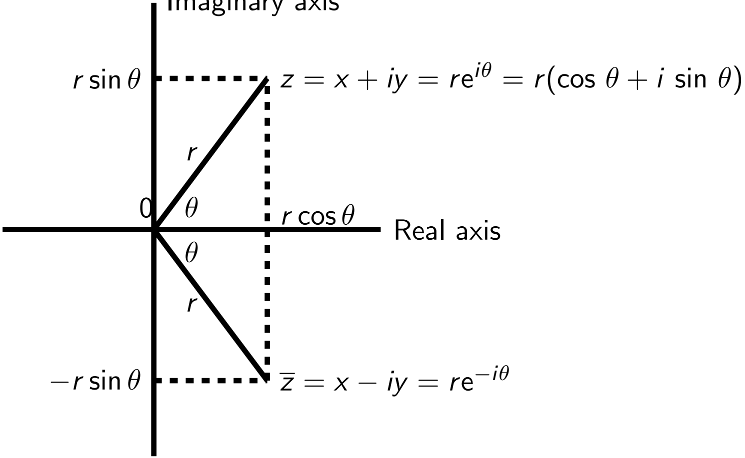
$z = x + iy = r\exp(i\theta)$, $\overline{z}$, $|z|$, **Arg**(z) **etc.**

Argand diagram



$\operatorname{Arg} z \in (-\pi, \pi]$ =principal argument. (arg z is multi-valued.)

Note $|z|^2 = z\,\overline{z}$. $|z|$ =absolute value.

Multiplication, powers, roots of unity

Suppose $z = re^{i\theta}$, $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$.

Multiplication: $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$.

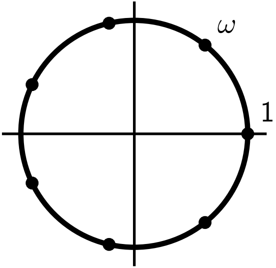
Powers: $z^n = r^n e^{in\theta}$, $n = 0, \pm 1, \pm 2, \ldots$.

Observe $e^{2\pi i} = \exp(2\pi i) = \cos(2\pi i) + i \sin(2\pi i) = 1$.

Roots of unity: Let $\omega = \exp(2\pi i/n)$.

$$1, \omega, \omega^2, \ldots, \omega^{n-1}$$

all satisfy $z^n - 1 = 0$ and are uniformly spaced on the unit circle.



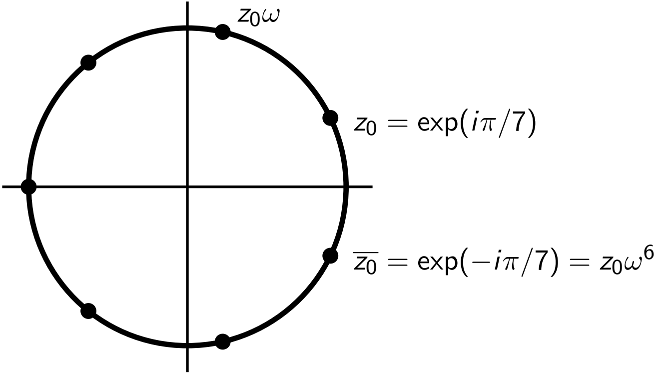
$n = 7$

Roots of any number, the case $z^n - \zeta = 0$

Roots of any number:

Let $\zeta = \rho \exp(i\phi)$, $\phi = \text{Arg}(\zeta)$ and let $z_0 = \sqrt[n]{\rho}\exp(i\phi/n)$. This is the principal value solution. All n roots are z_0, $z_0\omega$, ..., $z_0\omega^{n-1}$.

Example: $z^7 = -1 = \exp(i\pi)$.



$n = 7$, solutions of $z^7 = -1$. Note $z = -1$ is one solution.

How does the complex case help?

1. $$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{1+x^2}.$$

   This is bounded and infinitely differentiable on $\mathbb{R}$. To understand why its Maclaurin series only converges when $|x| < 1$ you need to consider $f(z)$ and the points $\pm i$. Later the term **simple poles** will be used to describe the singularities at $\pm i$.
   The Maclaurin series is just a geometric series in this case.

