Rational functions – definition and singularities

A polynomial can be factored. Suppose that

$$q(z) = (z - z_1)(z - z_2) \cdots (z - z_n).$$

The ratio of two polynomials is a rational function. Let

$$R(z) = \frac{p(z)}{q(z)},$$

The zeros $z_1, \ldots, z_n$ of $q(z)$ are singular points of $R(z)$.

If the limit exists as $z \to z_k$ then z_k is a **removable singularity**.

Otherwise $R(z)$ has a **pole singularity** at z_k. A **simple pole** is the case when $1/R(z)$ has a simple zero at z_k.

The order of the pole of $R(z)$ is the multiplicity of the zero of $1/R(z)$.

Rational functions – partial fractions representation

$$R(z) = \frac{p(z)}{q(z)}, \quad q(z) = (z - z_1)(z - z_2) \cdots (z - z_n).$$

When $\deg p(z) < \deg q(z)$ and the zeros of $q(z)$ are simple we have the partial fraction representation of the form

$$R(z) = \frac{p(z)}{q(z)} = \sum_{k=1}^{n} \frac{A_k}{z - z_k}.$$

When $\deg p(z) \geq \deg q(z)$ and the zeros of $q(z)$ are simple we have a representation of the form

$$R(z) = \frac{p(z)}{q(z)} = (\text{some polynomial}) + \sum_{k=1}^{n} \frac{A_k}{z - z_k}.$$

In either case A_k is the **residue** at z_k.

Getting the residues when we only have simple poles

$$R(z) = \frac{p(z)}{q(z)} = \text{(some polynomial)} + \sum_{k=1}^{n} \frac{A_k}{z - z_k}.$$

To get A_k we have

$$\begin{aligned} A_k &= \lim_{z\to z_k} (z - z_k) R(z) = \lim_{z\to z_k} \frac{(z - z_k)p(z)}{q(z)} \\ &= \lim_{z\to z_k} p(z) \lim_{z\to z_k} \frac{(z - z_k)}{q(z)} = \frac{p(z_k)}{q'(z_k)}. \end{aligned}$$

With

$$q(z) = (z - z_1)(z - z_2)\cdots(z - z_n) = (z - z_k) g_k(z).$$

Here $g_k(z)$ is the product of the other factors.

$$q'(z) = (z - z_k){g_k}'(z) + g_k(z), \quad q'(z_k) = g_k(z_k).$$

Multiple poles case

When $q(z)$ has a zero at z_0 of multiplicity $r \geq 1$ we need terms involving

$$\frac{1}{z - z_0}, \quad \frac{1}{(z - z_0)^2}, \quad \ldots, \quad \frac{1}{(z - z_0)^r}.$$

Usually there is more work to get the representation when $r > 1$.

The residue comes from the term involving $\frac{1}{z - z_0}$.

Partial fraction examples in week 6

$$
\begin{aligned}
f_1(z) &= \frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i}. \\
f_2(z) &= \frac{z^3}{z^2+1} = (\text{Degree 1 polynomial}) + \frac{A}{z+i} + \frac{B}{z-i}. \\
f_3(z) &= \frac{4}{(z^2+1)(z-1)^2} = \frac{A}{z+i} + \frac{B}{z-i} + \frac{C_1}{z-1} + \frac{C_2}{(z-1)^2}.
\end{aligned}
$$

In all cases we have $z^2+1=(z+i)(z-i)$ and we have pole singularities at $\pm i$. The residues are associated with the simple pole terms and are labelled as A and B in the case of f_1 and f_2 and are labelled as A, B and C_1 in the case of f_3.

In the calculation in the $f_3(z)$ case we used

$$
(z-1)^2 f_3(z) = \frac{4}{z^2+1},
$$

before differentiation and limits were considered.

