi}_5) = a(\hat{\phi}_9, \hat{\phi}_5) = 0$.
For the connection between $\hat{\phi}_5$ and $\hat{\phi}_2$ we need to consider two contributions given by $(K_r)_{1,2}$.

$$a(\hat{\phi}_2, \hat{\phi}_5) = a(\hat{\phi}_8, \hat{\phi}_5) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

For the connection between $\hat{\phi}_5$ and $\hat{\phi}_4$ we need to consider two contributions given by $(K_r)_{1,3}$.

$$a(\hat{\phi}_4, \hat{\phi}_5) = a(\hat{\phi}_6, \hat{\phi}_5) = -1 - 1 = -2.$$

For the diagonal entry we have contributions from 6 triangles. We need to consider each diagonal entry of K_r twice.

$$a(\hat{\phi}_5, \hat{\phi}_5) = \frac{5}{4} + \frac{5}{4} + 1 + 1 + \frac{1}{4} + \frac{1}{4} = 5.$$

As $\hat{\phi}_5$ is non-zero over the 6 triangles which have $\underline{x}_5$ as a node we have

$$(1, \hat{\phi}_5) = \sum_{r=1}^{8} (1, \hat{\phi}_5)_r = \frac{6}{3} = 2.$$

The equation is

$$5U_5 - 2(U_4 + U_6) - \frac{1}{2}(U_2 + U_8) = 2.$$

6 MARKS

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2. Derive the weak forms for the following problems involving Poisson's equation

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^2, \quad f \in C(\bar{\Omega}),$$

subject to the boundary conditions given below, stating in each case the appropriate space of functions involved (in terms of Sobolev spaces) and classify each boundary condition as an essential boundary condition or as a natural boundary condition.

(a) $u = 0$ on $\partial\Omega$.

(b) $u = 0$ on $\partial\Omega_D$ and $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega_N$ where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ and assuming that $\partial\Omega_D$ is not empty.

(c) $u = 0$ on $\partial\Omega_D$ and $\dfrac{\partial u}{\partial n} + \beta u = g$ on $\partial\Omega_N$ where $\beta \ge 0$, $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ and assuming that $\partial\Omega_D$ is not empty.

ANSWER

From the vector identity

$$\nabla \cdot (v\nabla u) = v\Delta u + \nabla u \cdot \nabla v$$

we obtain

$$-v\Delta u = \nabla u \cdot \nabla v - \nabla \cdot (v\nabla u).$$

Then by the divergence theorem

$$-\iint_\Omega v\Delta u \, \mathrm{d}x\mathrm{d}y = \iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y - \int_{\partial\Omega} v\frac{\partial u}{\partial n} \, \mathrm{d}s.$$

(a) If we take $v = 0$ on $\partial\Omega$ then the boundary integral term is 0. Hence we define

$$V = \{v \in H^1(\Omega) : \quad v = 0 \text{ on } \partial\Omega\}.$$

The weak form is find $u \in V$ such that

$$\iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega fv \, \mathrm{d}x\mathrm{d}y \quad \forall v \in V.$$

(b) As $\dfrac{\partial u}{\partial n} = 0$ on the part $\partial\Omega_N$ we only need v to vanish on the other part of $\partial\Omega$ to remove the boundary integral term. Hence we define

$$V = \{v \in H^1(\Omega) : \quad v = 0 \text{ on } \partial\Omega_D\}.$$

The weak form is find $u \in V$ such that

$$\iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega fv \, \mathrm{d}x\mathrm{d}y \quad \forall v \in V.$$

(c) On the part $\partial\Omega_N$ we have from the mixed boundary condition that

$$-v\frac{\partial u}{\partial n} = (\beta u - g)v.$$

If we define

$$V = \{v \in H^1(\Omega) : \quad v = 0 \text{ on } \partial\Omega_D\}$$

then

$$-\int_{\partial\Omega} v\frac{\partial u}{\partial n} \, \mathrm{d}s = \int_{\partial\Omega_N} (\beta u - g)v \, \mathrm{d}s.$$

The weak form is find $u \in V$ such that

$$\iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y + \int_{\partial\Omega_N} \beta uv \, \mathrm{d}s = \iint_\Omega fv \, \mathrm{d}x\mathrm{d}y + \int_{\partial\Omega_N} gv \, \mathrm{d}s \quad \forall v \in V.$$

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3. Let T denote the standard triangle with vertices $\underline{s}_1^T = (0,0)$, $\underline{s}_2^T = (1,0)$ and $\underline{s}_3^T = (0,1)$ and let Ω_r denote an actual triangle with vertices $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$.

   (a) State the linear basis functions ϕ_1, ϕ_2 and ϕ_3 defined on T which have the property that $\phi_i(\underline{s}_i) = 1$ and $\phi_i(\underline{s}_j) = 0$ for $j \ne i$ and give the affine mapping of T onto Ω_r such that $\underline{x}(\underline{s}_i) = \underline{x}_i$, $i = 1, 2, 3$.
   State the Jacobian matrix $\tilde{J}$ of this mapping.

ANSWER

The basis functions are

$$\begin{aligned}\phi_1(s,t) &= 1-s-t,\\ \phi_2(s,t) &= s,\\ \phi_3(s,t) &= t.\end{aligned}$$

The affine map is

$$\underline{x}(s,t) = \underline{x}_1\phi_1(s,t) + \underline{x}_2\phi_2(s,t) + \underline{x}_3\phi_3(s,t) = \underline{x}_1 + (\underline{x}_2-\underline{x}_1)s + (\underline{x}_3-\underline{x}_1)t.$$

With $\underline{x}_1^T = (x_1, y_1)$, $\underline{x}_2^T = (x_2, y_2)$ and $\underline{x}_3^T = (x_3, y_3)$ the Jacobian matrix (written in several different ways) is

$$\tilde{J} = \begin{pmatrix} \frac{\partial \underline{x}}{\partial s} & \frac{\partial \underline{x}}{\partial t}\end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\end{pmatrix} = \begin{pmatrix} \underline{x}_2-\underline{x}_1 & \underline{x}_3-\underline{x}_1\end{pmatrix} = \begin{pmatrix} x_2-x_1 & x_3-x_1 \\ y_2-y_1 & y_3-y_1\end{pmatrix}.$$

(b) Let $U(\underline{x})$ denote a linear function defined on T and let $U_i = U(\underline{x}_i)$, $i = 1,2,3$. Show that the gradient vector ∇U can be written as

$$\nabla U = \frac{1}{J}\begin{pmatrix} y_3-y_1 & y_1-y_2 \\ x_1-x_3 & x_2-x_1\end{pmatrix}\begin{pmatrix} U_2-U_1 \\ U_3-U_1\end{pmatrix}$$

where $J = (x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)$.

ANSWER

Now

$$\nabla U = \begin{pmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y}\end{pmatrix}.$$

By the chain rule of partial differentiation

$$\begin{pmatrix} \frac{\partial U(\underline{x}(s,t))}{\partial s} \\ \frac{\partial U(\underline{x}(s,t))}{\partial t}\end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t}\end{pmatrix}\begin{pmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y}\end{pmatrix} = \tilde{J}^T\begin{pmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y}\end{pmatrix}.$$

Thus

$$\nabla U = \begin{pmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y}\end{pmatrix} = \tilde{J}^{-T}\begin{pmatrix} \frac{\partial U(\underline{x}(s,t))}{\partial s} \\ \frac{\partial U(\underline{x}(s,t))}{\partial t}\end{pmatrix}.$$

Now

$$\begin{aligned} U(\underline{x})(s,t) &= U_1 + (U_2-U_1)s + (U_3-U_1)t, \\ \begin{pmatrix} \frac{\partial U(\underline{x}(s,t))}{\partial s} \\ \frac{\partial U(\underline{x}(s,t))}{\partial t}\end{pmatrix} &= \begin{pmatrix} U_2-U_1 \\ U_3-U_1\end{pmatrix}.\end{aligned}$$

As

$$\tilde{J}^{-1} = \frac{1}{J}\begin{pmatrix} y_3 - y_1 & -(x_3 - x_1) \\ -(y_2 - y_1) & x_2 - x_1 \end{pmatrix}, \quad \tilde{J}^{-T} = \frac{1}{J}\begin{pmatrix} y_3 - y_1 & -(y_2 - y_1) \\ -(x_3 - x_1) & x_2 - x_1 \end{pmatrix}$$

where $J = \det(\tilde{J}) = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$ we have the result.

(c) Given that we have reformulated Poisson's equation $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$ into weak form involving

$$a(u, v) = \iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega f v \, \mathrm{d}x\mathrm{d}y = (f, v),$$

explain what is meant by the 3×3 element stiffness matrix K_r for the triangle Ω_r and show that it can be written in the form

$$K_r = \frac{|J|}{2} B^T B \quad \text{where} \quad B = \tilde{J}^{-T}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

where, as above, $\tilde{J}$ is Jacobian matrix and $J = \det(\tilde{J})$.

