Loop integrals and analytic functions

Let f be a function which is analytic in a domain D and let Γ be a positively orientated loop in D and let z be a point inside D. Also suppose that $f(z)$ is analytic inside Γ.

Cauchy-Goursat theorem

$$\oint_\Gamma f(\zeta)\,\mathrm{d}\zeta = 0,$$

The Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

From this formula it followed that all the derivatives of f exist and we have similar formulas for the derivatives.

The generalised Cauchy integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta, \quad n = 0, 1, 2, \ldots$$

Remark about Taylor series representation

In chapter 7 the Taylor series representation of f is derived from the Cauchy integral formula. Before this is done note that the generalised version gives a representation for the Taylor series coefficients as

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z, \quad n = 0, 1, 2, \ldots$$

Remark about Laurent series representation

In chapter 7 the Laurent series representation of a function analytic in an annulus is also derived from the Cauchy integral formula.

Loop integrals of $f(z)/\left((z-z_0)^{n+1}\right)$

By directly using the generalised Cauchy integral formula we can evaluate integrals of the following form.

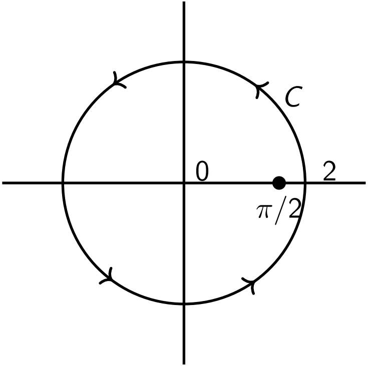
$$\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = (2\pi i)\frac{f^{(n)}(z_0)}{n!}.$$

We just need a function $f(z)$ which is analytic inside Γ, the function can have non-analytic points elsewhere. z_0 is the only point where the integrand has an isolated singularity.

An example of evaluating an integrals using the CIF

Let $C = \{z : |z| = 2\}$. We traverse once in the anti-clockwise direction.

$$I = \oint_C \frac{\sin(3z)}{z - \pi/2}\,\mathrm{d}z.$$



$$\frac{f(z)}{z - z_0} = \frac{\sin(3z)}{z - \pi/2},$$

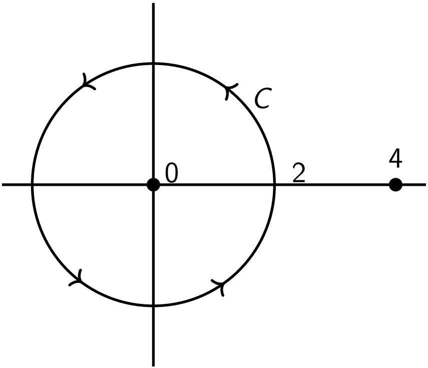
with $f(z) = \sin(3z), \quad z_0 = \pi/2.$

$$I = (2\pi i) f(\pi/2) = (2\pi i)\sin(3\pi/2) = -2\pi i.$$

An example using the generalised CIF

Let $C = \{z : |z| = 2\}$. We traverse once in the anti-clockwise direction.

$$I = \oint_C \frac{\sin z}{z^2(z-4)}\,\mathrm{d}z.$$



$$\frac{f(z)}{(z-z_0)^{n+1}} = \frac{\sin z}{z^2(z-4)},$$

$$\text{with } f(z) = \frac{\sin z}{z-4},$$

$$z_0 = 0 \quad n+1 = 2.$$

$f(z)$ is analytic inside C.

$$I = (2\pi i)f'(0) = \text{..some detail to do..} = -\frac{\pi}{2}i.$$

Loop integrals of $f(z)/q(z)$ where q=polynomial

Suppose $f(z)$ is analytic inside a loop and

$$q(z) = (z - z_1)^{r_1}(z - z_2)^{r_2} \cdots (z - z_n)^{r_n},$$

with $r_k \ge 1$ for $k = 1, \ldots, n$. Using partial fractions we get

$$\frac{f(z)}{q(z)} = f(z)\left(\cdots + \frac{A_{1,k}}{z - z_k} + \cdots + \frac{A_{r_k,k}}{(z - z_k)^{r_k}} + \cdots\right).$$

By using the generalised Cauchy integral formula on each term for each point z_k inside Γ we can determine

$$\oint_\Gamma \frac{f(z)}{q(z)}\,\mathrm{d}z.$$

In chapter 8 on residue theory we can extend further and have the ratio of any two analytic functions. We postpone until chapter 8 most of the examples with several poles inside the loop.

Functions defined by loop integrals

Let Γ denote a closed loop traversed once in the anti-clockwise direction and let $g(\zeta)$ denote any continuous function defined on Γ. If we define

$$G(z) = \oint_\Gamma \frac{g(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

then this defines an analytic function for z inside Γ and it also defines an analytic function for z outside Γ. (This is an exercise sheet question.) As an example

$$g(\zeta) = \frac{1}{2\pi i}\left(\frac{f(\zeta)}{\zeta - z_0}\right) \quad \text{gives } G(z) = \begin{cases} f'(z_0), & \text{if } z = z_0, \\ \dfrac{f(z) - f(z_0)}{z - z_0}, & \text{if } z \neq z_0. \end{cases}$$

In this case $g(\zeta)$ is defined inside the loop and it has a singularity at $\zeta = z_0$.

Morera's theorem

Cauchy's theorem: If f is analytic inside Γ then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

Morera's theorem: If

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0$$

for all loops Γ then f is analytic inside Γ.

The proof is very short. All loop integrals being 0 implies the existence of an analytic F s.t. $f = F'$. All derivatives of F are analytic and hence $f(z)$ and all its derivatives are also analytic.

