Topics in MA3614 in 2023/4

- Preliminaries (Chaps 1 and 2).
- Complex differentiation: Analytic functions, Cauchy Riemann equations, harmonic functions, ... (Chap 3).
- Elementary functions of a complex variable: Polynomials, rational functions, $\exp(z)$, $\mathrm{Log}(z)$, z^{α}, ... (Chap 4).
- Contour integrals, loop integrals, Cauchy integral theorem, Cauchy integral formula, ... (Chaps 5 and 6).
- Taylor series, Laurent series representations (Chap 7).
- Residue theory and its use in evaluating real integrals (Chap 8).

Analytic functions – definitions

- **Complex derivative:** Let f be a complex valued function defined in a neighbourhood of z_0. The **derivative of f at z_0** is given by

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

provided the limit exists.
Note that the limit must be independent of how $h \to 0$.
This was used later to justify the generalised Cauchy integral formula for $f'(z)$ at the start of term 2.
- A function f is **analytic at z_0** if f is differentiable at all points in some neighbourhood of z_0.
- A function f is **analytic in a domain** if f is analytic at all points in the domain.
- A function $f : \mathbb{C} \to \mathbb{C}$ is an **entire function** if it is analytic on the whole complex plane $\mathbb{C}$.

The Cauchy Riemann equations for $f(z) = u(x,y) + iv(x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

When u and v have continuous partial derivatives on a domain D the function $f = u + iv$ is analytic on D if and only if the Cauchy Riemann (CR) equations are satisfied throughout D.

If $f = u + iv$ is analytic then u and v are harmonic functions. v is said to be the **harmonic conjugate** of u. By one CR equation

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

and we partially integrate to get

$$v(x,y) = (\text{some function}) + g(y).$$

Then by partially differentiating and using the other CR equation

$$\frac{\partial v}{\partial y} = (\text{deriv of some function}) + g'(y) = \frac{\partial u}{\partial x}$$

This gives $g'(y)$.

Some representations of $f'(z)$

With the usual notation let $z = x + iy = re^{i\theta}$ and let

$$f(z) = u(x,y) + i\,v(x,y) = \tilde{u}(r,\theta) + i\,\tilde{v}(r,\theta)$$

be an analytic function. As we get the same value by differentiating in any direction we can represent the derivative in many different ways. Let h be real. We have the following as $h \to 0$.

$$\frac{f(z+h)-f(z)}{h} \to \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

$$\frac{f(z+he^{i\theta})-f(z)}{he^{i\theta}} \to \frac{1}{e^{i\theta}}\left(\frac{\partial \tilde{u}}{\partial r} + i\frac{\partial \tilde{v}}{\partial r}\right).$$

Analytic functions can be expressed in terms of z alone

In the case of a polynomial we can use a finite Maclaurin series representation. More generally we have a Taylor series or a Laurent series.

Example of a function which is not analytic

$$f(z) = \overline{z} = x - iy \qquad \text{is not analytic anywhere.}$$

This can be proved using the definition or by showing that the Cauchy Riemann equations are not satisfied when $u = x$, $v = -y$.

Examples of functions which are analytic (see Chap 4)

$$\begin{aligned}
z &= x + iy, \\
z^2 &= (x^2 - y^2) + 2ixy, \quad (\text{and } z^3, z^4, \ldots, \text{ all polynomials}), \\
\frac{1}{z} &= \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}, \qquad z \ne 0, \\
e^z &= e^x(\cos y + i \sin y), \\
\cos z &= \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \\
\cosh z &= \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}, \\
\operatorname{Log} z &= \ln |z| + i \operatorname{Arg} z, \qquad z \ne 0, \quad \operatorname{Arg} z \ne \pi, \\
z^\alpha &= \exp(\alpha \operatorname{Log} z), \qquad z \ne 0, \quad \operatorname{Arg} z \ne \pi, \quad \alpha \in \mathbb{C}.
\end{aligned}$$

Contour integrals: definition and anti-derivatives

Chap 5. With $\Gamma = \{z(t) : \; a \le t \le b\}$ describing a curve we have

$$\int_\Gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt.$$

In many places (e.g. chap 6 and chap 8) we used the following result.

$$\left| \int_\Gamma f(z)\,dz \right| \le ML, \quad M = \max\{|f(z)| : \; z \in \Gamma\}, \quad L = \text{length of } \Gamma.$$

When f has an **anti-derivative** F on Γ (i.e. $f = F'$) we have

$$\begin{aligned}
\int_\Gamma f(z)\,dz &= \int_a^b F'(z(t))z'(t)\,dt = \int_a^b \frac{dF(z(t))}{dt}\,dt \\
&= F(z(b)) - F(z(a)).
\end{aligned}$$

When an anti-derivative exists on a closed loop

$$\oint_\Gamma f(z)\,dz = 0.$$

Loop integrals and analytic functions

Here f is analytic in a simply connected domain D and Γ is any loop (i.e. a closed contour) in D.