ANSWER

Let $\tilde{\phi}_1$, $\tilde{\phi}_2$ and $\tilde{\phi}_3$ be the basis functions defined on Ω_r given by

$$\tilde{\phi}_i(\underline{x}(s, t)) = \phi_i(s, t), \quad i = 1, 2, 3.$$

The element stiffness matrix is the 3×3 matrix $(a(\tilde{\phi}_i, \tilde{\phi}_j))$.
Let

$$B = \begin{pmatrix} \frac{\partial \tilde{\phi}_1}{\partial x} & \frac{\partial \tilde{\phi}_2}{\partial x} & \frac{\partial \tilde{\phi}_3}{\partial x} \\ \frac{\partial \tilde{\phi}_1}{\partial y} & \frac{\partial \tilde{\phi}_2}{\partial y} & \frac{\partial \tilde{\phi}_3}{\partial y} \end{pmatrix}$$

which is constant on Ω_r. We have

$$(B^T B)_{ij} = \nabla \tilde{\phi}_i \cdot \nabla \tilde{\phi}_j \ .$$

Thus

$$K_r = (\text{area of } \Omega_r) B^T B.$$

The area of the triangle Ω_r is $|J|/2 = |\det(\tilde{J})|/2$.
To obtain an expression for B we use the chain rule of partial differentiation.

$$\begin{pmatrix} \frac{\partial \phi_1}{\partial s} & \frac{\partial \phi_2}{\partial s} & \frac{\partial \phi_3}{\partial s} \\ \frac{\partial \phi_1}{\partial t} & \frac{\partial \phi_2}{\partial t} & \frac{\partial \phi_3}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{\phi}_1}{\partial x} & \frac{\partial \tilde{\phi}_2}{\partial x} & \frac{\partial \tilde{\phi}_3}{\partial x} \\ \frac{\partial \tilde{\phi}_1}{\partial y} & \frac{\partial \tilde{\phi}_2}{\partial y} & \frac{\partial \tilde{\phi}_3}{\partial y} \end{pmatrix}.$$

Thus

$$\begin{aligned} \begin{pmatrix} \frac{\partial \tilde{\phi}_1}{\partial x} & \frac{\partial \tilde{\phi}_2}{\partial x} & \frac{\partial \tilde{\phi}_3}{\partial x} \\ \frac{\partial \tilde{\phi}_1}{\partial y} & \frac{\partial \tilde{\phi}_2}{\partial y} & \frac{\partial \tilde{\phi}_3}{\partial y} \end{pmatrix} &= \tilde{J}^{-T} \begin{pmatrix} \frac{\partial \phi_1}{\partial s} & \frac{\partial \phi_2}{\partial s} & \frac{\partial \phi_3}{\partial s} \\ \frac{\partial \phi_1}{\partial t} & \frac{\partial \phi_2}{\partial t} & \frac{\partial \phi_3}{\partial t} \end{pmatrix} \\ &= \tilde{J}^{-T} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \end{aligned}$$

(d) Explain why the matrix K_r given in part (c) is unchanged if we do any of the following.
We shift the vertices to $\underline{x}_i + \underline{c}$, $i = 1, 2, 3$, i.e. we do a translation.
We do a uniform scaling, i.e. $\underline{x}_i \to \alpha \underline{x}_i$, $i = 1, 2, 3$ where $\alpha > 0$.
We rotate the points $\underline{x}_i$ about a point or we reflect the points $\underline{x}_i$ about a line which corresponds to replacing the Jacobian matrix $\tilde{J}$ by $Q\tilde{J}$ where Q is a 2×2 orthogonal matrix.

ANSWER

From part (c) we have

$$K_r = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} C \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{where} \quad C = |\det(\tilde{J})|\tilde{J}^{-1}\tilde{J}^{-T}.$$

The Jacobian matrix of the map corresponding to the points $\underline{x}_i + \underline{c}$, $i = 1, 2, 3$ does not depend on $\underline{c}$ and hence C and K_r is independent of $\underline{c}$.
If we replace $\underline{x}_i$ by $\alpha \underline{x}_i$ then the Jacobian matrix $\tilde{J}_\alpha$ is given by

$$\tilde{J}_\alpha = \alpha \tilde{J} \quad \text{and} \quad \tilde{J}_\alpha^{-1} = \frac{1}{\alpha}\tilde{J}^{-1}.$$

As we have a 2×2 matrix

$$\det(\tilde{J}_\alpha) = \alpha^2 \det(\tilde{J}).$$

Hence C is independent of α and we have the same element stiffness matrix.
Let $\tilde{J}_Q = Q\tilde{J}$. As $Q^TQ = I$ we have $|\det(Q)| = 1$ and $|\det(\tilde{J}_Q)| = |\det(\tilde{J})|$. Also

$$\tilde{J}_Q^{-1} = \tilde{J}^{-1}Q^T \quad \text{and} \quad \tilde{J}_Q^{-T} = Q\tilde{J}^{-T}$$

so that

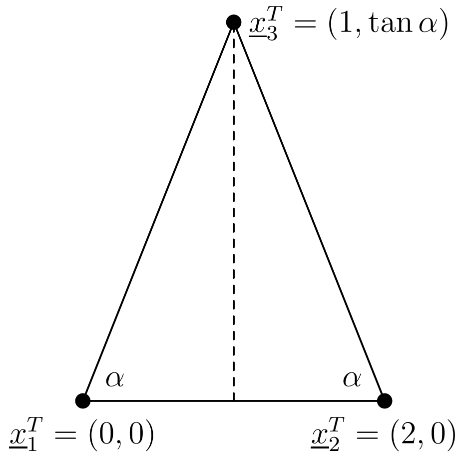
$$\tilde{J}_Q^{-1}\tilde{J}_Q^{-T} = \tilde{J}^{-1}\tilde{J}^{-T}$$

and again we get the same matrix C and element matrix K_r.

(e) In the case of the isosceles triangle Ω_r with vertices $\underline{x}_1^T = (0, 0)$, $\underline{x}_2^T = (2, 0)$ and $\underline{x}_3^T = (1, \tan\alpha)$ show that the element stiffness matrix is

$$K_r = \frac{1}{4\tan\alpha}\begin{pmatrix} \sec^2\alpha & 2 - \sec^2\alpha & -2 \\ 2 - \sec^2\alpha & \sec^2\alpha & -2 \\ -2 & -2 & 4 \end{pmatrix}.$$

[Note that we obtain from this the two specific cases covered in the notes by letting $\alpha = \pi/2$ (for the standard triangle) and letting $\alpha = \pi/3$ (for the equilateral triangle).]



ANSWER

The mapping and the Jacobian matrix are

$$\underline{x}(s,t) = \underline{x}_2 s + \underline{x}_3 t, \quad \tilde{J} = \begin{pmatrix} 2 & 1 \\ 0 & \tan\alpha \end{pmatrix}.$$

The determinant is $J = 2\tan\alpha$. The inverse of the Jacobian and its transpose are

$$\tilde{J}^{-1} = \frac{1}{J}\begin{pmatrix} \tan\alpha & -1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \tilde{J}^{-T} = \frac{1}{J}\begin{pmatrix} \tan\alpha & 0 \\ -1 & 2 \end{pmatrix}.$$

The matrix B is given by

$$B = \frac{1}{J}\begin{pmatrix} \tan\alpha & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\tan\alpha & \tan\alpha & 0 \\ -1 & -1 & 2 \end{pmatrix}.$$

Using $\sec^2\alpha = 1 + \tan^2\alpha$ we have

$$B^T B = \frac{1}{J^2}\begin{pmatrix} \sec^2\alpha & 1-\tan^2\alpha & -2 \\ 1-\tan^2\alpha & \sec^2\alpha & -2 \\ -2 & -2 & 4 \end{pmatrix} = \frac{1}{J^2}\begin{pmatrix} \sec^2\alpha & 2-\sec^2\alpha & -2 \\ 2-\sec^2\alpha & \sec^2\alpha & -2 \\ -2 & -2 & 4 \end{pmatrix}.$$

As

$$\frac{|J|}{2J^2} = \frac{1}{2J} = \frac{1}{4\tan\alpha}$$

we have the required result.

4. The following was question 4 of the June 2003 `MA3056S` paper.

   Let Ω denote a bounded domain with a polygonal boundary $\partial\Omega$ and consider the Poisson equation

   $$-\Delta u = 1 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

   (a) Show that the weak form for this problem involves the expression

   $$\iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega v \, \mathrm{d}x\mathrm{d}y$$

   and state an appropriate space of functions in which u and v should lie.

[5 MARKS]

(b) Suppose that the domain Ω is partitioned into n triangular elements $\Omega_1, \cdots, \Omega_n$ such that

$$\bar{\Omega} = \bigcup_{i=1}^{n} \bar{\Omega}_i, \quad \Omega_i \cap \Omega_j = \text{empty set for } i \neq j$$

and let T denote the standard triangle with vertices $\underline{s}_1^T = (0,0)$, $\underline{s}_2^T = (1,0)$ and $\underline{s}_3^T = (0,1)$ in the (s,t) plane.

(i) State the 3 standard Lagrange basis functions ϕ_1, ϕ_2 and ϕ_3 defined on T such that $\phi_i(\underline{s}_i) = 1$ and $\phi_i(\underline{s}_j) = 0$ if $j \neq i$.

[1 MARK]

(ii) Suppose that the triangle Ω_i has vertices $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$ and let $U_i = U(\underline{x}_i)$, $i = 1,2,3$ denote the values of the function $U(\underline{x})$ which is linear for $\underline{x} \in \Omega_i$. Describe the mapping $\underline{x} : T \to \Omega_i$ and describe the function $U(\underline{x})$ at any point $\underline{x} \in \Omega_i$ using ϕ_1, ϕ_2 and ϕ_3 of part (i). Also give the Jacobian matrix $\tilde{J}$ of the mapping $\underline{x} : T \to \Omega_i$ in terms of $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$.