This is used when sequences and series are considered as when we have a sequence of analytic functions (e.g. polynomials) which converges uniformly we deduce that the limit function is also analytic.

Harmonic functions — further results

Suppose that $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D with $u, v \in \mathbb{R}$. We have the following.

- As all derivatives of f exist and are analytic it follows that all partial derivatives of u and v exist and are continuous.
- Both u and v are harmonic, i.e. $\nabla^2 u = 0$ and $\nabla^2 v = 0$, with v being the harmonic conjugate of u.
- $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

  As f' is analytic the first partial derivatives of u and v are harmonic functions. All the partial derivatives of u and v are harmonic.

Creating an analytic function from a harmonic function

Suppose we have a function ϕ which is harmonic in a domain D. (There is no restriction here on D being simply connected.) Let

$$g(z) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}.$$

We show that $g(z)$ is analytic and let

$$u_1 = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v_1 = -\frac{\partial \phi}{\partial y}.$$

We can check if the Cauchy Riemann equations are satisfied.

$$\frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial y} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} = \frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial^2 \phi}{\partial x \partial y} = 0.$$

This is because ϕ is harmonic and as mixed partial derivatives can be done in any order.

Creating an analytic function from a harmonic function continued

When we have a function ϕ which is harmonic in a domain D the following function is hence analytic in D.

$$g(z) = \frac{\partial \phi}{\partial x} - i \frac{\partial \phi}{\partial y}.$$

To repeat, there was no restriction on the domain D in the above.

Now if we restrict D to be a simply connected domain then the property that g is analytic in D implies that g has an anti-derivative G in D, i.e. $G' = g$. If we represent G as $G = u + iv$, $u, v \in \mathbb{R}$ then as G is analytic we now have

$$g = G' = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{\partial \phi}{\partial x} - i \frac{\partial \phi}{\partial y}.$$

This implies that $\phi = u + C$, where C is a constant, and we know that a function v exists which is a harmonic conjugate of ϕ in the domain D.

Harmonic u and $u + iv$ when D is not simply connected

Let

$$D = \{z : \ 0 < |z| < \infty\}$$

and let

$$u(x, y) = \ln(r) = \ln|z| = \frac{1}{2}\ln(x^2 + y^2).$$

This function is harmonic in D but no harmonic conjugate v exists such that $u + iv$ is analytic in D. If we remove part of D to create the simply connected domain

$$D' = \{z \in D : \ \operatorname{Arg} z \neq \pi\}$$

then the harmonic conjugate

$$v = \operatorname{Arg} z$$

is such that

$$u + iv = \ln|z| + i\operatorname{Arg} z = \operatorname{Log}(z)$$

is analytic in D' but not in D due to the branch cut along the negative real axis.

Other versions of the formulae and entire functions

Cauchy integral formula when Γ=circle of radius R.

$$f^{(n)}(z_0) = \frac{n!}{2\pi R^n} \int_0^{2\pi} f(z_0 + R\mathrm{e}^{it})\mathrm{e}^{-int}\,\mathrm{d}t.$$

This is used to show that a bounded entire function is a constant (**Liouville's theorem**) by considering what happens when $n = 1$ and $R \to \infty$. The ML inequality is used.

The fundamental theorem of algebra

Theorem: Every non-constant polynomial with complex coefficients has at least one zero.

Outline of the proof

Let

$$p(z) = a_n z^n + \cdots + a_1 z + a_0, \quad n \ge 1, \quad a_n \ne 0,$$

denote the polynomial and let

$$f(z) = \frac{1}{p(z)}.$$

The proof is by contradiction which starts by assuming that $p(z)$ has no zeros and this implies that $f(z)$ is an entire function.

The bulk of the detail is in showing that $f(z)$ is bounded and a bounded entire function is a constant. $f(z)$ being constant implies that $p(z)$ is constant and we have our contradiction as the degree $n \ge 1$.

Bounding $1/p(z)$

$$\frac{p(z)}{z^n} = \left(a_n + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right) \to a_n \quad \text{as } |z| \to \infty.$$

This guarantees that $\exists R > 0$ such that for $|z| \ge R$

$$\left| \frac{p(z)}{z^n} \right| \ge \frac{|a_n|}{2} \quad \text{giving} \quad \left| \frac{1}{p(z)} \right| \le \frac{2}{|a_n z^n|} \le \frac{2}{|a_n| R^n}.$$

Hence we have a bound on $|f(z)|$ for $|z| \ge R$.

The analytic property in $|z| \le R$ implies that $|f(z)|$ is bounded here. It is itself a proof by contradiction to prove this step.

Thus $|f(z)|$ is bounded in the entire complex plane.

Depending on the time available another proof of the Fundamental theorem of algebra may be mentioned which uses Rouché theorem. If done then this will be in chapter 8.

Further results

Lemma: If $f(z)$ is analytic in a domain D and $|f(z)|$ is constant in D then $f(z)$ is a constant.

This was on an term 1 exercise sheet.

The mean value property

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\,\mathrm{d}t, \quad |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + Re^{it})|\,\mathrm{d}t.$$

Lemma: Suppose $f(z)$ is analytic in a disk centred at z_0. If the maximum value of $|f(z)|$ over the disk is the value at the centre, i.e. the value $|f(z_0)|$, then $f(z)$ is constant on the disk.

The maximum modulus theorem

Theorem: If f is analytic in a domain D and $|f(z)|$ achieves its maximum value at a point $z_0 \in D$ then f is a constant in D.