A summary of some key points of MA3614 so far

- **Complex derivative:** Let f be a complex valued function defined in a neighbourhood of z_0. The **derivative of f at z_0** is given by

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

provided the limit exists. This will be directly used again when the generalised Cauchy integral formula is derived.

- A function f is **analytic at z_0** if f is differentiable at all points in some neighbourhood of z_0.
- Suppose $z = x+iy$, $f(z) = u(x,y)+iv(x,y)$, $x,y,u,v \in \mathbb{R}$. This is analytic in a domain if and only if **Cauchy Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied in the domain.

A summary continued

- If $f = u+iv$ is analytic, with $u, v \in \mathbb{R}$, then u and v are harmonic functions. v is said to be the **harmonic conjugate** of u.
- Given a harmonic function u in a domain we can find a harmonic conjugate v such that $f = u+iv$ is analytic in the domain when the domain is simply connected.
  This will be justified in term 2.
  Examples have been done when the domain is $\mathbb{C}$.
- An analytic function cannot depend on $\overline{z}$ and can be written in terms of z only.
- When the analytic function $f(z)$ is a polynomial of degree n we have the finite Maclaurin expansion

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)}{n!}z^n.$$

A summary continued, rational functions

Suppose p and q are polynomials with $\deg(p(z)) < \deg(q(z))$ and with

$$q(z) = (z-z_1)^{r_1}(z-z_2)^{r_2}\cdots(z-z_n)^{r_n}$$

and let

$$R(z) = \frac{p(z)}{q(z)}.$$

This function has a partial fraction representation of the form

$$\left(\frac{A_{1,1}}{z-z_1} + \cdots + \frac{A_{r_1,1}}{(z-z_1)^{r_1}}\right) + \cdots + \left(\frac{A_{1,n}}{z-z_n} + \cdots + \frac{A_{r_n,n}}{(z-z_n)^{r_n}}\right)$$

$R(z)$ has poles at the zeros of $q(z)$. $R(z)$ is analytic at all other points and as such we say that the zeros of $q(z)$ are isolated singularities of $R(z)$.

The coefficients $A_{1,k}$, $k = 1, \ldots, n$ are the residues of $R(z)$. These are important later when we consider integration.

Some things that are generalised later

In term 2 one of the topics is series.

This will generalise somethings done with polynomials and rational functions which can be described with a finite number of parameters.

- If $f(z)$ is analytic in a disk $\{z : |z-z_0| < R\}$ then the Taylor series

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \\ &= f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots \end{aligned}$$

is valid in the disk.

- If $f(z)$ be analytic in an annulus $r < |z-z_0| < R$ then it has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}.$$

Exponential function

$$e^z \equiv \exp(z) := e^x\, e^{iy} = e^x(\cos y + i \sin y).$$

As in the real case we have for all $z, z_1, z_2 \in \mathbb{C}$,

$$\frac{d}{dz} e^z = e^z, \quad e^{-z} = \frac{1}{e^z}, \quad e^{z_1+z_2} = e^{z_1}\, e^{z_2}.$$

The function $w = \exp(z)$ is periodic with period $2\pi i$ and is one-to-one on

$$G = \{z = x + iy : \ -\pi < y \le \pi\}$$

with inverse

$$\operatorname{Log} w = \ln |w| + i \operatorname{Arg} w$$

which is the principal valued logarithm.

