Definition of a limit and continuity in $\mathbb{C}$

A **neighbourhood** of a point z_0 means a disk of the form $\{z \in \mathbb{C} : \ |z - z_0| < \rho\}$ for some $\rho > 0$.

Limit: Let f be a function defined in a neighbourhood of z_0 and let $f_0 \in \mathbb{C}$. If for every $\epsilon > 0$ there exists a real number $\delta > 0$ such that

$$|f(z) - f_0| < \epsilon \quad \text{for all } z \text{ satisfying } 0 < |z - z_0| < \delta$$

then we say that

$$\lim_{z \to z_0} f(z) = f_0.$$

Continuity: A function $w = f(z)$ is continuous at $z = z_0$ provided $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Examples of continuous functions

1. All the monomials 1, z, z^2, $\ldots$ are continuous on $\mathbb{C}$ and hence all polynomials are continuous at all points in $\mathbb{C}$.
2. Let $p(z)$ and $q(z)$ be polynomials and let
$$f(z) = \frac{p(z)}{q(z)},$$
which is rational function. This is continuous on $\mathbb{C}$ except at a finite number of points which are the roots of $q(z)$.
3. $$\exp(z) = \mathrm{e}^x(\cos y + i \sin y)$$
is continuous on $\mathbb{C}$.

All of the above are often classified as "elementary functions".

Points where limits do not exist

1. $$f(z) = \frac{1}{z}$$

   is unbounded as $z \to 0$.

2. $$f(z) = \operatorname{Arg} z \in (-\pi, \pi]$$

   is not defined at $z = 0$ and it does not have a limit on the negative real axis. As we cross the negative real axis the magnitude of the jump in the function value is 2π.

3. $$f(z) = \exp(-1/z^2)$$

   is unbounded as $z \to 0$ when $z \in \mathbb{C}$. It is however bounded when we restrict to $z \in \mathbb{R}$.

4. $$f(z) = \frac{\overline{z}}{z}$$

   does not have a limit as $z \to 0$ but it is bounded.

Points where limits do not exist, more jargon

We meet the term analytic this week. Later we meet the terms simple pole, isolated singularity and essential singularity.

1. $$f(z) = \frac{1}{z}, \quad \text{a simple pole at } z = 0\text{, an isolated singularity.}$$

2. $$f(z) = \operatorname{Arg} z \in (-\pi, \pi], \quad \text{this is not analytic anywhere.}$$

   The singularity on the negative real axis is not isolated.

3. $$f(z) = \exp(-1/z^2), \quad \text{an essential singularity at } z = 0.$$

4. $$f(z) = \frac{\overline{z}}{z}, \quad \text{this is not analytic anywhere.}$$

When some of the terms will be defined

1. $$\frac{1}{z}, \quad \exp(-1/z^2).$$

   These have isolated singularities at $z = 0$.
   The term isolated singularity will appear many times from about chapter 4 onwards.
   A formal definition will be when Laurent series is done in term 2.

2. $\operatorname{Arg} z$, and the jump discontinuity, will appear when the principal valued $\operatorname{Log} z$ and complex powers z^α are considered in chapter 4.

The definition of a derivative in the real case

If $f(x)$ denotes a real valued function defined in a neighbourhood of x_0 then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

If $g(x, y)$ denotes a real valued function defined in a neighbourhood of (x_0, y_0) then

$$\begin{aligned}
\frac{\partial g}{\partial x}(x_0, y_0) &= \lim_{h \to 0} \frac{g(x_0 + h, y_0) - g(x_0, y_0)}{h}, \\
\frac{\partial g}{\partial y}(x_0, y_0) &= \lim_{h \to 0} \frac{g(x_0, y_0 + h) - g(x_0, y_0)}{h}.
\end{aligned}$$

Note that in the above definitions the division is by h, which is real, and we are just considering "the change in one direction".

Analytic functions

- **Complex derivative:** Let f be a complex valued function defined in a neighbourhood of z_0. The **derivative of f at z_0** is given by

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

  provided the limit exists. Note that here $h \in \mathbb{C}$.
- A function f is **analytic at z_0** if f is differentiable at all points in some neighbourhood of z_0.
- A function f is **analytic in a domain** if f is analytic at all points in the domain.
- A function $f : \mathbb{C} \to \mathbb{C}$ is an **entire function** if it is analytic on the whole complex plane $\mathbb{C}$.

Continuity/analytic comments summary

$f(z)$ is continuous at z_0 if $f(z)$ is close to $f(z_0)$ whenever z is close to z_0.

Let

$$\lambda(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0), & z \neq z_0, \\ 0, & z = z_0 \end{cases}$$

If $f(z)$ is analytic at z_0 then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0)$$

with $\lambda(z)$ being continuous and $\lambda(z_0) = 0$. Continuity of $\lambda(z)$ implies that $\lambda(z) \approx 0$ when $|z - z_0|$ is small. Later in the module we show that actually $\lambda(z)$ is analytic and there is a Taylor series representation of $f(z)$ which is valid in a neighbourhood of z_0.