Finer points about the residue

Suppose

$$R(z) = \frac{2}{4z^2 - 1} = \frac{A}{2z + 1} + \frac{B}{2z - 1}.$$

To get A and B we have

$$A = \lim_{z \to -1/2} \frac{2(2z + 1)}{4z^2 - 1} = -1, \quad B = \lim_{z \to 1/2} \frac{2(2z - 1)}{4z^2 - 1} = 1.$$

The residues are however

$$\lim_{z \to -1/2} (z + 1/2) R(z) = \frac{A}{2} = -\frac{1}{2} \quad \text{and} \quad \lim_{z \to 1/2} (z - 1/2) R(z) = \frac{B}{2} = \frac{1}{2}.$$

$$R(z) = \frac{-1/2}{z + 1/2} + \frac{1/2}{z - 1/2}.$$

Special case of one multiple pole

Suppose

$$R(z) = \frac{p(z)}{(z - z_0)^n}, \quad p(z) \text{ being a polynomial of degree } m.$$

We use the Taylor series representation of $p(z)$ about z_0.

$$p(z) = p(z_0) + p'(z_0)(z - z_0) + \cdots + \frac{p^{(m)}(z_0)}{m!}(z - z_0)^m.$$

If $m < n - 1$ then the residue is 0. If $m \geq n - 1$ then

$$R(z) = \frac{p(z_0)}{(z - z_0)^n} + \frac{p'(z_0)}{(z - z_0)^{n-1}} + \cdots + \frac{p^{(n-1)}(z_0)/(n-1)!}{z - z_0} + \cdots$$

and the residue at z_0 is

$$\frac{p^{(n-1)}(z_0)}{(n-1)!}.$$

Is a partial fraction representation always possible?

Suppose $\deg(p(z)) < \deg(q(z))$ with

$$q(z) = (z - z_1)^{r_1}(z - z_2)^{r_2} \cdots (z - z_n)^{r_n},$$

$z_1 \ldots, z_n$ being distinct, and let

$$R(z) = \frac{p(z)}{q(z)}.$$

Assuming a representation is possible, i.e.

$$\left(\frac{A_{1,1}}{z - z_1} + \cdots + \frac{A_{r_1,1}}{(z - z_1)^{r_1}}\right) + \cdots + \left(\frac{A_{1,n}}{z - z_n} + \cdots + \frac{A_{r_n,n}}{(z - z_n)^{r_n}}\right)$$

we can get the coefficients as in the examples. We have a formula for each coefficient (see on the next slides).

General case ...comments on the validity

$$R(z) = \frac{p(z)}{(z-z_1)^{r_1}(z-z_2)^{r_2}\cdots(z-z_n)^{r_n}}.$$

With the procedures above we can get the coefficients in the following candidate representation of $R(z)$.

$$\left(\frac{A_{1,1}}{z-z_1}+\cdots+\frac{A_{r_1,1}}{(z-z_1)^{r_1}}\right)+\cdots+\left(\frac{A_{1,n}}{z-z_n}+\cdots+\frac{A_{r_n,n}}{(z-z_n)^{r_n}}\right).$$

The coefficients are

$$A_{i,j} = \frac{1}{(r_j-i)!}\lim_{z\to z_j}\left(\frac{\mathrm{d}^{r_j-i}}{\mathrm{d}z^{r_j-i}}(z-z_j)^{r_j}R(z)\right), \quad i=1,2,\ldots,r_j.$$

General case ...comments on the validity continued

How do we show that the following are the same function for all z?

Rational function

$$R(z) = \frac{p(z)}{q(z)} = \frac{p(z)}{(z - z_1)^{r_1}(z - z_2)^{r_2} \cdots (z - z_n)^{r_n}}.$$

Partial fraction representation denoted by $\tilde{R}(z)$ given by

$$\left(\frac{A_{1,1}}{z - z_1} + \cdots + \frac{A_{r_1,1}}{(z - z_1)^{r_1}} \right) + \cdots + \left(\frac{A_{1,n}}{z - z_n} + \cdots + \frac{A_{r_n,n}}{(z - z_n)^{r_n}} \right).$$

Let

$$g(z) = R(z) - \tilde{R}(z).$$

This is a rational function. $g(z) = 0$ because it can be shown that it has removable singularties at $z_1, \ldots, z_n$ and because it tends to 0 as $|z| \to \infty$. Details are long and are not examinable.