Note that e^z is not 0 for any z and we need to exclude $w = 0$ when we consider $\operatorname{Log} w$.

cosh z, sinh z, cos z, sin z

We define

$$\cosh z = \frac{1}{2}\left(e^z + e^{-z}\right), \quad \sinh z = \frac{1}{2}\left(e^z - e^{-z}\right),$$
$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right).$$

As in the real case

$$\frac{d}{dz}\cosh z = \sinh z, \quad \frac{d}{dz}\sinh z = \cosh z,$$
$$\frac{d}{dz}\cos z = -\sin z, \quad \frac{d}{dz}\sin z = \cos z.$$

We also have the identities

$$\cos^2 z + \sin^2 z = \cosh^2 z - \sinh^2 z = 1.$$

For all $z_1, z_2 \in \mathbb{C}$ we have the addition formulas

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2,$$
$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2.$$

cot **and** tan

As

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \frac{e^{-iz}}{2i}\left(e^{2iz} - 1\right)$$

it follows that $\sin z = 0$ if and only if $e^{2iz} = 1$ which is if and only if $z = k\pi$ where k is an integer.

$$\cot z = \frac{\cos z}{\sin z}, \quad \tan z = \frac{\sin z}{\cos z} = -\cot(z - \pi/2) = \frac{1}{\tan(\pi/2 - z)}.$$

$\cot z$ has simple poles at $k\pi$ and $\tan z$ has simple poles at $\pi/2 + k\pi$ where $k \in \mathbb{Z}$.

In term 2 we will define properly what we mean by poles of functions which have isolated singularities. At the moment we have only done this properly in the case of rational functions. The residue will also be defined in this context.

Let $z_k = k\pi$. The residue of $\cot z$ at z_k is

$$\lim_{z \to z_k} (z - z_k) \cot z = \cos(z_k) \lim_{z \to z_k} \frac{z - z_k}{\sin z} = 1.$$

More advanced representations of cot z and sin z

The following is beyond what will be covered in MA3614 but it can be shown that $\cot z$ has a partial fraction type representation in terms of its poles and $\sin z$ can be 'factorised in terms of its zeros' in the following sense.

$$\cot z = \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{z + n\pi} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}.$$

The Euler-Wallis formula for the sine function is

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n\pi}\right)^2\right).$$

$\mathrm{Log}\, z$ **and the multi-valued** $\log z$

The principal valued logarithm is

$$\mathrm{Log}\, z = \ln |z| + i\mathrm{Arg}\, z.$$

The multi-valued version $w = \log z$ means all complex numbers w such that

$$e^{w} = z$$

and the set of values is

$$\{\mathrm{Log}\, z + 2k\pi i : \ k \in \mathbb{Z}\}.$$

In both cases

$$e^{\mathrm{Log}\, z} = e^{\log z} = z.$$

(In the last case we technically get the set $\{z\}$ which just has one entry and we interpret this as z.)

Complex powers z^{α}

The principal value of z^{α} is defined as

$$e^{\alpha \mathrm{Log}\, z}.$$

The possibly multi-valued version is

$$e^{\alpha \log z}.$$

In the multi-valued case how many values depends on α. If $\alpha = n \in \mathbb{Z}$ then there is just 1 value and when $\alpha = 1/n$, n being an integer $w = z^{1/n}$ gives the n roots of z, i.e. all solutions of

$$w^{n} = z.$$

If α is irrational or not-real then the multi-valued version means infinitely many different numbers.

In all cases the principal value is one of the values that the multi-valued version gives.

