Where will chap 4 results appear again?

Let

$$R(z) = \frac{p(z)}{(z-\zeta_1)^{r_1}(z-\zeta_2)^{r_2}\cdots(z-\zeta_n)^{r_n}}.$$

When $\deg(p) < r_1 + \cdots + r_n$ we have a partial fraction representation

$$\left(\frac{A_{1,1}}{z-\zeta_1} + \cdots + \frac{A_{r_1,1}}{(z-\zeta_1)^{r_1}}\right) + \cdots + \left(\frac{A_{1,n}}{z-\zeta_n} + \cdots + \frac{A_{r_n,n}}{(z-\zeta_n)^{r_n}}\right).$$

The coefficients are

$$A_{i,j} = \frac{1}{(r_j - i)!} \lim_{z\to\zeta_j} \left(\frac{\mathrm{d}^{r_j - i}}{\mathrm{d}z^{r_j - i}} (z-\zeta_j)^{r_j} R(z)\right).$$

The residues $A_{1,1}, \ldots, A_{1,n}$ will appear at the end of chap 5 and in term 2. In term 2 we will see that in the $A_{1,j}$ case we can replace $R(z)$ by $f(z)$ for any function f with an isolated singularity at ζ_j.

Where will chap 4 results appear again continued?

Consider a real interval $-\pi < \theta \le \pi$. By the substitution $z = \mathrm{e}^{i\theta}$ we get the unit circle C for z considered once in the anti-clockwise direction.

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = i\mathrm{e}^{i\theta} = iz, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}.$$

Observe that

$$\cos\theta = \frac{1}{2}\left(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

$$\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{a + \cos\theta} = \oint_C \frac{\mathrm{d}\theta}{\mathrm{d}z}\left(\frac{1}{a + \frac{1}{2}\left(z + \frac{1}{z}\right)}\right)\mathrm{d}z, \quad |a| > 1.$$

We get the integration of a rational function around the unit circle. As we will see later that the answer depends on the residues at the poles which are inside the unit circle.

Poles for more general functions

Rational functions have a finite number of poles but other functions can have infinitely many poles, for example

$$\cot z = \frac{\cos z}{\sin z} = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z + n\pi}.$$

Series and the residue more generally

Taylor series: If $f(z)$ is analytic in the disk $|z - z_0| < R$ then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

and the series converges uniformly in $|z - z_0| \le R' < R$.

Laurent series: If $f(z)$ is analytic in $0 < r < |z - z_0| < R$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n},$$

Uniform convergence in $0 \le r < r_1 \le |z - z_0| \le R_1 < R$.

Both series are unique once z_0 is specified.

All the coefficients can be written as loop integrals.

The coefficient a_{-1} is the residue at z_0 when $r = 0$.

Key results about analytic function before series

Let f be a function which is analytic in a domain D and let Γ be a positively orientated loop in D and let z be a point inside D.

Cauchy-Goursat theorem (Near the end of chap 5)

$$\oint_\Gamma f(\zeta)\,\mathrm{d}\zeta = 0.$$

The Cauchy integral formula (Planned for chap 6)

$$f(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

The generalised Cauchy integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta, \quad n = 0, 1, 2, \ldots$$

The representation of functions by series is planned for chap 7.

