## Laurent series representation

Let $f(z)$ be analytic in an annulus $r < |z - z_0| < R$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}.$$

The series converge uniformly in any closed sub-annulus $r < \rho_1 \le |z - z_0| \le \rho_2 < R$. The coefficients $a_n$ are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, \mathrm{d}z,$$

where $C$ is any positively orientated simple closed curve lying in the annulus which has $z_0$ as an interior point.

---

This indicates that the representation is unique.

Also note that in none of the examples did we obtain $a_n$ by evaluating this integral as we had other ways to get them.



## Isolated zeros of non-zero analytic functions

When $f(z)$ has a **zero of multiplicity** $m \ge 1$ **at** $z_0$ we have

$$f(z) = a_m (z - z_0)^m + a_{m+1} (z - z_0)^{m+1} + \cdots = (z - z_0)^m g(z)$$

with $g(z)$ being analytic at $z_0$ and $g(z_0) = a_m \ne 0$. These properties of $g(z)$ imply that in a neighbourhood $\{z : |z - z_0| < \delta\}$, for some $\delta > 0$, $g(z)$ is non-zero and thus $f(z)$ is non-zero. The zeros of $f(z)$ are isolated.

---

As an example suppose that the Cauchy Riemann equations are used to show that the following is analytic.

$$f(x+iy) = (-2x^2 - 10xy + 6x + 2y^2 + 15y) + i(5x^2 - 4xy - 15x - 5y^2 + 6y).$$

$$f(x) = (-2x^2 + 6x) + i(5x^2 - 15x).$$

$$g(z) = (-2z^2 + 6z) + i(5z^2 - 15z).$$

$f(x + iy)$ and $g(z)$ are both analytic with $f(z) - g(z) = 0$ on the real line. Hence $f(z) = g(z)$ for all $z$.



## Complex identity and the related real relation

The isolated zeros property of non-zero analytic functions is a way to quickly explain why many identities are also true in the complex plane. For example,

$$\begin{aligned} \cos^2(x) + \sin^2(x) - 1 &= 0, \\ \sin(2x) - 2\sin(x)\cos(x) &= 0, \end{aligned}$$

being true for all $x \in \mathbb{R}$ also hold for all $z \in \mathbb{C}$, i.e.

$$\begin{aligned} \cos^2(z) + \sin^2(z) - 1 &= 0, \\ \sin(2z) - 2\sin(z)\cos(z) &= 0. \end{aligned}$$



## Laurent series: Classifying poles

If $f(z)$ has **a removable singularity** at $z_0$ then it has a Laurent series with no negative powers valid in $0 < |z - z_0| < R$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad \lim_{z \to z_0} f(z) = a_0.$$

Example: $\sin(z)/z$ has a removable singularity at $z = 0$.

---

If $f(z)$ has a **pole of order** $m$ then in $0 < |z - z_0| < R$ we have

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n = \frac{\phi(z)}{(z - z_0)^m}$$

with $\phi(z)$ being analytic at $z_0$ and $\phi(z_0) = a_{-m} \ne 0$.

---

An **essential singularity at** $z_0$ has infinitely many negative powers

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R.$$

Example: $\exp(1/z)$ with $z_0 = 0$.

# Integrating a Laurent Series

Let $f(z)$ be analytic in an annulus with the following Laurent series representation.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R.$$

The coefficient $a_{-1}$ is called the residue at $z_0$. We write $\mathrm{Res}(f, z_0)$.

Let $\Gamma$ denote a loop traversed once in the anti-clockwise sense with $z_0$ inside $\Gamma$. Then term-by-term integration gives

$$\oint_\Gamma f(z)\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n \oint_\Gamma (z - z_0)^n\,\mathrm{d}z = 2\pi i\, a_{-1}.$$



# Start of Chap 8 on Residue theory

We begin with a review of earlier results which involve the following.

- The definition of analytic at a point. (chap 3)
- Loop integrals in the following situations.
  - When we have an anti-derivative. (chap 5)
  - Cauchy's theorem. When $f(z)$ is analytic inside a loop. (chap 5)
  - The generalised Cauchy integral formula. (chap 6)
  - The use of partial fractions to express $1/Q(z)$, $Q(z)$ being a polynomial, to deal with $f(z)/Q(z)$. (chap 4)
- Taylor's series in a disk. (chap 7)
- Laurent series in an annulus. (chap 7)



# Definition of an analytic function

**Complex derivative:** Let $f$ be a complex valued function defined in a neighbourhood of $z_0$. The **derivative of $f$ at** $z_0$ is given by

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

provided the limit exists. Note that $h$ is complex.