2. $$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} \exp(-1/x^2), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

   This is infinitely differentiable on $\mathbb{R}$ but it is unbounded when we consider instead $f(z)$ with $z \in \mathbb{C}$.
   In term 2 the type of singularity at $z = 0$ is called an **essential singularity** of $f(z)$.

Further comments about an example

By the geometric series we have for $|z| < 1$

$$\frac{1}{1+z^2} = \frac{1}{1-\left(-z^2\right)} = 1 - z^2 + z^4 - z^6 + \cdots$$

When $|z| > 1$, $1/|z| < 1$ and if we write

$$1 + z^2 = z^2\left(1 + \left(\frac{1}{z^2}\right)\right)$$

then

$$\frac{1}{1+z^2} = \left(\frac{1}{z^2}\right)\left(1 - \left(\frac{-1}{z^2}\right)\right)^{-1} = \left(\frac{1}{z^2}\right)\left(1 - \frac{1}{z^2} + \frac{1}{z^4} + \cdots\right).$$

This is a Laurent series which is valid in $|z| > 1$.

Laurent series is a topic in term 2.

Both series do not converge when $|z| = 1$.

Further comments: the geometric series

In many places in this module you will see the geometric series or similar relations.

For example, we have the polynomial factorization

$$\begin{aligned} z^n - 1 &= (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1) \\ 1 - z^n &= (1-z)(1 + z + \cdots + z^{n-2} + z^{n-1}) \end{aligned}$$

as well as in the geometric series form

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots, \quad \text{when } |z| < 1.$$

Other versions of the factorization

Suppose $a \neq 0$.

$$\begin{aligned} a^n - z^n &= a^n \left(1 - \left(\frac{z}{a}\right)^n\right) \\ &= a^n \left(1 - \left(\frac{z}{a}\right)\right) \left(1 + \left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^2 + \cdots + \left(\frac{z}{a}\right)^{n-1}\right) \\ &= (a-z)(a^{n-1} + a^{n-2}z + \cdots + az^{n-2} + z^{n-1}). \end{aligned}$$

When did interest in complex numbers begin?
The roots of cubics

Cardano found that his method for finding roots of cubics needed complex numbers and a summary of the steps is given below.

Consider the problem of solving

$$x^3 + cx + d = 0.$$

Cardano's method involves letting

$$x = u + \frac{p}{u}, \quad p = -c/3, \quad \text{so that } 3p + c = 0.$$

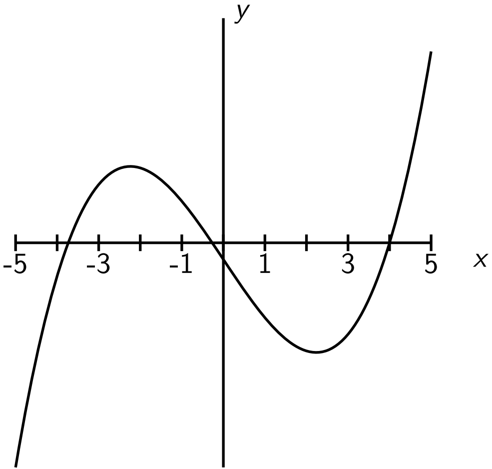
With a small amount of algebra we get

$$x^3 + cx = u^3 + \frac{p^3}{u^3} \quad \text{giving } x^3 + cx + d = \frac{1}{u^3}\left(u^6 + du^3 + p^3\right).$$

$$u^3 = \frac{-d \pm \sqrt{d^2 - 4p^3}}{2}. \quad \text{Often } d^2 - 4p^3 < 0 \text{ and } u^3 \text{ is not real.}$$

Often the intermediate quantity u is not real but the solution x is real.

We consider the following cubic



$$y = \frac{x^3 - 15x - 4}{10}.$$

Cardano method example

$$x^3 - 15x - 4 = (x-4)(x^2+4x+1)$$

Here $c = -15$, $p = 5$ and $d = -4$.

$$u^6 - 4u^3 + 5^3 = 0 \quad \text{and } d^2 - 4p^3 = 16 - 4 \times 5^3 = -4 \times 11^2.$$

$$u^3 = 2 \pm 11i.$$

One solution to $u^3 = 2 + 11i$ is $u = 2 + i$.

