Some formulae and some terms extracted from the module

Notation, ..., division.

$$z = x + iy = re^{i\theta} = r(\cos(\theta) + i\sin(\theta)), \quad x, y, r, \theta \in \mathbb{R}, \quad r \geq 0.$$

When $\theta \in (-\pi, \pi]$, $\mathrm{Arg}z = \theta$. $\overline{z} = x - iy$, $r^2 = z\,\overline{z} = x^2 + y^2 \geq 0$.

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{x - iy}{|z|^2} = \frac{x - iy}{x^2 + y^2}.$$

The finite geometric series.

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}. \quad \text{It tends to } \frac{1}{1-z} \text{ as } n \to \infty \text{ when } |z| < 1.$$

Roots of unity. $\omega = e^{2\pi i/n}$. $1, \omega, \ldots, \omega^{n-1}$ all satisfy $z^n = 1$ and are equally spaced.
Some functions of z.

$$\begin{aligned} e^z &= e^x e^{iy} = e^x(\cos(y) + i\sin(y)), \quad \mathrm{Log}(z) = \ln r + i\mathrm{Arg}(z), \\ z^\alpha &= \exp(\alpha \mathrm{Log}(z)), \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2}. \end{aligned}$$

Polynomials, rational functions. When $p(z)$ is a polynomial of degree m,

$$p(z) = p(z_0) + p'(z_0)(z - z_0) + \cdots + \frac{p^{(m)}(z_0)}{m!}(z - z_0)^m, \quad \text{for all points } z_0.$$

When $\deg p(z) \geq \deg q(z)$ and the zeros of $q(z)$ are simple we have a representation of the form

$$R(z) = \frac{p(z)}{q(z)} = (\text{some polynomial}) + \sum_{k=1}^{n} \frac{A_k}{z - z_k}, \quad A_k = \text{residue at } z_k.$$

When $q(z)$ has a zero at z_0 of multiplicity $r \geq 1$ the representation involves

$$\cdots + \frac{B_1}{z - z_0} + \frac{B_2}{(z - z_0)^2} + \cdots + \frac{B_r}{(z - z_0)^r} + \cdots, \quad B_1 = \text{residue at } z_0.$$

The Cauchy Riemann equations for $f(z) = u(x, y) + iv(x, y)$, $u, v \in \mathbb{R}$,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$f(z)$ is complex differentiable at a point if and only if these are satisfied. A function f is **analytic at** z_0 if f is differentiable at all points in some neighbourhood of z_0.
Harmonic: $\phi(x, y)$ is harmonic if

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

If $f = u + iv$ is analytic then u and v are harmonic functions. v is said to be the **harmonic conjugate** of u and satisfies

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.$$