Taylor series comment

In term 2 we show that when is analytic we have the Cauchy integral formula representation

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta.$$

Here Γ is a closed loop traversed once in the anti-clockwise direction and z is a point inside Γ.

It is essentially a re-write of this which gives the Taylor series representation in a neighbourhood of a point z_0.

$$\begin{aligned} f(z) &= f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k. \end{aligned}$$

The derivative of monomials

As in the real case when $n = 0, 1, \ldots$ we have

$$\frac{\mathrm{d}}{\mathrm{d}z} z^n = n\, z^{n-1}.$$

The proof is as in the real case and can be done using the binomial theorem with $f(z) = z^n$ and

$$f(z+h) - f(z) = (z+h)^n - z^n = nhz^{n-1} + \cdots + h^n.$$

Dividing by h and letting $h \to 0$ gives the result.

Alternatively the geometric series gives the factorization

$$f(z) - f(z_0) = (z - z_0)(z^{n-1} + z_0 z^{n-2} + \cdots + z_0^{n-1}).$$

Dividing by $z - z_0$ and letting $z \to z_0$ gives the result.

Later we define z^α for any $\alpha \in \mathbb{C}$ and it is shown that we have the corresponding result where z^α is differentiable.

Combining differentiable functions

Let f and g be differentiable at z_0. We have the following.

(i)
$$(f \pm g)'(z_0) = f'(z_0) \pm g'(z_0).$$

(ii)
$$(cf)'(z_0) = cf'(z_0) \quad \text{for all constants } c \in \mathbb{C}.$$

(iii)
$$(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0).$$

This is the product rule.

(iv)
$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}, \quad \text{if } g(z_0) \neq 0.$$

This is the quotient rule.

(v) Let now f be a function which is differentiable at $g(z_0)$. Then
$$\frac{\mathrm{d}}{\mathrm{d}z} f(g(z))\bigg|_{z=z_0} = f'(g(z_0))g'(z_0).$$

This is the chain rule.

The derivative of powers of z

For the negative power of -1 we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z}\right) = -\frac{1}{z^2}.$$

Hence if $n > 0$ is an integer then by the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z^n}\right) = -\left(\frac{1}{z^n}\right)^2 nz^{n-1} = -\frac{n}{z^{n+1}}.$$

Thus as in the real case we have that for all non-zero integers

$$\frac{\mathrm{d}}{\mathrm{d}z}z^n = n\,z^{n-1}.$$

Also

$$\frac{\mathrm{d}}{\mathrm{d}z}1 = 0.$$

A comment about an anti-derivative

We just had that for all integers n

$$\frac{\mathrm{d}}{\mathrm{d}z} z^n = n\, z^{n-1}.$$

Thus when $m \neq -1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{z^{m+1}}{m+1} \right) = z^m$$

When integration is done this means that z^m has an anti-derivative which is another monomial for all integers except $m = -1$.

Roughly speaking, many of the results of the module are concerned with the special case of $m = -1$.

Functions which are not analytic anywhere

There are several ways to show that a function is not analytic which include showing that the limit in the complex derivative expression does not exist and/or showing that the Cauchy Riemann equations are not satisfied (see later). In term 2 we also briefly describe Morera's theorem as yet another way of characterising when a function is analytic or not analytic.

Examples of functions which are not analytic include the following.

- $f(z) = \overline{z}$.
- $f(z) = x$ or $f(z) = y$ or $f(z) = |z|$.
- If $g(z)$ is analytic and not constant then $f(z) = g(\overline{z})$ is not analytic.

Later in the chapter 3 material we show that "analytic functions cannot depend on the complex conjugate $\overline{z}$" once we have defined more precisely what this means.

The Cauchy Riemann equations for $f(z) = u(x, y) + iv(x, y)$

When f is analytic at z_0 the following limit exists.

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}.$$

By considering the case when h is real and then purely imaginary we get

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \\ &= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \end{aligned}$$

Equating the real and imaginary parts gives the Cauchy Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Next week we show that the converse is true, i.e. when u and v have continuous first partial derivatives on a domain D and the Cauchy Riemann equations are satisfied then f is analytic on D.

The representation of f' when $f = u + iv$

When f is analytic we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

If $f(x)$ is real when x is real then

$$v(x,0) = 0, \quad \text{which implies that} \quad \frac{\partial v}{\partial x}(x,0) = 0.$$

Hence in this case on the real axis we have

$$f'(x) = \frac{\partial u}{\partial x}(x,0).$$

That is the expressions that you have met for the derivative in the real case are correct in the complex case when the derivative exists in the complex sense.