To verify

$$\begin{aligned} (2+i)^2 &= 3 + 4i, \\ (2+i)^3 &= (2+i)(2+i)^2 = (2+i)(3+4i) = 2 + 11i. \end{aligned}$$

Hence one solution is

$$x = u + \frac{p}{u} = (2+i) + \frac{5}{2+i} = (2+i) + (2-i) = 4.$$

The roots of polynomials

For a polynomial of any degree Gauss proved the following.

Fundamental theorem of algebra: A polynomial of degree n can always be factorised in the form

$$\begin{aligned} p_n(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \\ &= a_n (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n) \end{aligned}$$

where $a_0, \cdots, a_n, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $a_n \neq 0$. This will be proved in term 2 (the proof is not examinable).

When the polynomial coefficients are real the non-real roots occur in complex conjugate pairs as a consequence of

$$\overline{p(\overline{z})} = p(z).$$

This is on the first exercise sheet.

A related comment about $\overline{f(\overline{z})}$

Let $f : \mathbb{R} \to \mathbb{R}$.

When f is such that we can generalise and consider $f(z)$, with $z \in \mathbb{C}$, we will see that

$$\overline{f(\overline{z})} = f(z).$$

when $f(z)$ is analytic (analytic will appear from chapter 3).

Note for example

$$\begin{aligned} \exp(x+iy) &= e^x(\cos y + i \sin y), \\ \exp(x-iy) &= e^x(\cos y - i \sin y) = \overline{\exp(x+iy)}. \end{aligned}$$

Functions of a complex variable
Some comments to start

The chapter 2 material will have very few exercises but a few terms should be noted.

- The neighbourhood of a point z_0.
- What is meant by a domain and a region.
- Simply-connected. (Only loosely defined here.)
- The definition of a limit and continuity in $\mathbb{C}$.

Functions of a complex variable – some jargon

Let $A \subset \mathbb{C}$. We write

$$f : A \to \mathbb{C}$$

with A denoting the domain of definition of f.

We define here what we mean by 'domain' and 'region' in the context of this module and this requires some intermediate terms.

Open disk: A set of the form

$$\{z \in \mathbb{C} : \ |z - z_0| < \rho\}, \quad \rho > 0.$$

The boundary is the circle $|z - z_0| = \rho$ which is not part of the set.

Unit disk: This is the set

$$\{z \in \mathbb{C} : \ |z| < 1\}.$$

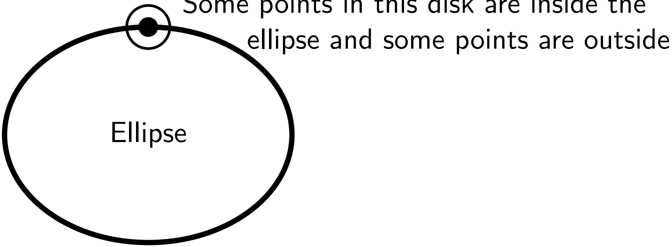
A **neighbourhood** of a point z_0 means a disk of the form $\{z \in \mathbb{C} : \ |z - z_0| < \rho\}$ for some $\rho > 0$.

Jargon continued

Interior point of A: A point $z_0 \in A$ such that a neighbourhood of z_0 is also in A.

Open set: A set such that every point is an interior point.

Boundary point of A: A point z_0 such that every neighbourhood of z_0 contains points which are in A and also contains points which are not in A.

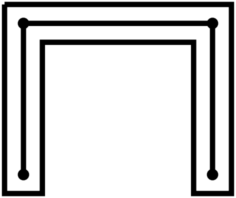


Boundary of A: the set of all boundary points.

Jargon continued

Polygonal path: Let $w_1, w_2, \ldots, w_{n+1}$ be points in $\mathbb{C}$ and let l_k be the straight line segment joining w_k to w_{k+1}. The successive line segments $l_1, l_2, \ldots, l_{n+1}$ is a **polygonal path** joining w_1 to w_{n+1}.

Connected: A set A is **connected** if every pair of points z_1 and z_2 in A can be joined by a polygonal path which is contained in A.



Domain: In this module a **domain** refers to an open connected set.

Region: A domain or a domain together with some or all of the boundary points.

Jargon continued

Bounded: A set A is bounded if there exists $R > 0$ such that the set is contained in the disk $\{z : \ |z| < R\}$.

