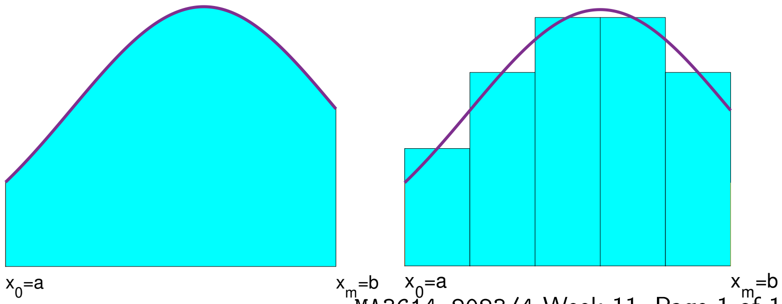
Real integrals – the area under a curve

Reminders about "an appropriate limit of a sum" definition of a definite integral.

Let $a = x_0 < x_1 < \cdots < x_m = b$.

$$A_m = \sum_{i=1}^{m} h_i f(x_{i-1/2}), \quad h_i = x_i - x_{i-1}, \quad x_{i-1/2} = \frac{x_{i-1} + x_i}{2}.$$

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\substack{m\to\infty \\ \max_i h_i \to 0}} A_m.$$



Extending to complex valued functions

If $f : [a,b] \to \mathbb{C}$ with $f = u + iv$, $u, v \in \mathbb{R}$ then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b u(x)\,\mathrm{d}x + i\int_a^b v(x)\,\mathrm{d}x.$$

Integrating a derivative

When

$$F'(x) = f(x)$$

then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a).$$

The interval $[a,b]$ of the real axis is an example of a directed smooth arc.

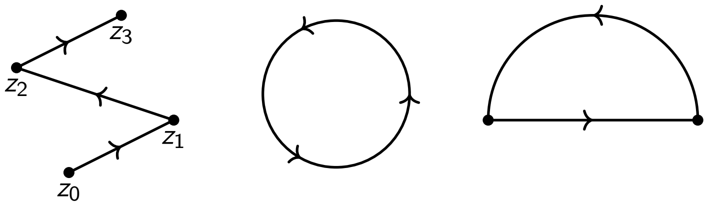
Smooth arcs and contours

A set $\gamma \subset \mathbb{C}$ is a smooth arc if the set can be described in the form

$$\{z(t) : \ a \le t \le b\}, \quad z'(t) \ne 0 \text{ being continuous on } [a,b].$$

A contour is 1 point or a finite sequence of directed smooth arcs γ_k with the end of γ_k being the start of arc γ_{k+1}.

Examples of contours



Definitions of integrals along an arc

A very small change Δt in the parameter t gives a small change

$$\Delta z \approx \frac{\mathrm{d}z}{\mathrm{d}t}\Delta t.$$

The length of γ is

$$L = \int_a^b |z'(t)|\,\mathrm{d}t.$$

The contour integral of $f(z)$ is

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t = \int_a^b (\tilde{u}(t) + i\tilde{v}(t))\,\mathrm{d}t.$$

where $f(z(t))z'(t) = \tilde{u}(t) + i\tilde{v}(t)$.

The ML inequality is

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le ML, \quad \text{where } M = \max_{z\in\gamma} |f(z)|.$$

Independence of the path when $f = F'$

The contour integral of $f(z)$ on $\gamma = \{z(t) : a \le t \le b\}$ is

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t$$

If there exists an anti-derivative F along the path then

$$\frac{\mathrm{d}}{\mathrm{d}t}F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t).$$

This is the integrand in the expression for the contour integral.

Key result:
Suppose that the function $f(z)$ is continuous in a domain D and has an anti-derivative $F(z)$ throughout D. Then for any arc γ contained in D with initial point $z(a)$ and an end point $z(b)$ we have

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b F'(z(t))z'(t)\,\mathrm{d}t = F(z(b)) - F(z(a)).$$

When we have a contour – a union of directed arcs

Suppose $F' = f$ throughout the contour and

$$\Gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n, \quad \text{with} \quad \gamma_k = \{z(t) : \tau_{k-1} \le t \le \tau_k\}.$$

The end point of γ_k is the starting point of γ_{k+1} for $k = 1, \ldots, n-1$.

$$\begin{aligned}\int_\Gamma f(z)\,\mathrm{d}z &= \sum_{k=1}^n \int_{\gamma_k} f(z)\,\mathrm{d}z = \sum_{k=1}^n \int_{\gamma_k} F'(z)\,\mathrm{d}z \\ &= \sum_{k=1}^n (F(z(\tau_k)) - F(z(\tau_{k-1}))) \\ &= F(z(\tau_n)) - F(z(\tau_0)).\end{aligned}$$

The last part is because we have a ‘telescoping’ sum. The answer just depends on the end points when F exists throughout Γ. The continuity of F is needed in the above.

Closed loops and powers of z

Let Γ denote a closed loop.

Let $n \in \mathbb{Z}$ and $z_0 \in \mathbb{C}$.