[4 MARKS]

(iii) Explain what is meant by the element stiffness matrix K_i in this case of Poisson's equation and piecewise linear basis functions and show that K_i can be written in the form

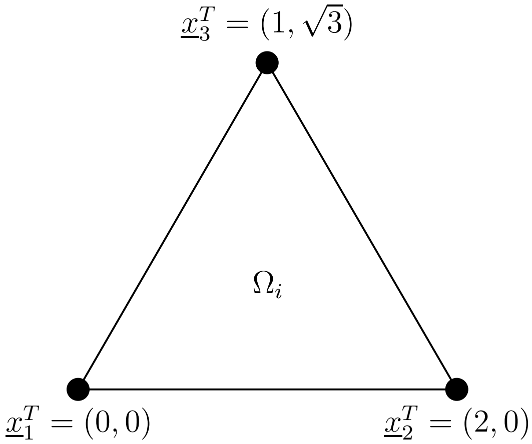
$$K_i = \frac{|\det \tilde{J}|}{2} \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{J}^{-1} \tilde{J}^{-T} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

[5 MARKS]

(iv) If Ω_i is the equilateral triangle shown below with vertices $\underline{x}_1^T = (0,0)$, $\underline{x}_2^T = (2,0)$ and $\underline{x}_3^T = (1,\sqrt{3})$ then give an affine mapping $\underline{x} : T \to \Omega_i$ and show that

$$K_i = \frac{1}{2\sqrt{3}} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

[3 MARKS]

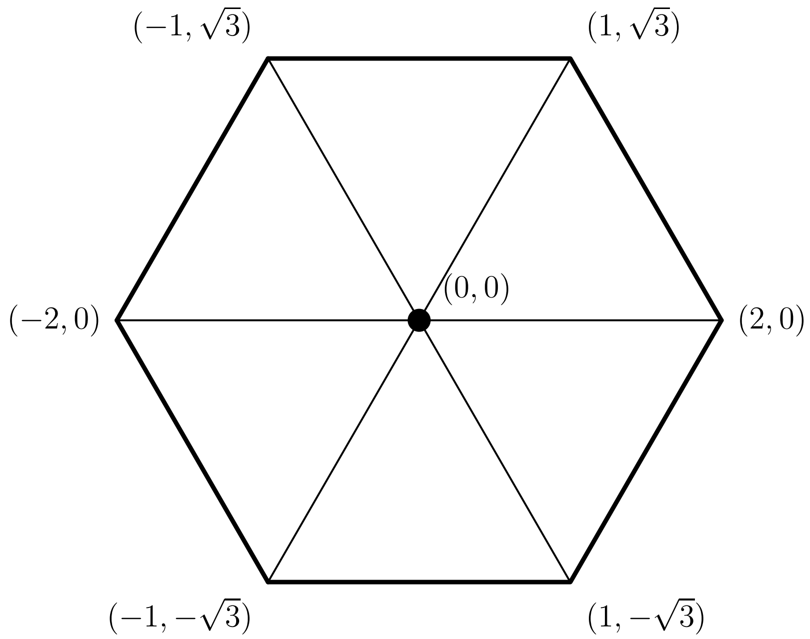


Suppose that Ω is the hexagonal domain which is the union of the 6 equilateral triangles shown below. Given that the element contribution to the right hand side vector for each triangle is

$$\iint_{\Omega_i} \begin{pmatrix} \tilde{\phi}_1(\underline{x}) \\ \tilde{\phi}_2(\underline{x}) \\ \tilde{\phi}_3(\underline{x}) \end{pmatrix} \mathrm{d}x\mathrm{d}y = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where $\tilde{\phi}_1$, $\tilde{\phi}_2$ and $\tilde{\phi}_3$ are the linear basis functions on Ω_i, and that all the element matrices are the same and are as given above, determine the finite element solution at the centre $(0,0)$ using piecewise linears on this mesh of 6 triangles.

[2 MARKS]



ANSWER

(a) From the vector identity

$$\nabla \cdot (v\nabla u) = v\Delta v + \nabla v \cdot \nabla u$$

we have by using the divergence theorem that

$$\begin{aligned} -\iint_\Omega v\Delta u \,\mathrm{d}x\mathrm{d}y &= \iint_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\mathrm{d}y - \int_{\partial\Omega} v\nabla u \cdot \underline{n} \,\mathrm{d}s \\ &= \iint_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\mathrm{d}y - \int_{\partial\Omega} v\frac{\partial u}{\partial n} \,\mathrm{d}s \\ &= \iint_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\mathrm{d}y \end{aligned}$$

if we restrict to functions v such that $v = 0$ on $\partial\Omega$. Hence if we take

$$V = \{v \in C^2(\bar{\Omega}) : \ v = 0 \ \text{ on } \partial\Omega\}$$

then the exact solution $u \in V$ satisfies

$$\iint_\Omega \nabla u \cdot \nabla v \; \mathrm{d}x\mathrm{d}y = \iint_\Omega v \; \mathrm{d}x\mathrm{d}y \quad \text{for all } v \in V.$$

5 MARKS

(b) (i) The 3 functions are $\phi_1(s,t) = 1 - s - t$, $\phi_2(s,t) = s$ and $\phi_3(s,t) = t$.

1 MARK

(ii) The mapping and the approximation are given by

$$\begin{aligned} \underline{x}(s,t) &= \underline{x}_1\phi_1(s,t) + \underline{x}_2\phi_2(s,t) + \underline{x}_3\phi_3(s,t) \\ &= \underline{x}_1 + (\underline{x}_2 - \underline{x}_1)s + (\underline{x}_3 - \underline{x}_1)t, \\ U(\underline{x}(s,t)) &= U_1\phi_1(s,t) + U_2\phi_2(s,t) + U_3\phi_3(s,t) \\ &= U_1 + (U_2 - U_1)s + (U_3 - U_1)t. \end{aligned}$$

The Jacobian matrix $\tilde{J}$ is

$$\tilde{J} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \bar{\underline{x}}_2 & \bar{\underline{x}}_3 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}$$

where $\bar{\underline{x}}_2 = \underline{x}_2 - \underline{x}_1$ and $\bar{\underline{x}}_3 = \underline{x}_3 - \underline{x}_1$.

4 MARKS

(iii) Let $\tilde{\phi}_1$, $\tilde{\phi}_2$ and $\tilde{\phi}_3$ denote the basis functions defined on Ω_i such that

$$\tilde{\phi}_j(\underline{x}(s,t)) = \phi_j(s,t), \quad j = 1,2,3.$$

Also let

$$a(u,v)_i = \iint_{\Omega_i} \nabla u \cdot \nabla v \; \mathrm{d}x\mathrm{d}y.$$

The element stiffness matrix is the 3×3 matrix given by $K_i = (a(\tilde{\phi}_j, \tilde{\phi}_k)_i)$. Now by the chain rule we have in matrix form

$$\begin{pmatrix} \frac{\partial \phi_i}{\partial s} \\ \frac{\partial \phi_i}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{\phi}_i}{\partial x} \\ \frac{\partial \tilde{\phi}_i}{\partial y} \end{pmatrix} = \tilde{J}^T \begin{pmatrix} \frac{\partial \tilde{\phi}_i}{\partial x} \\ \frac{\partial \tilde{\phi}_i}{\partial y} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \frac{\partial \tilde{\phi}_i}{\partial x} \\ \frac{\partial \tilde{\phi}_i}{\partial y} \end{pmatrix} = \tilde{J}^{-T} \begin{pmatrix} \frac{\partial \phi_i}{\partial s} \\ \frac{\partial \phi_i}{\partial t} \end{pmatrix}.$$

Let

$$\begin{aligned} B = \begin{pmatrix} \frac{\partial \tilde{\phi}_1}{\partial x} & \frac{\partial \tilde{\phi}_2}{\partial x} & \frac{\partial \tilde{\phi}_3}{\partial x} \\ \frac{\partial \tilde{\phi}_1}{\partial y} & \frac{\partial \tilde{\phi}_2}{\partial y} & \frac{\partial \tilde{\phi}_3}{\partial y} \end{pmatrix} &= \tilde{J}^{-T} \begin{pmatrix} \frac{\partial \phi_1}{\partial s} & \frac{\partial \phi_2}{\partial s} & \frac{\partial \phi_3}{\partial s} \\ \frac{\partial \phi_1}{\partial t} & \frac{\partial \phi_2}{\partial t} & \frac{\partial \phi_3}{\partial t} \end{pmatrix} \\ &= \tilde{J}^{-T} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The jk entry of B^TB is the integrand in $a(\tilde\phi_j,\tilde\phi_k)_i$. As the integrand is constant over Ω_i it follows that

$$K_i = (\text{area of } K_i) B^T B$$

and as the area of K_i is $|J|/2$ where $J = \det \tilde J$ the result follows.

5 MARKS

(iv) The mapping is

$$\underline{x}(s,t) = (\underline{x}_2 - \underline{x}_1)s + (\underline{x}_3 - \underline{x}_1)t = \begin{pmatrix} 2s+t \\ \sqrt{3}t \end{pmatrix}$$

and the Jacobian matrix is

$$\tilde J = \begin{pmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{pmatrix}.$$

$J = 2\sqrt{3}$,

$$\tilde J^{-1} = \frac{1}{J}\begin{pmatrix} \sqrt{3} & -1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \tilde J^{-T} = \frac{1}{J}\begin{pmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}.$$

Thus

$$B = \frac{1}{J}\begin{pmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{J}\begin{pmatrix} -\sqrt{3} & \sqrt{3} & 0 \\ -1 & -1 & 2 \end{pmatrix}.$$

Hence

$$B^TB = \frac{1}{J^2}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix}.$$

As $|J|/(2J^2) = 1/(2|J|) = 1/4\sqrt{3}$ the result follows.

3 MARKS

If $\hat\phi$ is the piecewise linear function associated with the point $(0,0)$ then the equation that $U(0,0)$ satisfies is

$$a(\hat\phi,\hat\phi)U(0,0) = (1,\hat\phi).$$

Because of the symmetry we get equal contributions to $a(\hat\phi,\hat\phi)$ and $(1,\hat\phi)$ from each of the 6 triangles. Hence $a(\hat\phi,\hat\phi) = 6/\sqrt{3}$ and $(1,\phi) = 6/\sqrt{3}$ and we get

$$U(0,0) = 1.$$

2 MARKS

5. The following was question 5 of the June 2003 MA3056S paper.
Let $\Omega = \{(x, y) :\ 0 < x < 2,\ 0 < y < 1\}$, a 2×1 rectangle, and consider Poisson's equation with the mixed boundary conditions

$$\begin{aligned}
&-\Delta u = f \quad \text{in } \Omega,\\
&u(0, y) = u(2, y) = 0, \quad 0 < y < 1,\\
&\frac{\partial u}{\partial y}(x, 0) = \frac{\partial u}{\partial y}(x, 1) = 0, \quad 0 < x < 2,
\end{aligned}$$

where f is a continuous function on $\bar{\Omega}$.