Unbounded: A set is unbounded if it is not bounded.

Simply-connected: A domain (which is thus connected) and does not have holes. A precise mathematical definition of this can be given.

The term simply connected will appear again when loop integrals are considered.

Examples of sets

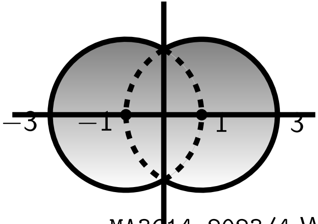
$\mathbb{C}$: an unbounded domain.

$\mathbb{R}$: is not a domain.

All neighbourhoods of a point in $\mathbb{R}$ contains points with non-zero imaginary part.

$$A = \{z : \ |z-1| < 2\} \cup \{z : \ |z+1| < 2\}.$$

is a simply-connected domain.

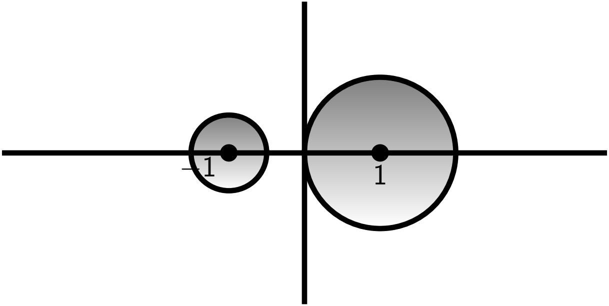


Examples continued

The set

$$A = \{z : \ |z - 1| < 1\} \cup \{z : \ |z + 1| < 0.5\}$$

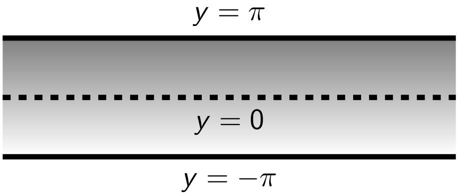
is not connected.



If f is defined on A then we have two separate problems.

An unbounded polygon

The infinite strip $A = \{z = x + iy :\ -\infty < x < \infty,\ -\pi < y \le \pi\}$ is an unbounded polygonal region.

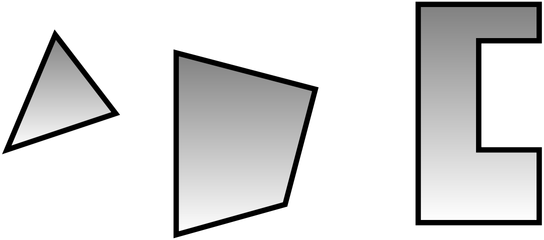


The function

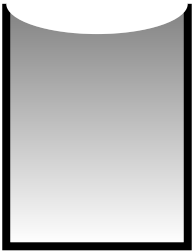
$$e^{x+iy} = \exp(x + iy) = \exp(x)(\cos y + i \sin y)$$

is one-to-one on this strip.

Examples of bounded polygons

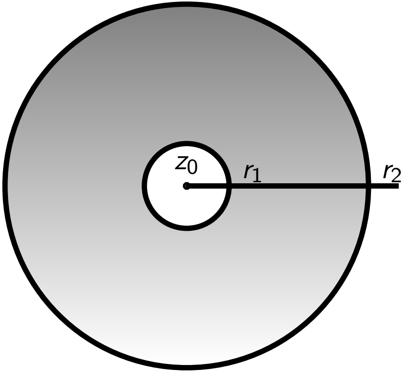


Another unbounded polygon



An annulus

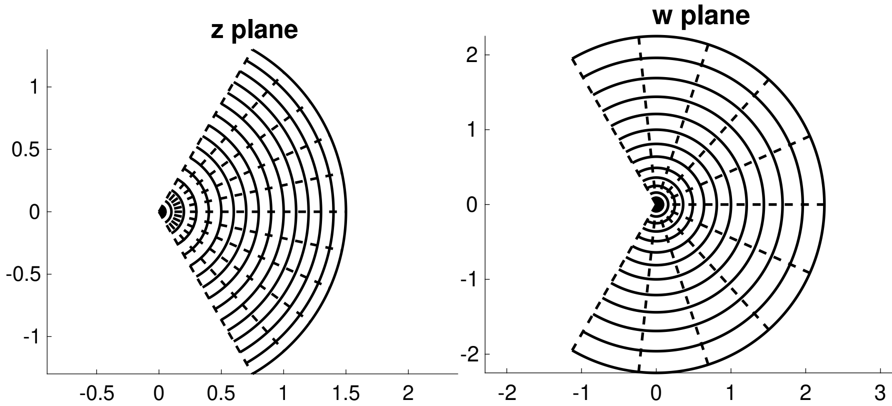
$$A = \{z \in \mathbb{C} : \; r_1 < |z - z_0| < r_2\}.$$