When $n \ne -1$ the anti-derivative of $(z - z_0)^n$ is $(z - z_0)^{n+1}/(n+1)$ and as a consequence

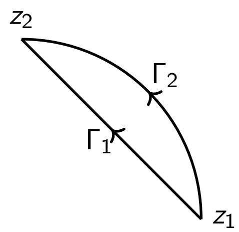
$$\oint_\Gamma (z - z_0)^n\,\mathrm{d}z = 0.$$

When $n = -1$ the function $1/(z - z_0)$ has an anti-derivative $\mathrm{Log}(z - z_0)$ but this function is discontinuous on a branch cut starting from z_0. The value of the integral depends on whether the branch cut intersects with Γ and this depends on whether z_0 is inside or outside the loop.

$$\oint_\Gamma \frac{\mathrm{d}z}{z - z_0}\,\mathrm{d}z = \begin{cases} 2\pi i, & \text{if } z_0 \text{ is inside } \Gamma, \\ 0, & \text{if } z_0 \text{ is outside } \Gamma. \end{cases}$$

The integral does not exist in the usual sense when z_0 is on Γ.

Equivalent statements relating to path independence, loop integrals and anti-derivatives



$\Gamma_2 \cup (-\Gamma_1)$ is a closed loop.

The following are equivalent statements involving the integral of f.

(i) All loop integrals of f are 0.
(ii) The value of the integral of f only depends on the end points.
(iii) There exists an anti-derivative F, i.e. $F' = f$.

(i) and (ii) are equivalent

Let z_I to z_E be points and suppose that Γ_1 and Γ_2 are two paths from z_I to z_E with $\Gamma_2 \cup (-\Gamma_1)$ being a closed loop.

(i) $\Longrightarrow$ (ii): As (i) is true and properties of the integral

$$0 = \oint_{\Gamma_2 \cup (-\Gamma_1)} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z - \int_{\Gamma_1} f(z)\,\mathrm{d}z.$$

All loops containing the two points generates all paths between the points.

(ii) $\Longrightarrow$ (i): As (ii) is true we have

$$\int_{\Gamma_2} f(z)\,\mathrm{d}z = \int_{\Gamma_1} f(z)\,\mathrm{d}z = -\int_{(-\Gamma_1)} f(z)\,\mathrm{d}z.$$

Let $\Gamma = \Gamma_2 \cup (-\Gamma_1)$ and note that this is a loop. Integrating on Γ gives

$$\oint_{\Gamma} f(z)\,\mathrm{d}z = \oint_{\Gamma_2 \cup (-\Gamma_1)} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z + \int_{-\Gamma_1} f(z)\,\mathrm{d}z = 0.$$

All ways of joining two points generates all loops containing the two points.

An expression for the anti-derivative

We have already shown that (iii) (F' existing) implies (ii) (path independence).

(ii) $\Longrightarrow$ (iii): Let D denote a simply connected domain, let $z_0 \in D$ and let $\Gamma(z)$ denote any path in D from z_0 to z.

When all contour integrals of f are path independent we can define

$$F(z) := \int_{\Gamma(z)} f(\zeta)\,\mathrm{d}\zeta$$

and from the definition of the derivative we can show that

$$F'(z) = f(z).$$

But when do we know that loop integrals are 0?

After the revision for the class test we consider a sufficient condition for this involving only properties of f.

The case of rational functions

Let

$$R(z) = \frac{p(z)}{q(z)}, \quad q(z) = (z - z_1)^{r_1}(z - z_2)^{r_2} \cdots (z - z_n)^{r_n}.$$

$$R(z) = \frac{p(z)}{q(z)} = (\text{some polynomial}) + \sum_{k=1}^{n} \frac{A_k}{z - z_k} + (\text{higher order poles}).$$

Here A_k is the **residue** at z_k.

The polynomial part has an anti-derivative (another polynomial) and a $(z - z_k)^{-j-1}$ term has an anti-derivative $(z - z_k)^{-j}/(-j)$ when $j \geq 1$ and hence loop integrals of these part are 0.

$1/(z - z_k)$ has an anti-derivative throughout a loop when z_k is outside the loop and hence loop integrals of such terms are 0.

Loop integrals and rational functions

If $z_1, \ldots, z_m$ are points inside Γ at which $R(z)$ has poles then

$$\oint_{\Gamma} R(z)\,\mathrm{d}z = \sum_{k=1}^{m} A_k \oint_{\Gamma} \frac{\mathrm{d}z}{z - z_k} = 2\pi i \sum_{k=1}^{m} A_k.$$

The answer just depends on the residues at the poles inside Γ.

The above is the residue theorem in the case of rational functions.

Towards the end of the module (in a chapter called "Residue Theory") we show that this holds more generally for any function $f(z)$ which is analytic inside Γ except for a finite number of isolated singularities. In the more general case we cannot give an additive decomposition of the integrand as above and other techniques covered in term 2 are needed to cope with this more general case.