A function $f$ is **analytic at** $z_0$ if $f$ is differentiable at all points in some neighbourhood of $z_0$.

# Key results before chap 6 about analytic functions

**The Cauchy Riemann equations for** $f = u + iv$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

**Cauchy-Goursat theorem**: If $f$ is analytic in a simply connected domain $D$ and $\Gamma$ is any loop (i.e. a closed contour) in $D$ then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$



# Results about analytic functions in term 2

**Generalised Cauchy integral formula**

With the same conditions as above and with $z_0$ inside $\Gamma$

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z, \quad n = 0, 1, 2, \ldots$$

Using the Cauchy integral formula we get series representations.

**Taylor series:** If $f(z)$ is analytic in the disk $|z - z_0| < R$ then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

**Laurent series:** If $f(z)$ is analytic in $0 \le r < |z - z_0| < R$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}, \quad a_m = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{m+1}}\,\mathrm{d}z,$$

where $C$ is simple closed loop in the annulus in the anti-clockwise sense. The series are unique once $z_0$ is specified.

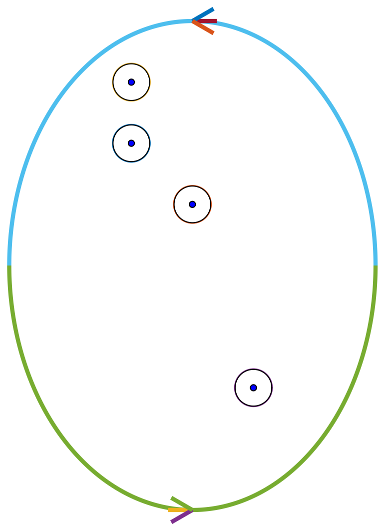
## A recap of some results about loop integrals

1. If $f$ has an anti-derivative continuous on $\Gamma$ then
$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$
$f$ need not be analytic inside $\Gamma$, e.g. $f(z) = 1/z^2$.
2. If $f(z)$ is analytic on and inside $\Gamma$, i.e. we have no isolated singularities, then by Cauchy's theorem
$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$
3. If the integrand is of the form
$$f(z) = \frac{g(z)}{(z-z_0)^{m+1}},$$
where $m$ is an integer and where $g(z)$ is analytic on and inside $\Gamma$, then by the generalised Cauchy integral formula
$$\oint_\Gamma f(z)\,\mathrm{d}z = \oint_\Gamma \frac{g(z)}{(z-z_0)^{m+1}}\,\mathrm{d}z = 2\pi i \frac{g^{(m)}(z_0)}{m!}$$



## What if there are several isolated singularities?

## Example: 4 isolated singularities of $f(z)$ inside $\Gamma$





## From chap 4: The case of rational functions

Let
$$R(z) = \frac{p(z)}{q(z)}, \quad q(z) = (z-z_1)^{r_1}(z-z_2)^{r_2}\cdots(z-z_n)^{r_n}.$$
$$R(z) = \frac{p(z)}{q(z)} = (\text{some polynomial}) + \sum_{k=1}^{n} \frac{A_k}{z-z_k} + (\text{higher order poles}).$$
Here $A_k$ is the **residue** at $z_k$.

The polynomial part has an anti-derivative (another polynomial) and a $(z-z_k)^{-j-1}$ term has an anti-derivative $(z-z_k)^{-j}/(-j)$ when $j \geq 1$ and hence loop integrals of these part are 0.

$1/(z-z_k)$ has an anti-derivative throughout a loop when $z_k$ is outside the loop and hence loop integrals of such terms are 0.



## Residue theorem for rational functions

If $z_1, \ldots, z_m$ are points inside $\Gamma$ at which $R(z)$ has poles then
$$\begin{aligned}\oint_\Gamma R(z)\,\mathrm{d}z &= \sum_{k=1}^{m} A_k \oint_\Gamma \frac{\mathrm{d}z}{z-z_k}\\ &= 2\pi i \sum_{k=1}^{m} A_k\\ &= 2\pi i \sum_{k=1}^{m} \mathrm{Res}(R,\, z_k).\end{aligned}$$

The answer just depends on the residues at the poles inside $\Gamma$.