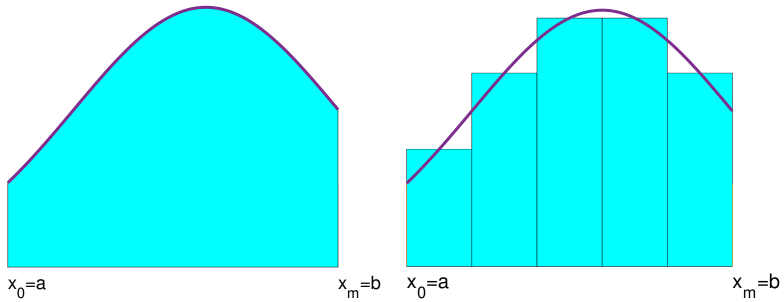
Real integrals – the area under a curve

Let $a = x_0 < x_1 < \cdots < x_m = b$, let $f : [a, b] \to \mathbb{R}$. Let

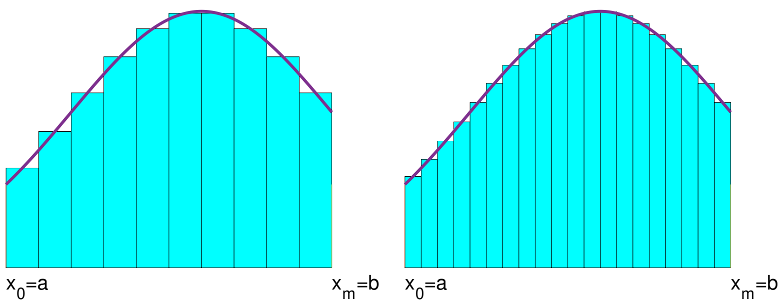
$$A_m = \sum_{i=1}^{m} h_i f(x_{i-1/2}), \quad h_i = x_i - x_{i-1}, \quad x_{i-1/2} = \frac{x_{i-1} + x_i}{2}.$$

When the following limit exists we have

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\substack{m\to\infty \\ \max_i h_i \to 0}} A_m.$$



The approximations with 10 and 20 strips



In this case it is visually reasonably clear that when we double the number of strips we get a more accurate approximation of the area under the curve.

A sufficient condition for the "limiting sum" to exist is that the function f is continuous on $[a, b]$. The limit also exists for many less smooth functions.

Extending to complex valued functions

If $f : [a, b] \to \mathbb{C}$ with $f = u + iv$, $u, v \in \mathbb{R}$ then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b u(x)\,\mathrm{d}x + i\int_a^b v(x)\,\mathrm{d}x.$$

In the following we extend this by replacing the real interval $[a, b]$ by a contour Γ in the complex plane and define what is meant by

$$\int_\Gamma f(z)\,\mathrm{d}z.$$

Integrating a complex valued function from the first exercise sheet

$$\int \mathrm{e}^{kx}\,\mathrm{d}x = \frac{\mathrm{e}^{kx}}{k} + \text{const.}$$

This is valid with $k = p + iq$, $p, q \in \mathbb{R}$.

Let $c = \cos(qx)$, $s = \sin(qx)$.

$$\begin{aligned}\frac{\mathrm{e}^{kx}}{k} &= \mathrm{e}^{px}\left(\frac{c+is}{p+iq}\right) = \mathrm{e}^{px}\left(\frac{(p-iq)(c+is)}{p^2+q^2}\right) \\ &= \mathrm{e}^{px}\left(\frac{(pc+qs)+i(ps-qc)}{p^2+q^2}\right).\end{aligned}$$

If we take the real and imaginary part then we get

$$\begin{aligned}\int \mathrm{e}^{px}\cos(qx)\,\mathrm{d}x &= \mathrm{e}^{px}\left(\frac{p\cos(qx)+q\sin(qx)}{p^2+q^2}\right) + \text{constant}, \\ \int \mathrm{e}^{px}\sin(qx)\,\mathrm{d}x &= \mathrm{e}^{px}\left(\frac{p\sin(qx)-q\cos(qx)}{p^2+q^2}\right) + \text{constant}.\end{aligned}$$

Integration is the reverse of differentiation

Let f denote a real valued function.

The fundamental theorem of calculus involves the following.

1. When an anti-derivative $F(x)$ of $f(x)$ exists, i.e.

$$F'(x) = f(x)$$

then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a).$$

2. When f is continuous

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(s)\,\mathrm{d}s = f(x).$$

Definition of a smooth arc

Smooth arc: A set $\gamma \subset \mathbb{C}$ is a smooth arc if the set can be described in the form

$$\{z(t) : \; a \le t \le b\}$$

where $z(t)$ is continuously differentiable on $[a, b]$, $z'(t) \ne 0$ on $[a, b]$ and the function $z(t)$ is one-to-one on $[a, b]$.

