MA3614 Complex variable methods and applications
Comments, topics and why it is taught

- **Will the module involve complex numbers?**
  Yes. The complex number material that you learned in MA1620 will be used.

  The module is more about functions of a complex variable. For many real valued functions $f(x)$, $x \in \mathbb{R}$ it makes sense to consider

$$f(z), \quad z = x + iy, \quad x, y \in \mathbb{R}, \quad i^2 = -1.$$

  The natural domain of many functions that you have considered is the complex plane. Hence you learn more about such functions.

Comments, topics and why it is taught continued

- **Why study something that is not real?**

  A brief answer to this is that it helps to understand the real case better. There are some examples of this in these slides.

  It is also a tool in solving real problems. This is the application part.

What previous study will be useful?

- Complex number manipulation from MA1620, e.g. $z = x + iy = re^{\imath\theta}$, $z^n = r^n e^{in\theta}$ etc.

- Partial differentiation from MA2612, e.g. for a sufficiently smooth function $u(x, y)$,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

- Geometric series from possibly several previous modules, i.e.

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots, \quad \text{when } |z| < 1.$$

Without detail what topics are involved?

- Differentiation in a complex sense.
- Integration in the complex plane.
- Power series and Laurent series representations of functions. (Term 2).
- Applications usually involving residue theory. (Term 2).

Which functions make sense with a complex variable?

1. **Polynomials**

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0.$$

This has n roots (counting multiplicities) in the complex plane. We need to study complex integration to explain this.

2. **Rational functions** (i.e. a ratio of polynomials).

$$f(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}.$$

When $n < m$ there is a partial fraction representation. You may have had rules to get this representation. Do you know why the rules work?

3. **Exponential function**.

$$\exp(z) = \mathrm{e}^x \mathrm{e}^{iy} = \mathrm{e}^x(\cos(y) + i\sin(y)).$$

The real case of e^x and the notation $\mathrm{e}^{i\theta} = \cos(\theta) + i\sin(\theta)$ are special cases.

Some examples of things the complex case explains

The following relate to things you possibly have met before.

1. Suppose that you have a real polynomial.

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad a_n \neq 0, \quad a_k \in \mathbb{R}.$$

   Non-real roots occur in complex conjugate pairs. This is a consequence of

$$p(\overline{z}) = \overline{p(z)}.$$

2. Why is the Maclaurin series for

$$f(x) = \frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \cdots$$

   only valid in $-1 < x < 1$? Note that the function is infinitely differentiable on $\mathbb{R}$. This is because $f(z)$ has singularities at $\pm i$. The series (which is a geometric series) is valid for $|z| < 1$.

What additional properties will be covered?
Differentiation and analytic

In the real case differentiation is considered. In this module we consider when the functions are also differentiable in a complex sense and a related **analytic** property. Many additional results will depend on where $f(z)$ is analytic and where it is not. The complex differentiable property at z_0 is concerned with when the following limit exists.

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}.$$

We have the same expression as in the real case but now we are dividing by a complex number and we must get the same value however h tends to 0 to be complex differentiable at z_0.

$f(z)$ is analytic at z_0 if it is complex differentiable at z_0 and in a neighbourhood of z_0. This will probably first be done in about week 3.

Why is MA2612 a prerequisite?

With $z = x + iy$, $x, y \in \mathbb{R}$ a function of a complex variable

$$w = f(z), \quad w = u + iv, \quad u, v \in \mathbb{R}$$

is in full

$$f(x + iy) = u(x, y) + iv(x, y).$$

We have real valued functions u and v of 2-variables x, y. When $f(z)$ is complex differentiable we can express $f'(z)$ in terms of the partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y}.$$

We will see that $f(z)$ is analytic in a domain if and only if the following hold in the domain.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are the Cauchy Riemann equations.

Contour integrals

In the real case you consider definite integrals of the form

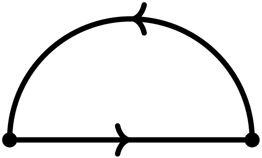
$$\int_a^b f(x)\,\mathrm{d}x.$$

Generalising to the complex case involves an arc Γ in the complex plane and we write

$$\int_\Gamma f(z)\,\mathrm{d}z.$$

Examples of Γ

A line segment.



A union of a line segment and a half circle to give a loop.

Taylor series will be explained?

With integration introduced a key result early in term 2 is to show that when $f(z)$ is analytic in a domain, Γ is a closed loop traversed once in the anti-clockwise direction and z is inside Γ we have the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

This implies the generalised Cauchy integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta, \quad n = 0, 1, 2, \ldots$$

Using both gives the Taylor series

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k.$$

If $f(z)$ is analytic in $|z - z_0| < R$ then the series representation is valid in this disk.

Early jargon: Laurent series

A Laurent series is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n.$$

When it converges the region is an annulus $\{z : \ r < |z - z_0| < R\}$.

Laurent series representation

Let $f(z)$ be analytic in an annulus $r < |z - z_0| < R$. Then it has the representation

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}.$$

Early jargon: A residue

This will first be met when considering partial fractions. Consider

$$R(z) = \frac{p(z)}{q(z)}, \quad q(z) = (z - z_1)(z - z_2) \cdots (z - z_n).$$

When $\deg p(z) < \deg q(z)$ and the zeros of $q(z)$ are simple we have the partial fraction representation of the form

$$R(z) = \frac{p(z)}{q(z)} = \sum_{k=1}^{n} \frac{A_k}{z - z_k}.$$

Here A_k is the **residue** at z_k. This will be covered in term 1.

More generally, when

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

converges in $0 < |z - z_0| < R$ the coefficient a_{-1} is the residue at z_0. This will be covered in term 2.