## A more general numerator

Suppose $Q(z)$ is a polynomial and $g(z)$ is analytic on and inside $\Gamma$.

$$f(z) = \frac{g(z)}{Q(z)}, \quad \text{with } Q(z) = (z - z_1)^{r_1} \cdots (z - z_n)^{r_n}.$$

By partial fractions we have the form

$$\frac{1}{Q(z)} = \sum_{k=1}^{n} \left( \frac{A_{1,k}}{z - z_k} + \cdots + \frac{A_{r_k,k}}{(z - z_k)^{r_k}} \right).$$

We can then separately determine

$$\oint_{\Gamma} \frac{g(z)}{(z - z_k)^{r_j}} \, \mathrm{d}z.$$

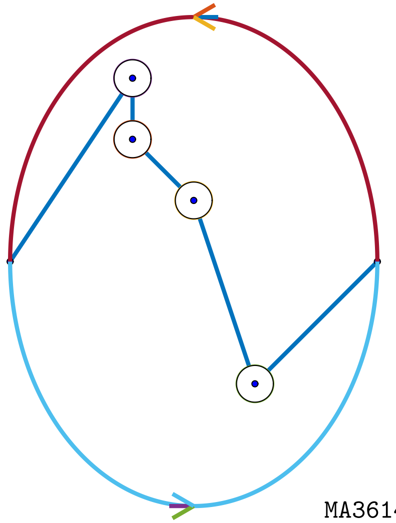
using the generalised Cauchy integral formula.



## Dealing with a more general denominator

The following slides enable us to deal with any denominator which is analytic and has zeros.

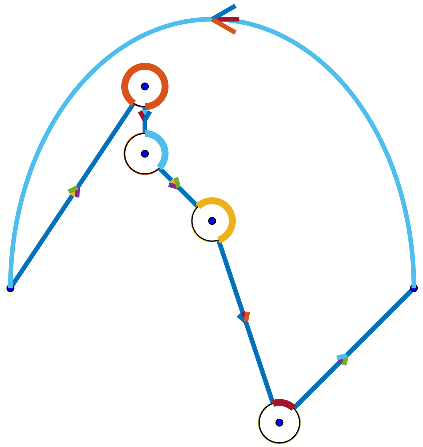
### Joining the points and dividing the domain



We have two simply connected domains, a top part and a bottom part. $f(z)$ is analytic in both simply connected domains.



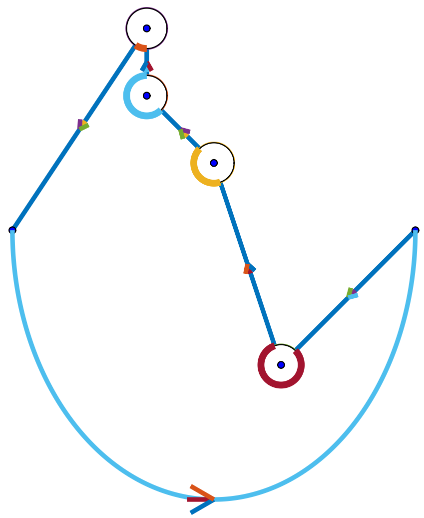
## The path $\Gamma_t$ of the top part



$$\int_{\Gamma_t} f(z) \, \mathrm{d}z = 0.$$



## The path $\Gamma_b$ of the bottom part



$$\int_{\Gamma_b} f(z) \, \mathrm{d}z = 0.$$

## The Residue theorem

If $z_1, z_2, \ldots, z_n$ are isolated singularities inside $\Gamma$ and $C_1, C_2, \ldots, C_n$ are non-intersecting circles traversed once in the anti-clockwise direction then $\Gamma \cup (-C_1) \cup \cdots \cup (-C_n)$ is the boundary of a region in which $f(z)$ is analytic and

$$\begin{aligned} \oint_\Gamma f(z)\,\mathrm{d}z &= \sum_{k=1}^{n} \oint_{C_k} f(z)\,\mathrm{d}z \\ &= 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f,\, z_k). \end{aligned}$$

---

Previously this was just shown to be true when we could split the integrand up using partial fractions for integrands which had a polynomial in the denominator. With the knowledge of Laurent series to describe the behaviour in the vicinity of each point we have now generalised to the above.



## Techniques to calculate the residue

In the case of a **simple pole** of $f(z)$ at $z_0$ most examples for calculating the residue have involved calculating the limit

$$\mathrm{Res}(f,\, z_0) = \lim_{z\to z_0} (z - z_0) f(z).$$

In many of the examples L'Hopital's rule has been used.