The case i^{i}

As

$$|i| = 1 \quad \text{and} \quad \mathrm{Arg}\, i = \frac{\pi}{2}$$

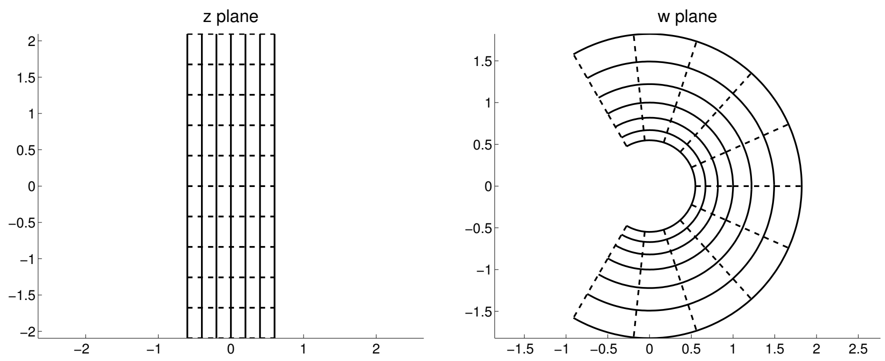
we have

$$\mathrm{Log}\, i = i\,\frac{\pi}{2}.$$

The principal value is

$$i^{i} = \exp(i \mathrm{Log}\, i) = e^{-\pi/2}.$$

Mapping of $w = \exp(z)$, level curves of $z = \mathrm{Log}(w)$

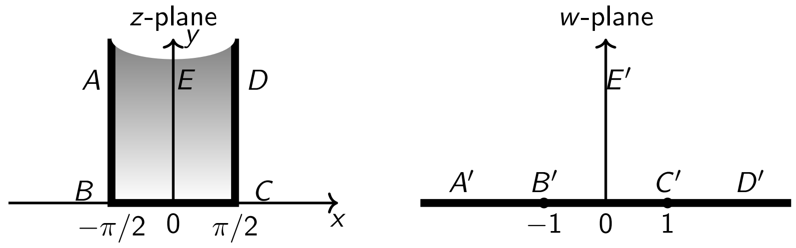


The circles and radial lines are curves where the real and imaginary parts of $\mathrm{Log}(w)$ are constant. These are orthogonal.

The mapping $w = \sin z$

$$G' = \{z = x + iy : -\pi/2 \le x \le \pi/2,\ 0 \le y < \infty\}.$$

The image of the semi-infinite strip G' under the mapping $w = \sin z$, $z = x + iy$ is the upper half plane shown below.



$$\frac{d}{dz} \sin z = \cos z.$$

At points B and C the derivative is 0.

The mapping $w = \sin z$ on a rectangle

Let

$$G_b = \{z = x + iy : -\pi/2 \le x \le \pi/2,\ 0 \le y < b\}.$$

The image of the rectangle G_b under the mapping $w = \sin z$, $z = x + iy$ is half an ellipse.

The points

$$\sin(x + ib) = \sin x \cosh b + i \cos x \sinh b, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

are on an upper part of an ellipse. The semi-axes of the ellipse have lengths $\cosh b$ and $\sinh b$ respectively.

The mapping $w = \tan z$ on lines $x = 0$ and $x = \pi/4$

Let $z = x + iy$ and $w = u + iv$ where $x, y, u, v \in \mathbb{R}$.

$$w = \tan z = \frac{\sin z}{\cos z} = (-i)\left(\frac{1 - e^{-2iz}}{1 + e^{-2iz}}\right).$$

$$\tan(iy) = i \tanh y = i\frac{\sinh y}{\cosh y}.$$

The image of $\{iy : y \in \mathbb{R}\}$ is $\{iv : v \in (-1, 1)\}$ which is a segment of the imaginary axis.

In the exercises you are asked to show that

$$|\tan(\pi/4 + iy)| = 1.$$

The image of $\{\pi/4 + iy : y \in \mathbb{R}\}$ is $\{e^{i\phi},\ -\pi/2 < \phi < \pi/2\}$ which is a half circle.

Rational functions – where they will appear later

Consider a real interval $-\pi < \theta < \pi$. By the substitution $z = e^{i\theta}$ we get the unit circle C for z.

$$\frac{dz}{d\theta} = ie^{i\theta} = iz, \quad \frac{d\theta}{dz} = \frac{1}{iz}.$$

Observe that

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

$$\int_{-\pi}^{\pi} \frac{d\theta}{a + \cos\theta} = \oint_C \frac{d\theta}{dz} \left(\frac{1}{a + \frac{1}{2}\left(z + \frac{1}{z}\right)} \right) dz.$$

We get the integration of a rational function around the unit circle.

As we will see later that the answer depends on the residues at the poles which are inside the unit circle.