(a) Show that the weak form for the problem can be written as follows.
Find $u \in V$ such that

$$a(u, v) := \iint_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\mathrm{d}y = \iint_\Omega fv \,\mathrm{d}x\mathrm{d}y =: (f, v)$$

for all $v \in V$. In your answer you should give the space of functions V taking into account any natural boundary conditions.

[5 MARKS]

(b) Suppose that the weak form is to be approximately solved by the finite element method using a piecewise bilinear function defined on a uniform mesh of square elements each of length h. In an implementation each square element is mapped to a standard element $S = \{(s, t) :\ -1 < s, t < 1\}$ with vertices $\underline{s}_1^T = (-1, -1)$, $\underline{s}_2^T = (1, -1)$, $\underline{s}_3^T = (1, 1)$ and $\underline{s}_4^T = (-1, 1)$ in the (s, t) plane.

(i) State the 4 standard basis functions ϕ_1, ϕ_2, ϕ_3 and ϕ_4 defined on S such that $\phi_i(\underline{s}_i) = 1$ and $\phi_i(\underline{s}_j) = 0$ for $j \neq i$.

[2 MARKS]

(ii) Let $\underline{x}_1^T = (c, d)$, $\underline{x}_2^T = (c + h, d)$, $\underline{x}_3^T = (c + h, d + h)$ and $\underline{x}_4^T = (c, d + h)$ be the vertices of an actual element Ω_i and let U_1, U_2, U_3 and U_4 denote respectively the value of the approximation at these points. Let $\underline{x} : S \to \Omega_i$ denote the mapping of the standard element to Ω_i. Explain why this can be written in the form

$$\underline{x} = \begin{pmatrix} c \\ d \end{pmatrix} + \frac{h}{2} \begin{pmatrix} 1 + s \\ 1 + t \end{pmatrix}.$$

[1 MARK]

Further show that the gradient vector can be written in the form

$$\nabla U(\underline{x}(s, t)) = \frac{1}{2h} \begin{pmatrix} t - 1 & 1 - t & t + 1 & -(t + 1) \\ s - 1 & -(s + 1) & s + 1 & 1 - s \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}.$$

[3 MARKS]

(iii) It can be shown that the element stiffness matrix for this problem has the form

$$\begin{pmatrix} 2\beta & \gamma & -\beta & \gamma \\ \gamma & 2\beta & \gamma & -\beta \\ -\beta & \gamma & 2\beta & \gamma \\ \gamma & -\beta & \gamma & 2\beta \end{pmatrix}.$$

Show that $\beta = 1/3$ and $\gamma = -1/6$.
[Hint: First explain why $\beta + 2\gamma = 0$.]

[4 MARKS]

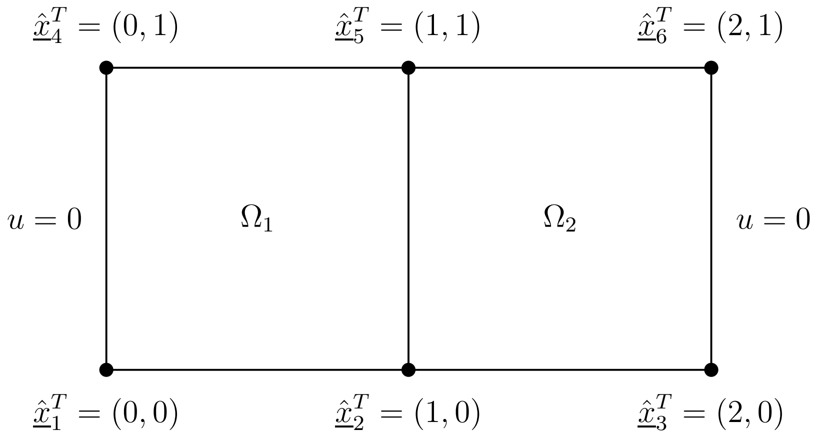
(iv) Consider the mesh involving the two squares shown in the figure. Let $\hat\phi_1$, $\hat\phi_2$, $\hat\phi_3$, $\hat\phi_4$, $\hat\phi_5$ and $\hat\phi_6$ be the 6 piecewise bilinear functions defined on $\bar\Omega$ which satisfy $\hat\phi_i(\underline{\hat x}_i) = 1$ and $\hat\phi_i(\underline{\hat x}_j) = 0$ for $j \ne i$. Describe $\hat\phi_2(\underline{x})$ and $\hat\phi_5(\underline{x})$ for all $\underline{x} \in \Omega$.

[3 MARKS]

By using the element stiffness matrix given in part (iii), determine the entries $a(\hat\phi_2, \hat\phi_2)$, $a(\hat\phi_2, \hat\phi_5)$ and $a(\hat\phi_5, \hat\phi_5)$ in the linear system

$$\begin{pmatrix} a(\hat\phi_2, \hat\phi_2) & a(\hat\phi_2, \hat\phi_5) \\ a(\hat\phi_2, \hat\phi_5) & a(\hat\phi_5, \hat\phi_5) \end{pmatrix} \begin{pmatrix} U(\underline{\hat x}_2) \\ U(\underline{\hat x}_5) \end{pmatrix} = \begin{pmatrix} (f, \hat\phi_2) \\ (f, \hat\phi_5) \end{pmatrix}.$$

[2 MARKS]



ANSWER

(a) From the vector identity

$$\nabla \cdot (v \nabla u) = v \Delta v + \nabla v \cdot \nabla u$$

we have using the divergence theorem that

$$\begin{aligned} -\iint_\Omega v \Delta u \, \mathrm{d}x\mathrm{d}y &= \iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y - \int_{\partial\Omega} v \nabla u \cdot \underline{n} \, \mathrm{d}s \\ &= \iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y - \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, \mathrm{d}s \\ &= \iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y \end{aligned}$$

if the integrand of the boundary term vanishes on $\partial\Omega$. We are given that the normal derivative of u vanishes on the top and bottom sides of the rectangle and thus to remove the boundary term we only need to restrict to functions v such that $v = 0$ of the other two sides. Hence if we take

$$V = \{v \in C^2(\bar{\Omega}) : \ v(0, y) = v(2, y) = 0, \ 0 \le y \le 1\}$$

then the exact solution $u \in V$ satisfies

$$\iint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega f v \, \mathrm{d}x\mathrm{d}y \quad \text{for all } v \in V.$$

The boundary condition $\partial u/\partial x = 0$ on two of the sides is a natural boundary condition for this problem.

5 MARKS

(b) (i)

$$\begin{aligned} \phi_1(s,t) &= \frac{1}{4}(1-s)(1-t), \\ \phi_2(s,t) &= \frac{1}{4}(1+s)(1-t), \\ \phi_3(s,t) &= \frac{1}{4}(1+s)(1+t), \\ \phi_4(s,t) &= \frac{1}{4}(1-s)(1+t). \end{aligned}$$

2 MARKS

(ii) Since $\phi_1 + \phi_2 + \phi_3 + \phi_4 = 1$ the mapping $\underline{x}(s,t)$ is given by

$$\begin{aligned} \underline{x} &= \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} h \\ 0 \end{pmatrix} \phi_2(s,t) + \begin{pmatrix} h \\ h \end{pmatrix} \phi_3(s,t) + \begin{pmatrix} 0 \\ h \end{pmatrix} \phi_4(s,t) \\ &= \begin{pmatrix} c \\ d \end{pmatrix} + \frac{h}{2} \begin{pmatrix} 1+s \\ 1+t \end{pmatrix}. \end{aligned}$$

1 MARK

Thus

$$\frac{\partial x}{\partial s} = \frac{\partial y}{\partial t} = \frac{h}{2} \quad \text{and} \quad \frac{\partial x}{\partial t} = \frac{\partial y}{\partial s} = 0.$$

Hence

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} = \frac{2}{h}.$$

Now

$$U(\underline{x}(s,t)) = U_1\phi_1(s,t) + U_2\phi_2(s,t) + U_3\phi_3(s,t) + U_4\phi_4(s,t).$$

Thus

$$\begin{aligned} \begin{pmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{\partial U}{\partial y} \end{pmatrix} &= \frac{2}{h} \begin{pmatrix} \dfrac{\partial U}{\partial s} \\ \dfrac{\partial U}{\partial t} \end{pmatrix} = \frac{2}{h} \begin{pmatrix} \dfrac{\partial \phi_1}{\partial s} & \dfrac{\partial \phi_2}{\partial s} & \dfrac{\partial \phi_3}{\partial s} & \dfrac{\partial \phi_4}{\partial s} \\ \dfrac{\partial \phi_1}{\partial t} & \dfrac{\partial \phi_2}{\partial t} & \dfrac{\partial \phi_3}{\partial t} & \dfrac{\partial \phi_4}{\partial t} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} \\ &= \frac{1}{2h} \begin{pmatrix} t-1 & 1-t & t+1 & -(t+1) \\ s-1 & -(s+1) & s+1 & 1-s \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}. \end{aligned}$$

3 MARKS

(iii) Since $\phi_1+\phi_2+\phi_3+\phi_4=1$ we get $\nabla(\phi_1+\phi_2+\phi_3+\phi_4)=\underline{0}$ which implies that the entries on each row of the stiffness matrix add to 0. Thus $\beta+2\gamma=0$. Let $\tilde{\phi}_j$ be such that $\tilde{\phi}_j(\underline{x}(s,t))=\phi_j(s,t)$. From part (ii) we have

$$\nabla\tilde{\phi}_j=\frac{2}{h}\begin{pmatrix}\dfrac{\partial\phi_j}{\partial s}\\ \dfrac{\partial\phi_j}{\partial t}\end{pmatrix}\quad\text{and}\quad J=h^2/4$$

where J is the determinant of the Jacobian matrix.
The entries of the element stiffness matrix are

$$\iint_{\Omega_i}\nabla\tilde{\phi}_k\cdot\nabla\tilde{\phi}_l\,\mathrm{d}x\mathrm{d}y=\int_{-1}^{1}\int_{-1}^{1}\frac{\partial\phi_k}{\partial s}\frac{\partial\phi_l}{\partial s}+\frac{\partial\phi_k}{\partial t}\frac{\partial\phi_l}{\partial t}\,\mathrm{d}s\mathrm{d}t\quad 1\le k,l\le 4.$$

Hence

$$2\beta=\frac{1}{16}\int_{-1}^{1}\int_{-1}^{1}(t-1)^2+(s-1)^2\,\mathrm{d}s\mathrm{d}t$$

We have

$$\int_{-1}^{1}\int_{-1}^{1}(t-1)^2\,\mathrm{d}s\mathrm{d}t=\int_{-1}^{1}\int_{-1}^{1}(s-1)^2\,\mathrm{d}s\mathrm{d}t=2\left[\frac{(s-1)^3}{3}\right]_{-1}^{1}=\frac{16}{3}.$$

Hence

$$2\beta=\frac{1}{16}\left(\frac{16}{3}+\frac{16}{3}\right)=\frac{2}{3}.$$

Thus $\beta=1/3$ and $\gamma=-1/6$.