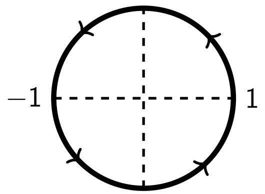
More generally when we have a **pole of order** $m \ge 1$ we can calculate the residue by using

$$\mathrm{Res}(f,\, z_0) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left((z - z_0)^m f(z)\right).$$

We need to know the order of the pole to use the above.



## Trig integrals evaluated using residue theory




$$I = \int_0^{2\pi} R(\cos\theta, \sin\theta)\,\mathrm{d}\theta = \oint_C \frac{1}{i} F(z)\,\mathrm{d}z = 2\pi \sum_k \mathrm{Res}(F,\, z_k).$$

Here $C$ is the unit circle and $F(z)$ is obtained by using

$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad \cos\theta = \frac{z + z^{-1}}{2}, \quad \sin\theta = \frac{z - z^{-1}}{2i}.$$

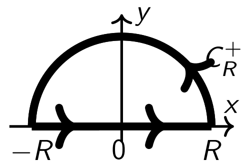
We determine $I$ by the Residue theorem involving the residues of $F(z)$ at the poles $z_k$ which are inside $C$. $F(z)$ is a rational function of $z$. Examples of these first appeared in chap 5.



## Integrals on $(-\infty, \infty)$ evaluated using residue theory

With $P(z)$and $Q(z)$ being polynomials we consider

$$f(z) = \frac{P(z)}{Q(z)} \quad \text{(weeks 23/24)} \quad \text{and} \quad f(z) = \frac{P(z)}{Q(z)}\mathrm{e}^{miz}. \quad \text{(week 24)}$$




Suppose that $f(z)$ has poles at points $z_1, \ldots, z_n$ in the upper half plane. Suppose that $Q(z)$ has no zeros on the real axis.

With $\Gamma_R = [-R, R] \cup C_R^+$ denoting the closed contour

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f,\, z_k).$$

When the integral involving $C_R^+$ tends to 0 as $R \to \infty$ we get

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \quad \text{or} \quad \text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x.$$

## Want does an infinite integral mean?

Let $a \in \mathbb{R}$ we define

$$\int_a^\infty f(x)\,\mathrm{d}x := \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x.$$

$$\int_{-\infty}^a f(x)\,\mathrm{d}x := \lim_{c\to-\infty} \int_c^a f(x)\,\mathrm{d}x.$$

When both limits exist

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x.$$

The principal value version only needs that the following exists.

$$\text{p.v.} \int_{-\infty}^\infty f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^R f(x)\,\mathrm{d}x.$$

When our workings give the principal value a comment is made to justify when we do not put the p.v. notation as the integral exists in the "other sense" as well.



## Example: When only the principal value exists

Consider

$$f(x) = \frac{i}{x+i}, \quad \int_{-R}^0 f(x)\,\mathrm{d}x, \quad \int_0^R f(x)\,\mathrm{d}x, \quad \int_{-R}^R f(x)\,\mathrm{d}x.$$

$$f(x) = \frac{i}{x+i} = \frac{i(x-i)}{x^2+1} = \frac{1+ix}{x^2+1} \quad \text{when } x \in \mathbb{R}.$$

Let

$$F(z) = i\mathrm{Log}(z+i), \quad F'(z) = f(z) = \frac{i}{z+i}.$$

$$\int_{-R}^0 f(x)\,\mathrm{d}x = F(0) - F(-R), \quad \int_0^R f(x)\,\mathrm{d}x = F(R) - F(0)$$

$$\int_{-R}^R f(x)\,\mathrm{d}x = F(R) - F(-R) = \mathrm{Arg}(-R+i) - \mathrm{Arg}(R+i) \to \pi$$

as $R \to \infty$. Both $|F(-R)|$ and $|F(R)|$ are unbounded as $R \to \infty$.

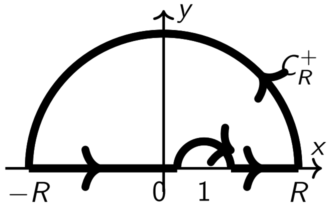


## Singularities on $\mathbb{R}$ and Cauchy principal values

In the lectures and in the exercises of about weeks 24/25 we will also consider integrals of the form

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x$$

when $f(x)$ has poles on the real axis. The integrals need to be considered in a principal valued sense. In the case of a singularity at 1 the indented contour is illustrated below.