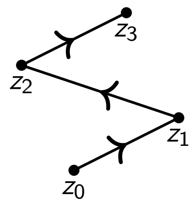
Smooth closed curve. Similar to the above but with now $z(a) = z(b)$ and the one-to-one property only needs to hold on $a \le t < b$ and for smoothness $z'(b) = z'(a)$.

Directed smooth arc: A smooth arc with a specific ordering of the points is known as a directed smooth arc.

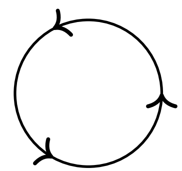
A contour

This is 1 point or a finite sequence of directed smooth arcs γ_k with the end of γ_k being the start of arc γ_{k+1}.

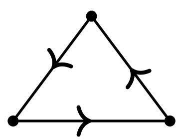
Examples of contours



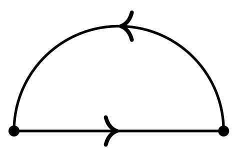
Polygonal path with $n = 3$ arcs.



Circle, anti-clockwise.

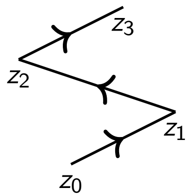


Closed polygonal path with $n = 3$ arcs.



Closed path, $n = 2$ arcs.

The length of an arc



The length of this contour is

$$|z_1 - z_0| + |z_2 - z_1| + |z_3 - z_2|.$$

Let $\gamma = \{z(t) : \ a \le t \le b\}$ and let $a = t_0 < t_1 < \cdots < t_m = b$.
The length of the arc is approximately

$$\sum_{i=1}^{m} |z(t_i) - z(t_{i-1})|.$$

Now when $t_i - t_{i-1}$ is small

$$z(t_i) - z(t_{i-1}) \approx z'(t_{i-1/2})(t_i - t_{i-1}), \quad t_{i-1/2} := \frac{t_i + t_{i-1}}{2}.$$

Take the limit as $m \to \infty$ with $\max_i (t_i - t_{i-1}) \to 0$ to give

$$l(\gamma) = \text{length of } \gamma = \int_a^b |z'(t)| \, \mathrm{d}t.$$

Definition of the contour integral on γ

Let $a = t_0 < t_1 < \cdots < t_m = b$ and let

$$A_m = \sum_{i=1}^{m} h_i f(z(t_{i-1/2})), \quad h_i = z(t_i) - z(t_{i-1}).$$

$$\begin{aligned} h_i f(z(t_{i-1/2})) &= (z(t_i) - z(t_{i-1})) f(z(t_{i-1/2})) \\ &\approx f(z(t_{i-1/2})) z'(t_{i-1/2})(t_i - t_{i-1}). \end{aligned}$$

$$\int_\gamma f(z) \, \mathrm{d}z = \lim_{\substack{m \to \infty \\ \max_i |h_i| \to 0}} A_m = \int_a^b f(z(t)) z'(t) \, \mathrm{d}t.$$

The value here does not depend on which particular valid parameterization $z(t)$ that we use to describe γ.

The ML inequality

Let M and L be defined by

$$M = \max_{z \in \Gamma} |f(z)| \quad \text{and} \quad L = \text{length of } \Gamma.$$

From the bound on $|f(z)|$ and the triangle inequality we have

$$\left| \sum_{i=1}^{m} h_i f(z(t_{i-1/2})) \right| \le \sum_{i=1}^{m} |h_i| \, |f(z(t_{i-1/2}))| \le M \sum_{i=1}^{m} |h_i| \le ML.$$

As the bound above is independent of m and as the integral is an appropriate limit of such a sum we have

$$\left| \int_\gamma f(z) \, \mathrm{d}z \right| = \left| \lim_{\substack{m \to \infty \\ \max_i |h_i| \to 0}} \sum_{i=1}^{m} h_i f(z(t_{i-1/2})) \right| \le ML.$$