4 MARKS

(iv)

$$\hat{\phi}_2(x)=\begin{cases}x(1-y), & \text{in }\Omega_1,\\ (2-x)(1-y), & \text{in }\Omega_2,\end{cases}$$

$$\hat{\phi}_5(x)=\begin{cases}xy, & \text{in }\Omega_1,\\ (2-x)y, & \text{in }\Omega_2.\end{cases}$$

3 MARKS

$a(\hat{\phi}_2,\hat{\phi}_2)$ and $a(\hat{\phi}_5,\hat{\phi}_5)$ involve adding a diagonal entry of the matrix for Ω_1 with a diagonal entry of the matrix for Ω_2. We have $a(\hat{\phi}_2,\hat{\phi}_2)=a(\hat{\phi}_5,\hat{\phi}_5)=4/3$. The $a(\hat{\phi}_2,\hat{\phi}_5)$ entry has a contribution of $\gamma=-1/6$ from each element. Thus $a(\hat{\phi}_2,\hat{\phi}_5)=-1/3$.

2 MARKS

6. Apart from a few minor changes and re-typing this was question 2 of the 2001 MA5056S paper. (MA5156S was previously labelled as MA5056S).

For the boundary value problem in which $u(\underline{x}) = u(x,y)$ satisfies $-\Delta u(\underline{x}) = f(\underline{x})$, $\underline{x} \in \Omega$; $u(\underline{x}) = 0$, $\underline{x} \in \partial\Omega$, where $\Omega \subset \mathbb{R}^2$ has a polygonal boundary $\partial\Omega$ derive the weak formulation

$$a(u,v) = \iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \iint_\Omega fv \; \mathrm{d}x\mathrm{d}y \quad \text{for all } v \in S,$$

giving the appropriate space of functions S.

[4 MARKS]

ANSWER

From the vector identity

$$\nabla \cdot (v\nabla u) = v\Delta u + \nabla u \cdot \nabla v$$

we obtain

$$-v\Delta u = \nabla u \cdot \nabla v - \nabla \cdot (v\nabla u).$$

Then by the divergence theorem

$$-\iint_\Omega v\Delta u \; \mathrm{d}x\mathrm{d}y = \iint_\Omega \nabla u \cdot \nabla v \; \mathrm{d}x\mathrm{d}y - \int_{\partial\Omega} v\frac{\partial u}{\partial n} \; \mathrm{d}s.$$

If we take $v = 0$ on $\partial\Omega$ then the boundary integral term is 0. Hence we define

$$V = \{v \in H^1(\Omega) : \quad v = 0 \text{ on } \partial\Omega\}.$$

The weak form is find $u \in V$ such that

$$\iint_\Omega \nabla u \cdot \nabla v \; \mathrm{d}x\mathrm{d}y = \iint_\Omega fv \; \mathrm{d}x\mathrm{d}y \quad \forall v \in V.$$

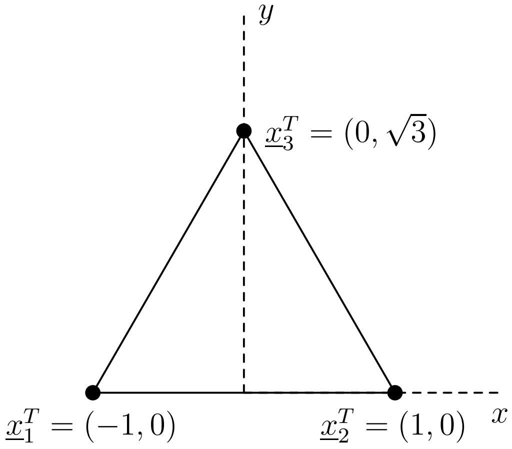
The weak problem is to be approximated using a finite element method based on piecewise linear functions defined over a triangular partition of Ω. The finite element method approximation is $U_h(\underline{x})$. What is meant by a finite element method being conforming and what conditions must $U_h(\underline{x})$ satisfy in order to be a conforming approximation?

[2 MARKS]

ANSWER

The finite element method is conforming if the finite element space V_h is a subspace of the space V in the weak formulation. A piecewise polynomial function is in V if it is continuous, i.e. $V_h \subset C(\bar{\Omega})$.

Suppose the standard triangular element is taken as the equilateral triangle T shown in the Figure with nodes $\underline{x}_1^T = (-1, 0)$, $\underline{x}_2^T = (1, 0)$, $\underline{x}_3^T = (0, \sqrt{3})$.



What are meant by the Lagrange basis functions on an element and what conditions must they satisfy?

[1 MARK]

ANSWER

The linear Lagrange basis function are linear polynomials which are 1 at one of the nodes and 0 at the other nodes.

(a) Write down the three linear Lagrange basis functions $\phi_1(x, y)$, $\phi_2(x, y)$ and $\phi_3(x, y)$.

[4 MARKS]

ANSWER

We can arrange for $\phi_1(\underline{x})$ to be 0 at $\underline{x}_2$ and at $\underline{x}_3$ if is is 0 along the line passing through points $\underline{x}_2$ and $\underline{x}_3$. This line is

$$y = \sqrt{3}(1 - x).$$

Hence

$$\phi_1(x, y) = C(y + \sqrt{3}(x - 1)) \quad \text{with} \quad 1 = \phi_1(-1, 0) = C(-2\sqrt{3}).$$

Thus

$$\phi_1(x, y) = \frac{1}{2\sqrt{3}}(\sqrt{3}(1 - x) - y).$$

Similarly for ϕ_2 we get

$$\phi_2(x, y) = \frac{1}{2\sqrt{3}}(\sqrt{3}(1 + x) - y)$$

and for $\phi_3(x, y)$ we get

$$\phi_3(x, y) = \frac{y}{\sqrt{3}}.$$

(b) Hence write down the approximation $U_h(x, y)$ for (x, y) in T.

[1 MARK]

ANSWER

Let $U_i = U_h(\underline{x}_i)$ for $i = 1, 2, 3$. Then

$$\begin{aligned} U_h(x, y) &= U_1\phi_1(x, y) + U_2\phi_2(x, y) + U_3\phi_3(x, y) \\ &= U_1\frac{1}{2\sqrt{3}}(\sqrt{3}(1-x) - y) + U_2\frac{1}{2\sqrt{3}}(\sqrt{3}(1+x) - y) + U_3\frac{y}{\sqrt{3}}. \end{aligned}$$

(c) Derive the 3×3 element matrix arising from $a(u, v)|_T$.

[6 MARKS]

ANSWER

The gradient of each of the basis functions is constant on the triangle and the area of the triangle is $\sqrt{3}$. Thus the i, j entry of element matrix is

$$\sqrt{3}\nabla\phi_i \cdot \nabla\phi_j.$$

Now for the gradients we have

$$B = \begin{pmatrix} \nabla\phi_1 & \nabla\phi_2 & \nabla\phi_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} \end{pmatrix}.$$

The element matrix is

$$\sqrt{3}B^TB = \frac{\sqrt{3}}{4}\begin{pmatrix} \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{4}{3} \end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

(d) Without evaluating the integrals give the 3×1 element vector arising from $(f, v)|_T$.

[1 MARK]

ANSWER

The element vector is

$$\iint_T f(x, y)\begin{pmatrix} \phi_1(x, y) \\ \phi_2(x, y) \\ \phi_3(x, y) \end{pmatrix} \mathrm{d}x\mathrm{d}y.$$

(e) Explain how the linear form for $U_h(x, y)$ as in (b) will produce a conforming finite element approximation.