Cauchy-Goursat theorem (near end of chap 5)

$$\oint_\Gamma f(z)\,dz = 0.$$

The Cauchy integral formula (chap 6)

Let z be a point inside a closed loop Γ traversed once in the anti-clockwise direction.

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

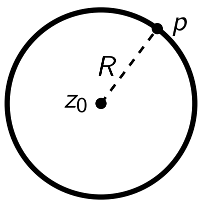
The generalised Cauchy integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta.$$

Taylor's series – the circle of convergence (chap 7)

If $f(z)$ is analytic at z_0 then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$



If p is the nearest non-analytic point of $f(z)$ to z_0 then $R = |p - z_0|$ is the **radius of convergence**, $|z - z_0| = R$ is the **circle of convergence** and the series converges uniformly in $|z - z_0| \le R'$ for all $R' < R$. The series diverges for all z satisfying $|z - z_0| > R$.

Example:

$$f(z) = \frac{1}{1 - z} = 1 + z + z^2 + \cdots + z^n + \cdots$$

The simple pole at $p = 1$ gives the circle of convergence as $|z| = 1$.

Power series define analytic functions when $R > 0$

Let a function $f(z)$ and let R be defined by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad R = \frac{1}{\limsup |a_n|^{1/n}} \ge 0.$$

When $R > 0$ this defines a function analytic in $|z - z_0| < R$ and R is the radius of convergence. Thus

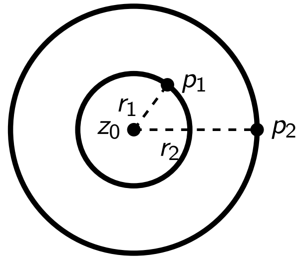
$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z.$$

Often R can be determined using the ratio test or the root test.

$$b_n = a_n (z - z_0)^n, \quad \left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{a_{n+1}}{a_n} \right| |z - z_0|, \quad |b_n|^{1/n} = |a_n|^{1/n} |z - z_0|.$$

If $|a_{n+1}/a_n| \to \alpha$ or if $|a_n|^{1/n} \to \alpha$ as $n \to \infty$ then we get a condition on $|z - z_0|$ for convergence and for divergence.

Laurent series (near the end of chap 7)



Suppose $f(z)$ has non-analytic points at p_1 and p_2 and

$$r_1 = |p_1 - z_0|, \quad r_2 = |p_2 - z_0|.$$

If $f(z)$ is analytic in $r_1 < |z - z_0| < r_2$ then it has a Laurent series representation

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n.$$

Example: $f(z) = \dfrac{1}{1 - z}$ has a pole at $z = 1$. Take $z_0 = 0$.

Power series in $|z| < 1$.

Laurent series in $1 < |z|$ only involving negative powers.

Laurent series – expanding in negative powers

Example: When $|z| > 2$ we have

$$2 - z = -z \left(1 - \frac{2}{z} \right)$$

$$f(z) = \frac{1}{2 - z} = \left(\frac{-1}{z} \right) \left(1 - \frac{2}{z} \right)^{-1} = \left(\frac{-1}{z} \right) \left(1 + \frac{2}{z} + \left(\frac{2}{z} \right)^2 + \cdots \right).$$

Laurent series – classifying isolated singularities

Suppose

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R.$$

$\mathrm{Res}(f, z_0) = a_{-1}$ is the **residue** at z_0.

If $a_n = 0$ for $n < 0$ then $f(z)$ has a **removable singularity**.

If $m < 0$, $a_m \ne 0$ and $a_n = 0$ for $n < m$, then $f(z)$ has a **pole of order** $|m|$.