We will consider cases when $r_1 = 0$ when we have an isolated singularity at $z = z_0$ and we consider cases when $r_2 = \infty$.

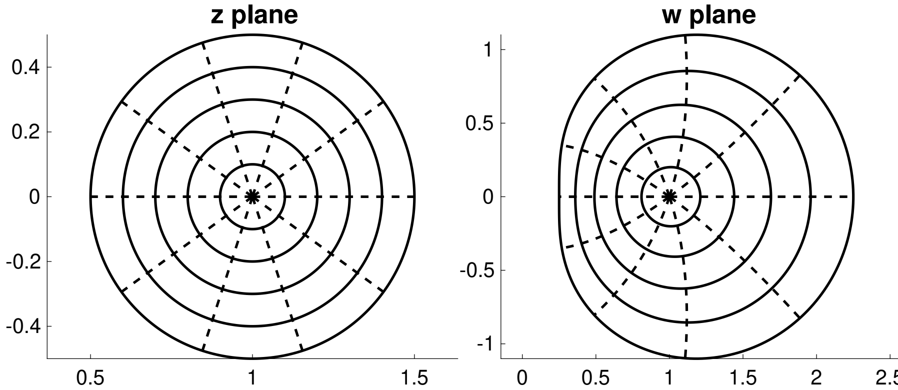
An attempt to graphically represent $f : \mathbb{C} \to \mathbb{C}$

Radial mesh of $w = f(z) = z^2$ centred about $z = 0$



$w = f(z) = z^2$, $z = r e^{i\theta}$, $|\theta| \le \pi/3$, $r \le 1.5$.

Radial mesh of $w = f(z) = z^2$ centred about $z = 1$



$$w = f(z) = z^2,\ |z - 1| \leq 0.5.$$

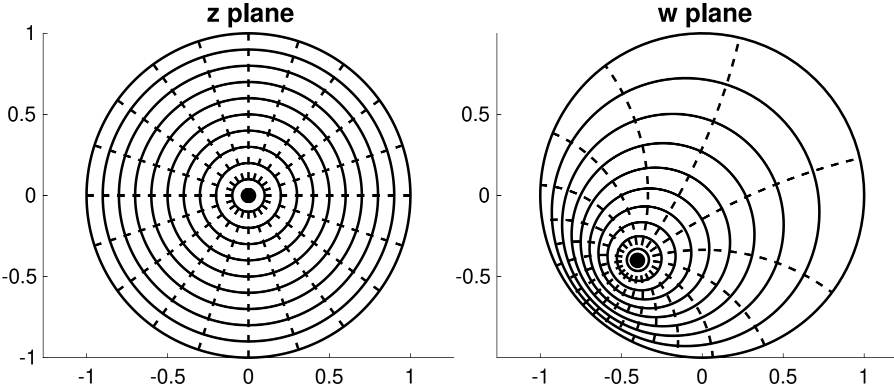
The situation close to $z = 1$

With a polynomial of degree 2 the Taylor series representation is finite and we have

$$\begin{aligned} f(z) = z^2 &= f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2, \\ &= 1 + 2(z-1) + (z-1)^2, \\ &\approx 1 + 2(z-1), \quad \text{when } |z-1| \text{ is small.} \end{aligned}$$

The image of the circles close to $z = 1$ are nearly circles.

Bilinear function: $f(z) = (z - z_0)/(1 - \overline{z_0}z)$, $|z_0| < 1$.



$$w = f(z) = \frac{z - z_0}{1 - \overline{z_0}z}, \text{ with } z_0 = 0.4(1+i) \text{ and } |z| \le 1.$$

It can be shown that this maps the unit disk onto itself (see the exercise sheet as to why the unit circle maps to the unit circle).