[1 MARK]

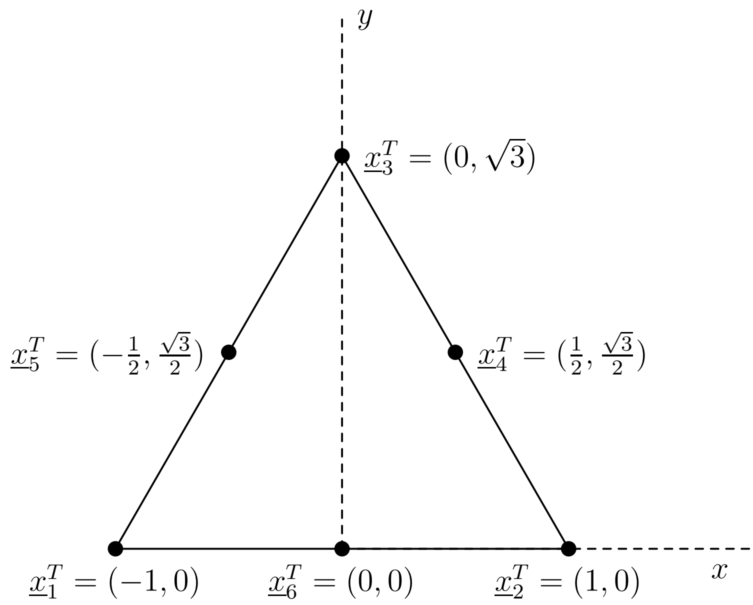
ANSWER

On the edge $\underline{x}_1$ to $\underline{x}_2$ the basis function ϕ_3 is 0 and thus $U_h(\underline{x})$ depends only on U_1 and U_2. Similarly on the other two edges the approximation depends only the value of U_h at the vertices of that edge. Thus at a point $\underline{x}_p$ on an edge between two adjacent triangles we obtain the same value $U_h(\underline{x}_p)$ as $\underline{x} \to \underline{x}_p$ from either of these triangles indicating that $U_h(\underline{x})$ is continuous across element edges and this sufficient to ensure a conforming approximation.

(f) For the equilateral triangle shown below with the six nodes

$$\underline{x}_1^T = (-1, 0), \quad \underline{x}_2^T = (1, 0), \quad \underline{x}_3^T = (0, \sqrt{3}),$$
$$\underline{x}_4^T = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}), \quad \underline{x}_5^T = (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}), \quad \underline{x}_6^T = (0, 0)$$

write down the six quadratic Lagrange basis functions $N_j(x, y)$, $i = 1, 2, 3, 4, 5, 6$. [Hint: The functions N_j can be expressed in terms of the ϕ_i of part (a).]



[5 MARKS]

ANSWER

We can arrange for the function N_i to be quadratic and 0 at 5 of the points if it is zero on two different straight lines which contain these 5 points.

In the case of N_1 we have the line passing through $\underline{x}_2$, $\underline{x}_4$ and $\underline{x}_3$ (used in the construction of ϕ_1) and we have the line passing through $\underline{x}_5$, and $\underline{x}_6$. The line passing through $\underline{x}_5$ and $\underline{x}_6$ is

$$y = -\sqrt{3}x.$$

This gives a factor

$$-\frac{1}{\sqrt{3}}(\sqrt{3}x + y).$$

which is 1 at the point $\underline{x}_1$. Thus

$$N_1(x, y) = -\frac{1}{\sqrt{3}}(\sqrt{3}x + y)\phi_1(x, y).$$

Similarly

$$\begin{aligned} N_2(x, y) &= \frac{1}{\sqrt{3}}(\sqrt{3}x - y)\phi_2(x, y), \\ N_3(x, y) &= \left(\frac{2y}{\sqrt{3}} - 1\right)\phi_3(x, y). \end{aligned}$$

Now $\phi_2(\frac{1}{2}, \frac{\sqrt{3}}{2}) = \phi_3(\frac{1}{2}, \frac{\sqrt{3}}{2}) = \frac{1}{2}$. Simlarly for the other ϕ_i functions when evaluated at the appropriate mid-side points. Hence

$$\begin{aligned} N_4(x, y) &= 4\phi_2(x, y)\phi_3(x, y), \\ N_5(x, y) &= 4\phi_1(x, y)\phi_3(x, y), \\ N_6(x, y) &= 4\phi_1(x, y)\phi_2(x, y). \end{aligned}$$

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7. This question is taken from the last part of question 2 of the 1998 MA3056S paper.

   The context of the question is that of Poisson's equation $-\Delta u = f$ which has been reformulated into the weak form

$$a(u, v) = (f, v), \ \forall v \in S$$

   where

$$a(u, v) = \iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) \mathrm{d}x\mathrm{d}y \quad \text{and} \quad (f, v) = \iint_\Omega fv \, \mathrm{d}x\mathrm{d}y.$$

   An earlier part of the question has also involved deriving the element matrix for the standard triangle T with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Using the basis functions $\phi_1(x, y) = 1 - x - y$, $\phi_2(x, y) = x$ and $\phi_3(x, y) = y$ the matrix is

$$K_i = \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

   (i) Give the 3×1 element vector arising from $(f, v)|_T$.

ANSWER

$$\iint_T f(x, y)\begin{pmatrix} \phi_1(x, y) \\ \phi_2(x, y) \\ \phi_3(x, y) \end{pmatrix} \mathrm{d}x\mathrm{d}y.$$

   (ii) Give the approximation to the element vector obtained when a one-point quadrature rule is used to evaluate the integrals.

ANSWER

The one-point quadrature rule involves evaluating at the centroid which is the point $(1/3, 1/3)$. We have $\phi_i(1/3, 1/3) = 1/3$, $i = 1, 2, 3$. As the area of the standard triangle is $1/2$ the approximation of the element vector is

$$\frac{1}{6} f(1/3, 1/3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let Ω be the unit square $\{(x, y) : 0 < x, y < 1\}$ and $\partial\Omega_D \equiv \{(x, 0) : 0 \le x \le 1\}$. Consider the triangular meshes shown in Fig. A and B, and in each case construct the 4×4 global stiffness matrix, by assembling the appropriate 3×3 element stiffness matrices, and the 4×1 global vector assuming that the one-point quadrature rule has been used. Then from the relevant linear equation systems satisfied by $(u_h(0, 1), u_h(1, 1))^T$, show that both meshes lead to the same solution only when

$$f(1/3, 1/3) + f(2/3, 2/3) = f(1/3, 2/3) \quad \text{and} \quad f(2/3, 1/3) + f(1/3, 1/3) = f(2/3, 2/3)\ .$$

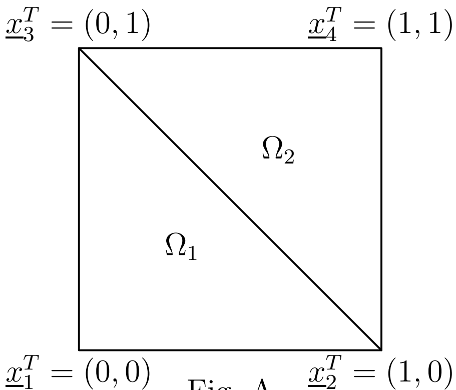

Fig. A

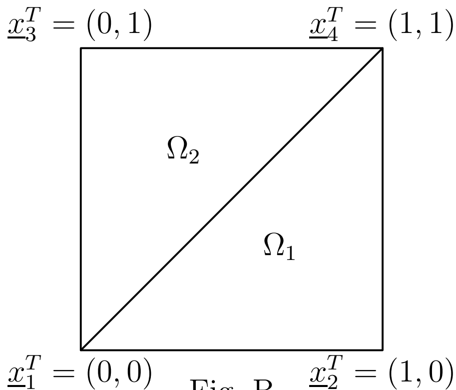

Fig. B

ANSWER

If we assemble the element matrices and just consider the part relating to nodes $\underline{x}_3$ and $\underline{x}_4$ then using the mesh of Fig. A the matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and using the mesh of Fig.B the matrix is

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

We have the same matrix from both meshes.

Assembling the element vectors for the mesh of Fig. A and retaining only the terms relating to nodes $\underline{x}_3$ and $\underline{x}_4$ we get

$$\frac{1}{6} f(1/3, 1/3) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{6} f(2/3, 2/3) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For the mesh of Fig. B the element centroids are instead $(1/3, 2/3)$ and $(2/3, 1/3)$ and we get

$$\frac{1}{6}f(2/3,1/3)\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{6}f(1/3,2/3)\begin{pmatrix}1\\1\end{pmatrix}.$$

To obtain the same solution from both meshes we need the same right hand side vector. That is we need

$$\begin{aligned} f(1/3,1/3)+f(2/3,2/3) &= f(1/3,2/3) \\ f(2/3,2/3) &= f(2/3,1/3)+f(1/3,2/3). \end{aligned}$$

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8. Apart from a few minor changes and re-typing this was question 2 of the 1999 `MA3056S` paper.

   For the mixed boundary value problem in which $u(x, y)$ satisfies

$$\begin{aligned} &-\Delta(u(x,y)) = f(x,y), && (x,y)\in\Omega, \\ &u(x,y)=0, && (x,y)\in\partial\Omega_D, \\ &\frac{\partial u}{\partial n}(x,y)=0, && (x,y)\in\partial\Omega_N \end{aligned}$$

   where $\Omega \subset \mathbb{R}^2$ is a rectangle with boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ and $\partial\Omega_D \cap \partial\Omega_N$ =empty set and $\frac{\partial}{\partial n}$ is the derivative in the outward normal direction to $\partial\Omega_N$, derive the weak formulation: find $u \in V$ such that

$$a(u,v)=\iint_\Omega\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)\,\mathrm{d}x\mathrm{d}y=\iint_\Omega fv\,\mathrm{d}x\mathrm{d}y=(f,v)\quad\forall v\in V,$$

   describing the appropriate space V.

ANSWER

From the vector identity

$$\nabla\cdot(v\nabla u)=v\Delta u+\nabla u\cdot\nabla v$$

we obtain

$$-v\Delta u=\nabla u\cdot\nabla v-\nabla\cdot(v\nabla u).$$

Then by the divergence theorem

$$-\iint_\Omega v\Delta u\,\mathrm{d}x\mathrm{d}y=\iint_\Omega\nabla u\cdot\nabla v\,\mathrm{d}x\mathrm{d}y-\int_{\partial\Omega}v\frac{\partial u}{\partial n}\,\mathrm{d}s.$$

As $\frac{\partial u}{\partial n}=0$ on the part $\partial\Omega_N$ we only need v to vanish on the other part of $\partial\Omega$ to remove the boundary integral term. Hence we define

$$V=\{v\in H^1(\Omega):\quad v=0\text{ on }\partial\Omega_D\}.$$

The weak form is find $u \in V$ such that

$$\iint_\Omega\nabla u\cdot\nabla v\,\mathrm{d}x\mathrm{d}y=\iint_\Omega fv\,\mathrm{d}x\mathrm{d}y\quad\forall v\in V.$$

The weak form is to be approximated using the finite element method. Explain what is meant by the conforming condition for the finite element approximation, giving sufficient conditions for this when piecewise polynomial functions are used.