Manipulations with power series and Laurent series

With series with the same expansion point we can add them term-by-term, differentiate term-by-term and integrate term-by-term. We can also multiply two series together. Examples:

$$f(z) = \tan z = \frac{\sin z}{\cos z} = b_1 z + b_3 z^3 + b_5 z^5 + \cdots \quad |z| < \pi/2.$$

As $\sin z = (\tan z)(\cos z)$ we have

$$z - \frac{z^3}{6} + \frac{z^5}{120} + \cdots = \left(1 - \frac{z^2}{2} + \frac{z^4}{24} + \cdots \right) (b_1 z + b_3 z^3 + b_5 z^5 + \cdots).$$

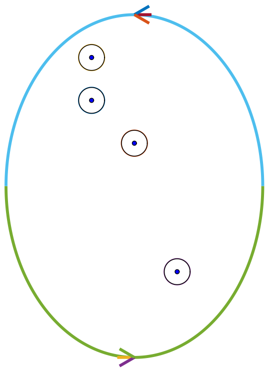
By equating coefficients we can get b_1, b_3 and b_5 etc.

$$g(z) = \frac{1}{\mathrm{e}^z - 1} = \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots, \quad 0 < |z| < 2\pi, \quad \mathrm{e}^{\pm 2\pi i} = 1.$$

$$1 = g(z)(\mathrm{e}^z - 1) = \left(\frac{c_{-1}}{z} + c_0 + c_1 z + \cdots \right) \left(z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \right).$$

By equating coefficients we can get c_{-1}, c_0 and c_1 etc.

The Residue theorem (chap 8)

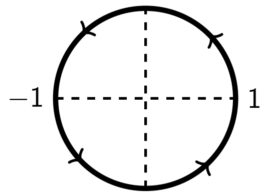


Let $f(z)$ be analytic inside the outer contour Γ except at 4 isolated points at the centres of the disks shown. $f(z)$ is analytic between Γ and the circles. A set-up such as this was used to explain residue theorem stated below.

Cauchy residue theorem: If Γ is a simple closed positively orientated contour and f is analytic inside and on Γ, except at points $z_1, \ldots, z_n$ inside Γ, then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f,\, z_k).$$

Trig integrals evaluated using residue theory



$$I = \int_0^{2\pi} R(\cos\,\theta, \sin\,\theta)\,\mathrm{d}\theta = \oint_C \frac{1}{i} F(z)\,\mathrm{d}z.$$

Here C is the unit circle and $F(z)$ is obtained by using

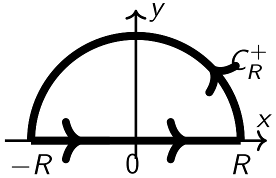
$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad \cos\,\theta = \frac{z + z^{-1}}{2}, \quad \sin\,\theta = \frac{z - z^{-1}}{2i}.$$

We determine I by the Residue theorem involving the residues of $F(z)$ at the poles which are inside C, i.e. have magnitude less than 1. ($F(z)$ is a rational function of z and examples were in chap 5.)

Integrals on $(-\infty, \infty)$ evaluated using residue theory

With $P(z)$ and $Q(z)$ being polynomials we considered

$$f(z) = \frac{P(z)}{Q(z)} \quad \text{and} \quad f(z) = \frac{P(z)}{Q(z)}\mathrm{e}^{miz}.$$



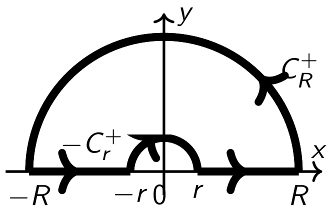
$f(z)$ has poles at points $z_1, \ldots, z_n$ in the upper half plane. $Q(z)$ has no zeros on the real axis.

With $\Gamma_R = [-R, R] \cup C_R^+$ denoting the closed contour

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f,\, z_k).$$

Using the ML inequality we show that the integral on C_R^+ tends to 0 as $R \to \infty$.

Indented contours and principal values



When $f(z)$ has a pole on the real axis then we use an indented contour. There may be a contribution as $r \to 0$. The limit as $r \to 0$ and $R \to \infty$ is known as the principal value.

We typically get $\mathrm{Res}(f,\, z_k)$ by using L'Hopitals's rule or with manipulations involving the Laurent series.

The ML inequality is used to explain why integrals involving C_R^+ tend to 0 as $R \to \infty$ and it is used as part of the explanation to get the contribution from C_r^+ as $r \to 0$.

When $z = x + iy$, $miz = -my + imx$ and

$$\left|\mathrm{e}^{miz}\right| = \mathrm{e}^{-my} \le 1, \quad \text{when } m \ge 0 \text{ and } y \ge 0.$$

Jordans' lemma is needed when $\deg(Q) = \deg(P) + 1$.