Comments about the bilinear example

Recall that $|z_0| < 1$.

$$\begin{aligned} w &= \frac{z - z_0}{1 - \overline{z_0} z}, \\ (1 - \overline{z_0} z) w &= z - z_0, \\ w + z_0 &= z(1 + \overline{z_0} w), \\ z &= \frac{w + z_0}{1 + \overline{z_0} w}. \end{aligned}$$

$w = f(z)$ and the inverse $z = g(w)$ have a similar form.

Observe the following limits:

As $z \to 1/\overline{z_0}$, $|w| \to \infty$.
As $|z| \to \infty$, $w \to -1/\overline{z_0}$.

As $|w| \to \infty$, $z \to 1/\overline{z_0}$.
As $w \to -1/\overline{z_0}$, $|z| \to \infty$.

Limits and continuity

Limit: Let f be defined in a neighbourhood of z_0 and let $f_0 \in \mathbb{C}$. If for every $\epsilon > 0$ there exists a real number $\delta > 0$ such that

$$|f(z) - f_0| < \epsilon \quad \text{for all } z \text{ satisfying } 0 < |z - z_0| < \delta$$

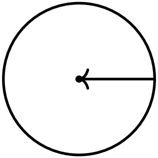
then we say that

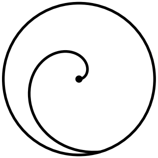
$$\lim_{z \to z_0} f(z) = f_0.$$

Continuity: A function $w = f(z)$ is continuous at $z = z_0$ provided $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Different possibilities of how $z \to z_0$





Limit at ∞

Let f be defined in a region of the form $\{z : \ |z| > \rho\}$. If for every $\epsilon > 0$ there exists a real number $r > 0$ such that

$$|f(z) - f_0| < \epsilon \quad \text{for all } z \text{ satisfying } |z| > r$$

then we say that

$$\lim_{z \to \infty} f(z) = f_0.$$

Examples:

$$\frac{1}{z} \to 0 \quad \text{as } z \to \infty \qquad \text{and} \quad \frac{z+1}{2z+1} = \frac{1+(1/z)}{2+(1/z)} \to \frac{1}{2} \quad \text{as } z \to \infty.$$

Combining continuous functions

Suppose that $f(z)$ and $g(z)$ are continuous at z_0.

$f(z) \pm g(z)$ and $f(z)g(z)$ are continuous at z_0.
$f(z)/g(z)$ is continuous at z_0 provided $g(z_0) \neq 0$.

Suppose that $f(z)$ is continuous at z_0 and $g(z)$ is continuous at $f(z_0)$ then $g(f(z))$ is continuous at z_0.

Continuity of the real and imaginary parts

Let $f(z) = u(x, y) + iv(x, y)$. If f is continuous at $z_0 = x_0 + iy_0$ then u and v are both continuous as functions on $\mathbb{R}^2$ at (x_0, y_0). Conversely, if u and v are both continuous at (x_0, y_0) then f is continuous at $z_0 = x_0 + iy_0$.

Examples of continuous functions

1. All the monomials 1, z, z^2, ... are continuous on $\mathbb{C}$ and hence all polynomials are continuous everywhere.
2. Let $p(z)$ and $q(z)$ be polynomials and let
$$f(z) = \frac{p(z)}{q(z)}$$
which is rational function. This is continuous on $\mathbb{C}$ except at a finite number of points which are the roots of $q(z)$.
3. $$\exp(z) = e^x(\cos y + i \sin y)$$
is continuous on $\mathbb{C}$.

Points where limits do not exist

1.
$$f(z) = \frac{1}{z}$$
is unbounded as $z \to 0$.

2.
$$f(z) = \operatorname{Arg} z \in (-\pi, \pi]$$
is not defined at $z = 0$ and it does not have a limit on the negative real axis. The function jumps by 2π as we cross the negative real axis.

3.
$$f(z) = \exp(-1/z^2)$$
is unbounded as $z \to 0$ when $z \in \mathbb{C}$.

4.
$$f(z) = \frac{\overline{z}}{z}$$
does not have a limit as $z \to 0$ but it is bounded.