ANSWER

The finite element method is conforming if the finite element space V_h is a subspace of the space V in the weak formulation. A piecewise polynomial function is in V if it is continuous, i.e. $V_h \subset C(\bar{\Omega})$.

The finite element approximation $U(x,y)$ is a piecewise bilinear function defined over a partition of Ω into square elements. In the case of the standard square S where $S = \{(x,y) : \ -1 \le x, y \le 1\}$ with nodes $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$ write down the four Lagrange basis functions each of which is associated with one of the nodes.

ANSWER

Each of the 4 basis functions can be written as the product of a linear basis function in s and a linear basis function in t. We can write them in the matrix form as follows.

$$\begin{pmatrix} \phi_4 & \phi_3 \\ \phi_1 & \phi_2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1+y \\ 1-y \end{pmatrix} \begin{pmatrix} 1-x & 1+x \end{pmatrix}.$$

Then for this element do the following.

(i) It can be shown that the 4×4 element stiffness matrix arising from $a(u,v)|_S$ when bilinears is used has the form

$$\begin{pmatrix} \alpha & \beta & \beta & \gamma \\ \beta & \alpha & \gamma & \beta \\ \beta & \gamma & \alpha & \gamma \\ \gamma & \beta & \beta & \alpha \end{pmatrix}.$$

Evaluate α, β and γ.

ANSWER

With an element stiffness matrix the sum of the entries on each row must add to 0 because $\nabla(\phi_1 + \phi_2 + \phi_3 + \phi_4) = \underline{0}$. Hence

$$\alpha + 2\beta + \gamma = 0.$$

Now

$$\nabla \phi_1 = -\frac{1}{4} \begin{pmatrix} 1-y \\ 1-x \end{pmatrix} \quad \text{and} \quad \nabla \phi_3 = \frac{1}{4} \begin{pmatrix} 1+y \\ 1+x \end{pmatrix}.$$

$$\begin{aligned} \alpha &= \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} (\nabla \phi_1)^2 \, \mathrm{d}x\mathrm{d}y \\ &= \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} (1-y)^2 + (1-x)^2 \, \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Now

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1}(1-y)^2\,\mathrm{d}x\mathrm{d}y &= \int_{-1}^{1}\int_{-1}^{1}(1-x)^2\,\mathrm{d}x\mathrm{d}y \\ &= 2\int_{-1}^{1}(1-x)^2\,\mathrm{d}x = 2\left[\frac{1}{3}(x-1)^3\right]_{-1}^{1} = \frac{16}{3}.\end{aligned}$$

Thus

$$\alpha = \frac{2}{3}.$$

$$\begin{aligned}\gamma &= \frac{1}{16}\int_{-1}^{1}\int_{-1}^{1}\nabla\phi_1\cdot\nabla\phi_3\,\mathrm{d}x\mathrm{d}y \\ &= -\frac{1}{16}\int_{-1}^{1}\int_{-1}^{1}(1-y^2)+(1-x^2)\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Now

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1}(1-y^2)\,\mathrm{d}x\mathrm{d}y &= \int_{-1}^{1}\int_{-1}^{1}(1-x^2)\,\mathrm{d}x\mathrm{d}y \\ &= 2\int_{-1}^{1}(1-x^2)\,\mathrm{d}x = 4\int_{0}^{1}(1-x^2)\,\mathrm{d}x = 4\left(1-\frac{1}{3}\right) = \frac{8}{3}.\end{aligned}$$

Thus

$$\gamma = -\frac{1}{3}.$$

From the relation $\alpha + 2\beta + \gamma = 0$ this gives $\beta = -1/6$.

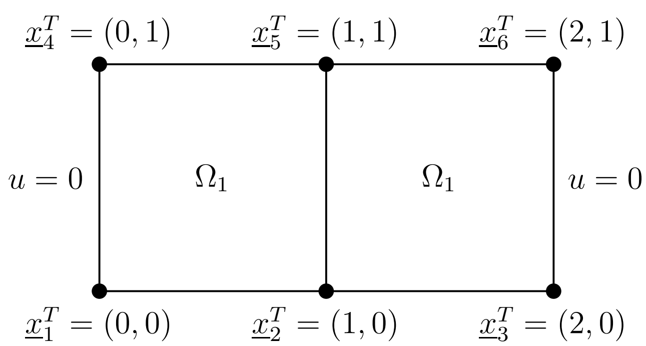
(ii) Give the 4×1 element vector arising from S, leaving each component in integral form.

ANSWER

The element vector is

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\begin{pmatrix}\phi_1(x,y)\\ \phi_2(x,y)\\ \phi_3(x,y)\\ \phi_4(x,y)\end{pmatrix}\mathrm{d}x\mathrm{d}y = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} f(x,y)\begin{pmatrix}(1-x)(1-y)\\ (1+x)(1-y)\\ (1+x)(1+y)\\ (1-x)(1+y)\end{pmatrix}\mathrm{d}x\mathrm{d}y.$$

Let Ω be the rectangle $\{(x,y):\ 0<x<2,\ 0<y<1\}$ and let $\partial\Omega_D = \{(0,y):\ 0<y<1\}\cup\{2,y):\ 0<y<1\}$ as in the Figure. Let $\underline{x}_i$, $i=1,\cdots,6$ be the nodes of the elements Ω_1 and Ω_2 of the mesh of Ω and let $U_i = U(\underline{x}_i)$.



By constructing the 4×4 element stiffness matrix for each of the elements Ω_1 and Ω_2 assemble the 6×6 global stiffness matrix associated with the given mesh of Ω. Apply the essential boundary conditions of the boundary value problem to obtain the 2×2 matrix K in the system

$$K \begin{pmatrix} U_2 \\ U_5 \end{pmatrix} = \begin{pmatrix} b_2 \\ b_5 \end{pmatrix}$$

that determine U_2 and U_5.

ANSWER

The element matrix for both Ω_1 and Ω_2 is

$$\frac{1}{6} \begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix}.$$

Let $\hat{K}$ denote the 6×6 matrix. Local nodes 1,2,3,4 on element Ω_1 correspond to global nodes 1,2,5,4 respectively. Local nodes 1,2,3,4 on element Ω_2 correspond to global nodes 2,3,6,5 respectively. Thus

$$6\hat{K} = \begin{pmatrix} 4 & -1 & \cdot & -1 & -2 & \cdot \\ -1 & 4 & \cdot & -2 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -2 & \cdot & 4 & -1 & \cdot \\ -2 & -1 & \cdot & -1 & 4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 4 & -1 & \cdot & -1 & -2 \\ \cdot & -1 & 4 & \cdot & -2 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & -2 & \cdot & 4 & -1 \\ \cdot & -2 & -1 & \cdot & -1 & 4 \end{pmatrix}.$$

Extracting the entries corresponding to nodes $\underline{x}_2$ and $\underline{x}_5$ gives

$$K = \frac{1}{6} \begin{pmatrix} 8 & -2 \\ -2 & 8 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}.$$

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9. This question is taken from the last part of question 2 of the 1997 `MA5056S` paper.

   The context of the question is that of Poisson's equation $-\Delta u = f$ in Ω with boundary conditions $u = 0$ on $\partial\Omega_D$ and $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega_N$ where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ with $\partial\Omega_D$ being non-empty. In an earlier part of the question this has been reformulated into the weak form

$$a(u, v) = (f, v), \ \forall v \in S$$

   where

$$a(u, v) = \iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) \, \mathrm{d}x\mathrm{d}y \quad \text{and} \quad (f, v) = \iint_\Omega fv \, \mathrm{d}x\mathrm{d}y.$$

   and where $S = \{v \in H^1(\Omega) : \ v = 0 \text{ on } \partial\Omega_D\}$.

An earlier part of the question has also been concerned with constructing the element matrix for the standard triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ and with the unit square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$. These matrices are

$$\frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{6}\begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix}.$$

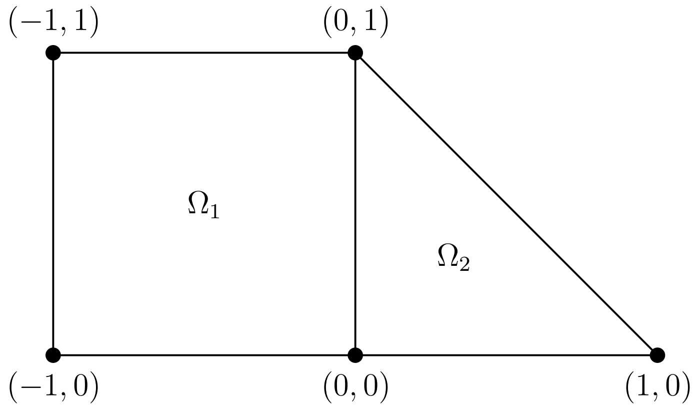
The last part of the question follows.