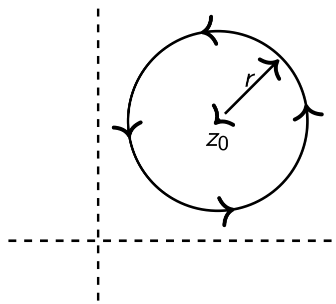
Integrals involving $(z - z_0)^n$, $n = 0, \pm 1, \pm 2, \ldots$

Let

$$C_r = \{z = z(\theta) = z_0 + r e^{i\theta} : \ 0 \le \theta \le 2\pi\}$$

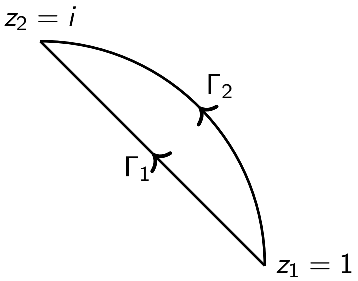
and note that

$$z'(\theta) = i r e^{i\theta}.$$



$$\int_{C_r} (z - z_0)^n \, \mathrm{d}z = \begin{cases} 2\pi i, & \text{if } n = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Examples with path independence



$$\begin{aligned} \Gamma_1 &= \{z_1 + t(z_2 - z_1) : \ 0 \le t \le 1\}, \quad z_1 = 1, \quad z_2 = i, \\ \Gamma_2 &= \{e^{it} : \ 0 \le t \le \pi/2\}. \end{aligned}$$

By direct computation, if $n \ne -1$ then we have

$$\int_{\Gamma_1} z^n \,\mathrm{d}z = \int_{\Gamma_2} z^n \,\mathrm{d}z = \frac{1}{n+1}(i^{n+1} - 1).$$

If $n = -1$ then we have

$$\int_{\Gamma_1} \frac{\mathrm{d}z}{z} = \int_{\Gamma_2} \frac{\mathrm{d}z}{z} = i\frac{\pi}{2}.$$

Independence of path when $f = F'$

If there exists an anti-derivative F along the path then

$$\frac{\mathrm{d}}{\mathrm{d}t} F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t).$$

This is the integrand in the expression for the contour integral.

Key result:

Suppose that the function $f(z)$ is continuous in a domain D and has an anti-derivative $F(z)$ throughout D. Then for any contour Γ contained in D with initial point z_I and an end point z_E we have

$$\int_\Gamma f(z) \,\mathrm{d}z = F(z_E) - F(z_I).$$

When we have a contour – a union of directed arcs

Suppose $F' = f$ throughout the contour and

$$\Gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$$

with the end point of γ_k being the starting point of γ_{k+1} for $k = 1, \ldots, n-1$ and with

$$\gamma_k = \{z(t) : \ \tau_{k-1} \le t \le \tau_k\}.$$

$$\begin{aligned} \int_\Gamma f(z) \,\mathrm{d}z &= \sum_{k=1}^n \int_{\gamma_k} f(z) \,\mathrm{d}z = \sum_{k=1}^n \int_{\gamma_k} F'(z) \,\mathrm{d}z \\ &= \sum_{k=1}^n (F(z(\tau_k)) - F(z(\tau_{k-1}))) \\ &= F(z(\tau_n)) - F(z(\tau_0)). \end{aligned}$$

The last part is because we have a ‘telescoping’ sum. The answer just depends on the end points when F exists throughout Γ.

Some anti-derivatives – powers of z

When $n \in \mathbb{Z}$ and $n \ne -1$.

$$f(z) = z^n, \quad F(z) = \frac{z^{n+1}}{n+1}.$$

Let $\beta \in \mathbb{R}$.

$$F(z) = \mathrm{Log}(e^{i\beta} z), \quad F'(z) = \frac{1}{z}.$$

In the context of contour integrals and integrating $1/z$ along a contour which is not closed we may be able to choose β so that we have an anti-derivative along the path.

For the principle value complex power for any $\alpha \in \mathbb{C}$, $\alpha \ne -1$, we have

$$f(z) = z^\alpha, \quad F(z) = \frac{z^{\alpha+1}}{\alpha+1}.$$

(There is an exercise sheet question to show this.) Care is needed depending on the path of the contour and the branch cut of the functions involved.