Exponential function

$$e^z \equiv \exp(z) := e^x\, e^{iy} = e^x(\cos\, y + i \sin\, y).$$

As in the real case we have for all $z, z_1, z_2 \in \mathbb{C}$,

$$\frac{d}{dz} e^z = e^z, \quad e^{-z} = \frac{1}{e^z}, \quad e^{z_1 + z_2} = e^{z_1}\, e^{z_2}.$$

The function $w = \exp(z)$ is periodic with period $2\pi i$ and is one-to-one on

$$G = \{z = x + iy :\ -\pi < y \le \pi\}$$

with inverse

$$\mathrm{Log}\, w = \mathrm{Log}\, |w| + i \mathrm{Arg}\, w$$

which is the principal valued logarithm.

The principal valued logarithm will be discussed more after the reading week break.

$\cosh z$, $\sinh z$, $\cos z$, $\sin z$

We define

$$\begin{aligned} \cosh z &= \frac{1}{2}\left(e^{z} + e^{-z}\right), \quad \sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right), \\ \cos z &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right). \end{aligned}$$

As in the real case

$$\frac{d}{dz}\cosh z = \sinh z, \quad \frac{d}{dz}\sinh z = \cosh z,$$
$$\frac{d}{dz}\cos z = -\sin z, \quad \frac{d}{dz}\sin z = \cos z.$$

We also have the identities

$$\cos^2 z + \sin^2 z = \cosh^2 z - \sinh^2 z = 1.$$

For all $z_1, z_2 \in \mathbb{C}$ we have the addition formulas

$$\begin{aligned} \sin(z_1 \pm z_2) &= \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2, \\ \cos(z_1 \pm z_2) &= \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2. \end{aligned}$$

Further comments about the complex versions

Let

$$f(z) = \cos^2 z + \sin^2 z - 1,$$
$$g(z) = \cosh^2 z - \sinh^2 z - 1.$$

From the definitions these are entire functions and from the identities in the case $z = x \in \mathbb{R}$ we have that they are zero on the real line.

As we see in term 2, the zeros of an analytic function which is not identically zero everywhere are isolated. As $f(x) = 0$ and $g(x) = 0$ for all $x \in \mathbb{R}$ this implies that $f(z) = 0$ and $g(z) = 0$ for all z in the complex plane. Of course, in these two examples we can verify that $f(z) = 0$ and $g(z) = 0$ without too much effort by just using the definitions.

The real and imaginary parts of $\sin(z)$ and $\cos(z)$

With $z = x + iy$, $x, y, \in \mathbb{R}$ we have

$$\begin{aligned} \sin(x + iy) &= \sin x \cosh y + i \cos x \sinh y, \\ \cos(x + iy) &= \cos x \cosh y - i \sin x \sinh y. \end{aligned}$$

The real and imaginary parts of these functions are hence harmonic functions.

Representing a function in terms of its zeros

A polynomial of degree n with zeros at $z_1, \ldots, z_n$ can be expressed in the form

$$p_n(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n).$$

Some of the standard functions with an infinite number of zeros can also be written as a product of an infinite number of terms. The following is beyond what will be covered in MA3614 but for interest the Euler-Wallis formula for the sine function is

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n\pi} \right)^2 \right).$$

The infinite product converges slowly.

More advanced representations of $\cot z$

Let $z_1, z_2, \ldots, z_n$ be points in the complex plane and let

$$p_n(z) = (z - z_1)(z - z_2)\cdots(z - z_n).$$

In the exercise sheet there was a question about showing that

$$\frac{p_n'(z)}{p_n(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n}.$$

In the case of $\cot z$ we similarly have

$$\cot z = \frac{\cos z}{\sin z} = \frac{\frac{\mathrm{d}}{\mathrm{d}z}\sin z}{\sin z}.$$

The following is beyond what will be covered in `MA3614` but it can be shown that $\cot z$ has a partial fraction type representation in terms of its simple poles in the following sense.

$$\cot z = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z + n\pi} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}.$$