The knowledge of the Laurent series enables us to determine the contribution from the smaller half circle as this shrinks to a point.



## Examples this week

$$I = \int_{-\infty}^\infty \frac{\mathrm{d}x}{x^2+2x+2} = \pi.$$

Let

$$f(z) = \frac{1}{z^2+2z+2}.$$

The magnitude on $|z| = R$ is of order $1/R^2$ when $R$ is large. By the $ML$ inequality the parts on $C_R^+$ tends to 0 as $R \to \infty$.

One simple pole at $z_1 = -1+i$ in the upper half plane.

$$\mathrm{Res}(f, z_1) = \frac{1}{2i}.$$

## Examples this week continued

$$I = \int_{-\infty}^{\infty} \frac{1}{x^4 + 16}\,\mathrm{d}x = \frac{\pi\sqrt{2}}{16}.$$

Let

$$f(z) = \frac{1}{z^4 + 16}.$$

$f(z)$ has 4 simple poles in the complex plane and 2 of these are in the upper half plane at the points

$$z_1 = 2\mathrm{e}^{i\pi/4} = \sqrt{2}(1+i), \quad z_2 = 2\mathrm{e}^{3i\pi/4} = \sqrt{2}(-1+i).$$

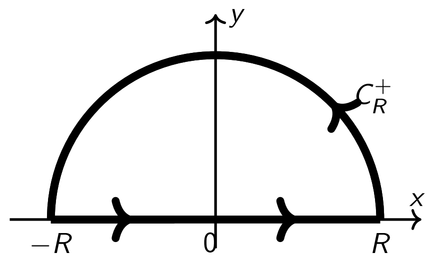
The magnitude on $|z| = R$ is of order $1/R^4$ when $R$ is large. By the $ML$ inequality the parts on $C_R^+$ tends to 0 as $R \to \infty$. In this case

$$I = 2\pi i \left(\mathrm{Res}(f,\, z_1) + \mathrm{Res}(f,\, z_2)\right).$$



## Use of the $ML$ inequality

Consider again the following contour.




The length of the half circle $C_R^+$ is $\pi R$. Suppose $M_R = \max\{|f(z)| :\ z \in C_R^+\}$. By the $ML$ inequality.

$$\left| \int_{C_R^+} f(z)\,\mathrm{d}z \right| \le \pi R M_R.$$

This tends to 0 as $R \to \infty$ when $M_R \to 0$ sufficiently rapidly.



## A sufficient condition for the $C_R^+$ part

Suppose that $f(z)$ is a rational function of the form

$$f(z) = \frac{P(z)}{Q(z)},$$

with $P(z) = a_p z^p + \cdots + a_1 z + a_0$ and $Q(z) = b_q z^q + \cdots + b_1 z + b_0$ where $a_p \ne 0$, $b_q \ne 0$. When $|z| = R$ is large

$$|f(z)| = \mathcal{O}\left(R^{p-q}\right) = \mathcal{O}\left(\frac{1}{R^{q-p}}\right).$$

$RM_R \to 0$ as $R \to \infty$ when $q - p \ge 2$, i.e. $q \ge p + 2$.



## The integrals on $C_R^+$ when we have a $\mathrm{e}^{imz}$ term

After this week.

With $z = x + iy$, $imz = -my + imx$, $\mathrm{e}^{imz} = \mathrm{e}^{-my}\mathrm{e}^{imx}$. When $m > 0$, $|\mathrm{e}^{imz}| = \mathrm{e}^{-my} \le 1$ when $y \ge 0$.

When $\deg(Q) \ge \deg(P) + 2$ we have

$$\int_{C_R^+} \frac{P(z)}{Q(z)}\,\mathrm{d}z \to 0 \quad \text{and} \quad \int_{C_R^+} \frac{P(z)}{Q(z)}\mathrm{e}^{imz}\,\mathrm{d}z \to 0$$

as $R \to \infty$ by using the $ML$ inequality as in the case when $m = 0$.

When $\deg(Q) = \deg(P) + 1$ Jordan's lemma also gives

$$\int_{C_R^+} \frac{P(z)}{Q(z)}\mathrm{e}^{imz}\,\mathrm{d}z \to 0$$

as $R \to \infty$. Jordan's lemma should be covered next week.

If $m < 0$ then the lower half circle needs to be used for a similar result.