Let $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ denote the region shown in the Figure where Ω_1 is a unit square and Ω_2 is the standard triangle and suppose that $\partial\Omega_D = \{(x,y) : \ -1 \le x \le 1, \ y = 0\}$. If the finite element method is used to approximate the solution using a bilinear function on Ω_1 and a linear function on Ω_2 with

$$U(x,y) = U(-1,1)B_1(x,y) + U(0,1)B_2(x,y)$$

describing the approximation then give the functions B_1 and B_2 and derive the 2×2 matrix K in the linear system

$$K\begin{pmatrix} U(-1,1) \\ U(0,1) \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad b_i = \iint_\Omega f(x,y)B_i(x,y)\,\mathrm{d}x\mathrm{d}y, \quad i = 1,2.$$



ANSWER

The functions B_1 and B_2 are as follows.

$$B_1(x,y) = \begin{cases} -xy & \text{for } (x,y) \in \Omega_1, \\ 0 & \text{for } (x,y) \in \Omega_2. \end{cases}$$

$$B_2(x,y) = \begin{cases} (x+1)y & \text{for } (x,y) \in \Omega_1, \\ y & \text{for } (x,y) \in \Omega_2. \end{cases}$$

The matrix K is given by

$$K = \begin{pmatrix} a(B_1,B_1) & a(B_1,B_2) \\ a(B_1,B_2) & a(B_2,B_2) \end{pmatrix}.$$

All the entries involving B_1 are obtained directly from the element matrix for the bilinear element as B_1 is 0 on Ω_2. For $a(B_2, B_2)$ we have a contribution from both elements.

$$a(B_2, B_2) = \iint_{\Omega_1} (\nabla B_2)^2 \,\mathrm{d}x\mathrm{d}y + \iint_{\Omega_2} (\nabla B_2)^2 \,\mathrm{d}x\mathrm{d}y = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}.$$

Thus

$$K = \frac{1}{6}\begin{pmatrix} 4 & -1 \\ -1 & 7 \end{pmatrix}.$$

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10. Apart from a few minor changes and re-typing this was question 1 of the 2002 `MA5156S` paper.

    Let S be the square element in the (s,t) plane, as shown in the figure, with vertices $(-1,1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$, labelled respectively as $\hat{P}_1$, $\hat{P}_2$, $\hat{P}_3$ and $\hat{P}_4$. The points $(0,-1)$, $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,0)$ are labelled respectively as $\hat{P}_5$, $\hat{P}_6$, $\hat{P}_7$, $\hat{P}_8$ and $\hat{P}_9$. For $i = 1, \cdots, 9$ write down the biquadratic basis functions $\phi_i(s,t)$, $(s,t) \in S$ which are such that

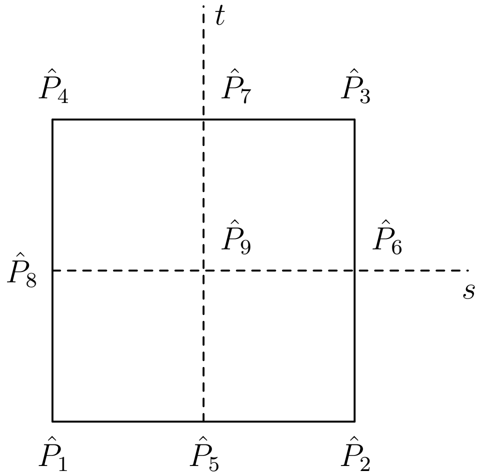
$$\phi_i(\hat{P}_j) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

[7 MARKS]

ANSWER

Each basis function is the product of a quadratic in s and a quadratic in t and the complete set of functions can be neatly written in matrix form as

$$\begin{pmatrix} \phi_4 & \phi_7 & \phi_3 \\ \phi_8 & \phi_9 & \phi_6 \\ \phi_1 & \phi_5 & \phi_2 \end{pmatrix} = \begin{pmatrix} \dfrac{t(t+1)}{2} \\ 1 - t^2 \\ \dfrac{t(t-1)}{2} \end{pmatrix} \begin{pmatrix} \dfrac{s(s-1)}{2} & 1 - s^2 & \dfrac{s(s+1)}{2} \end{pmatrix}.$$



Consider the points (nodes) P_i of the element Ω_r in the (x, y) plane with coordinates (x_i, y_i), $i = 1, \cdots, 9$, where

$$x_1 = x_8 = x_4 = -1, \quad y_1 = y_5 = y_2 = -1,$$
$$x_5 = x_9 = 0, \quad y_6 = y_8 = y_9 = 0,$$
$$x_2 = x_3 = x_6 = 1, \quad y_3 = y_4 = 1$$

and where (x_7, y_7) is such that $|x_7| < 1$ and $y_7 > 0$. With $\underline{x}_i^T = (x_i, y_i)$ and $\underline{x}^T = (x, y)$, the biquadratic mapping which takes S onto Ω_r is given by

$$\underline{x}(s,t) = \sum_{i=1}^{9} \underline{x}_i \phi_i(s,t).$$

What is the form of the mapping in the particular case $x_7 = 0$ and $y_7 = 1$.

[2 MARKS]

ANSWER

If $x_7 = 0$ and $y_7 = 1$ then $\hat{P}_i = P_i$ for $i = 1, \cdots, 9$ and we have the identity map, i.e. $x = s$ and $y = t$.

Hence explain why for a general point (x_7, y_7), $|x_7| < 1$, $y_7 > 0$ we have

$$\begin{aligned} x &= s + x_7\phi_7(s,t) \\ y &= t + (y_7 - 1)\phi_7(s,t). \end{aligned}$$

[3 MARKS]

ANSWER

In this case $\hat{P}_i = P_i$ for all i except $i = 7$. Hence the mapping is the identity map plus a term depending on the difference of P_7 from the position $(0, 1)$. The mapping is

$$\underline{x}(s,t) = \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} x_7 \\ y_7 - 1 \end{pmatrix} \phi_7(s,t).$$

Hence sketch the form of the element Ω_r which is the image of S under the mapping.

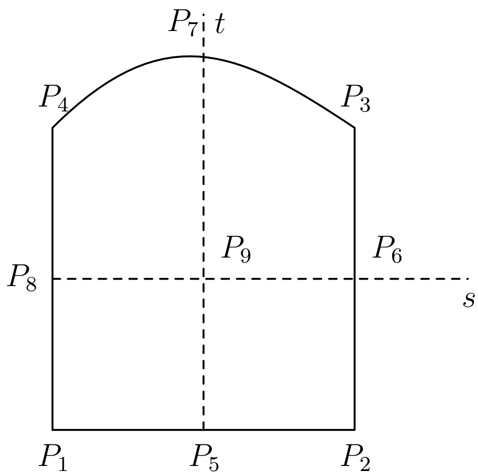
[2 MARKS]

ANSWER

The images of the sides of S except the top side are unchanged by the mapping but the top side may be curved and is described parametrically by

$$\underline{x}(s,1) = \begin{pmatrix} s \\ 1 \end{pmatrix} + \begin{pmatrix} x_7 \\ y_7 - 1 \end{pmatrix} \phi_7(s,1).$$

In the case $y_7 > 1$ a possible image is shown below.



Give the partial derivatives $\dfrac{\partial \phi_7}{\partial s}$ and $\dfrac{\partial \phi_7}{\partial t}$ and show that the determinant of the Jacobian of the mapping is given by

$$J = 1 + x_7 \frac{\partial \phi_7}{\partial s} + (y_7 - 1)\frac{\partial \phi_7}{\partial t}.$$

[6 MARKS]

ANSWER

$$\begin{aligned} \phi_7(s,t) &= \frac{1}{2} t(t+1)(1-s^2), \\ \frac{\partial \phi_7}{\partial s}(s,t) &= -t(t+1)s, \\ \frac{\partial \phi_7}{\partial t}(s,t) &= \frac{1}{2}(2t+1)(1-s^2). \end{aligned}$$

Thus

$$\tilde{J} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} x_7 \dfrac{\partial \phi_7}{\partial s} & x_7 \dfrac{\partial \phi_7}{\partial t} \\ (y_7 - 1)\dfrac{\partial \phi_7}{\partial s} & (y_7 - 1)\dfrac{\partial \phi_7}{\partial t} \end{pmatrix}.$$

The determinant is

$$\begin{aligned} J &= \left(1 + x_7 \frac{\partial \phi_7}{\partial s}\right)\left(1 + (y_7 - 1)\frac{\partial \phi_7}{\partial t}\right) - x_7(y_7 - 1)\frac{\partial \phi_7}{\partial s}\frac{\partial \phi_7}{\partial t}, \\ &= 1 + x_7 \frac{\partial \phi_7}{\partial s} + (y_7 - 1)\frac{\partial \phi_7}{\partial t}. \end{aligned}$$

Show that for $(s, t) \in S$

$$-2 \le \frac{\partial \phi_7}{\partial s} \le 2; \quad -\frac{1}{2} \le \frac{\partial \phi_7}{\partial t} \le \frac{3}{2}.$$

Also show that if (x_7, y_7) satisfies

$$-\frac{1}{4} < x_7 < \frac{1}{4} \quad \text{and} \quad \frac{2}{3} < y_7 < 2,$$

then $J > 0$.

[5 MARKS]

ANSWER

$$|t + 1| \le 2 \quad \text{for} \quad -1 \le t \le 1.$$

Thus as $|s| \le 1$ and $|t| \le 1$ we have

$$-2 \le \frac{\partial \phi_7}{\partial s} \le 2.$$

For $(s, t) \in T$ we have $0 \le 1 - s^2 \le 1$. For $-1 \le t \le 1$ we have $-1 \le 2t + 1 \le 3$. Together these imply that for all $(s, t) \in T$ we have

$$-\frac{1}{2} \le \frac{\partial \phi_7}{\partial s} \le \frac{3}{2}.$$

On T if $|x_7| \le 1/4$ we have

$$x_7 \frac{\partial \phi_7}{\partial s} > -\frac{2}{4} = -\frac{1}{2}.$$

$2/3 < y_7 < 2$ gives $-1/3 < y_7 - 1 < 1$. Hence on T we have

$$(y_7 - 1)\frac{\partial \phi_7}{\partial t} > -\frac{1}{3}\left(\frac{3}{2}\right) = -\frac{1}{2}.$$

Combining these last two results gives

$$J = 1 + x_7 \frac{\partial \phi_7}{\partial s} + (y_7 - 1)\frac{\partial \phi_7}{\partial t} > 1 - \frac{1}{2} - \frac{1}{2} = 0.$$