Exercises involving series

1. Find the circle of convergence of the following power series justifying your answer in each case.

   (a)
   $$\sum_{n=0}^{\infty} \frac{(z-i)^n}{2^n}.$$

   (b)
   $$\sum_{n=0}^{\infty} (z+4i)^{2n}(n-1)^2.$$

Solution

(a) This is a geometric series with common ratio

$$\frac{z-i}{2}.$$

The series converges for

$$\left|\frac{z-i}{2}\right| < 1$$

and diverges for

$$\left|\frac{z-i}{2}\right| > 1$$

and the circle of convergence is

$$|z-i| = 2.$$

(b) Let

$$b_n = (z+4i)^{2n}(n-1)^2.$$

Use the ratio test.

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{(z+4i)^2 n^2}{(n-1)^2}\right| \to |z+4i|^2 \quad \text{as } n \to \infty$$

as

$$\frac{n^2}{(n-1)^2} = \left(\frac{1}{1-(1/n)}\right)^2 \to 1 \quad \text{as } n \to \infty.$$

The circle of convergence is $|z+4i| = 1$.

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2. Suppose that a pseudo random number generator generates a sequence a_0, a_1, $\cdots$ with $|a_n| \le 1$. Find the circle of convergence of the power series

$$\sum_{n=0}^{\infty} (2 + a_n) z^n.$$

Solution

Let

$$b_n = (2 + a_n) z^n.$$

The ratio test does not help here on its own as we can only deduce that $|b_{n+1}/b_n|$ remains bounded. However with the root test we have

$$|b_n|^{1/n} = |2 + a_n|^{1/n} |z|.$$

As

$$1 \le |2 + a_n| \le 3$$

and

$$1 \le |2 + a_n|^{1/n} \le 3^{1/n} \to 1.$$

Thus

$$|b_n|^{1/n} \to |z| \quad \text{as } n \to \infty.$$

By the root test the circle of convergence is $|z| = 1$.

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3. For what values of z, if any, does the following series converge.

(a)

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2} + \sum_{n=1}^{\infty} \frac{n^2}{z^n}.$$

(b)

$$\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n + \sum_{n=1}^{\infty} \frac{1}{z^n}.$$

Solution

(a) With $b_n = z^n/n^2$ we have

$$\frac{b_{n+1}}{b_n} = z \left(\frac{n}{n+1}\right)^2 \to z \quad \text{as } n \to \infty.$$

By the ratio test the part with non-negative powers converges when $|z| < 1$ and diverges when $|z| > 1$.

With now $b_n = n^2/z^n$ we have

$$\frac{b_{n+1}}{b_n} = \frac{1}{z} \left(\frac{n+1}{n}\right)^2 \to \frac{1}{z} \quad \text{as } n \to \infty.$$

By the ratio test the part with negative powers converges when $|z| > 1$ and diverges when $|z| < 1$. When $|z| = 1$ the terms are not bounded.

There are no values of z where both parts converge.

(b) We have two parts and each part is geometric series.

$$\sum_0^\infty \left(\frac{z}{2}\right)^n$$

converges when $|z| < 2$.

$$\sum_1^\infty \frac{1}{z^n}.$$

converges when $|1/z| < 1$, i.e. when $|z| > 1$.

The expression converges when both parts converge which is $| < |z| < 2$.

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4. Prove that the Taylor series of $1/(\zeta - z)$ about $z_0 \neq \zeta$ is given by

$$\frac{1}{\zeta - z} = \sum_{n=0}^\infty \frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}} \quad \text{for } |z - z_0| < |\zeta - z_0|.$$

Solution

$$\zeta - z = (\zeta - z_0) - (z - z_0) = (\zeta - z_0)\left(1 - \left(\frac{z - z_0}{\zeta - z_0}\right)\right).$$

When

$$\left|\frac{z - z_0}{\zeta - z_0}\right| < 1$$

we have the geometric series

$$\begin{aligned}(\zeta - z)^{-1} &= (\zeta - z_0)^{-1}\left(1 + \left(\frac{z - z_0}{\zeta - z_0}\right) + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \left(\frac{z - z_0}{\zeta - z_0}\right)^3 + \cdots\right) \\ &= \sum_{n=0}^\infty \frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}}.\end{aligned}$$

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5. Find the first three non-zero terms of the Laurent series for each of the following functions in the specified domains.

(a)

$$\frac{e^{1/z}}{z^2 - 1}, \quad |z| > 1.$$

(b)
$$\operatorname{cosec}(z) = \frac{1}{\sin(z)}, \quad 0 < |z| < \pi.$$

Solution

(a) From the Maclaurin series

$$e^w = 1 + w + \frac{w}{2!} + \cdots + \frac{w^n}{n!} + \cdots$$

we get the Laurent series

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots + \frac{1}{n!}\frac{1}{z^n} + \cdots$$

which is valid for $|z| > 0$.
Let now $w = 1/z$ with $|z| > 1$.

$$z^2 - 1 = \left(\frac{1}{w}\right)^2 - 1 = \frac{1 - w^2}{w^2},$$

giving

$$\frac{1}{z^2 - 1} = \frac{w^2}{1 - w^2} = w^2(1 + w^2 + w^4 + \cdots) = \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^6 + \cdots$$

Hence

$$\begin{aligned} \frac{e^{1/z}}{z^2 - 1} &= \left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots\right)\left(\left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^6 + \cdots\right) \\ &= \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \left(1 + \frac{1}{2}\right)\left(\frac{1}{z}\right)^4 + \cdots \\ &= \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \frac{3}{2}\left(\frac{1}{z}\right)^4 + \cdots \end{aligned}$$

(b) $\sin z$ has a simple zero at $z = 0$ and it is an odd function. Thus the series is of the form

$$f(z) = \frac{1}{\sin z} = \frac{a_{-1}}{z} + a_1 z + a_3 z^3 + \cdots .$$

$$\begin{aligned} 1 &= f(z)\sin z \\ &= \left(\frac{a_{-1}}{z} + a_1 z + a_3 z^3 + \cdots\right)\left(z - \frac{z^3}{6} + \frac{z^5}{120} + \cdots\right). \end{aligned}$$

Equating the constant term gives $a_{-1} = 1$.

Equating the coefficient of z^2 gives

$$a_1 - \frac{a_{-1}}{6} = 0, \quad a_1 = \frac{1}{6}.$$

Equating the coefficient of z^4 gives

$$a_3 - \frac{a_1}{6} + \frac{a_{-1}}{120} = 0, \quad a_3 = \frac{1}{36} - \frac{1}{120} = \frac{7}{360}.$$

6. Let
$$f(z) = \frac{\tan z}{(z^2+1)(z^2+4)}$$
What is the radius of convergence of the following series.

   (a) The Maclaurin series.

   (b) The Taylor series about the point $z_0 = 1$.

   (c) The Taylor series about the point $z_0 = \pi(1+i)$.

Solution

(a) The function has isolated singularities at $\pm i$, $\pm 2i$ and at the equally spaced points $\pi/2 + k\pi$, $k \in \mathbb{Z}$.

The nearest singularities to the origin are at $\pm i$ and thus the radius of convergence of Maclaurin's series is 1.

(b) The nearest singularity to $z_0 = 1$ is at $\pi/2$ and thus the radius of convergence is $\pi/2 - 1$.

(c) When $z_0 = \pi(1+i)$,
$$|z_0 - 2i|^2 = |\pi + (\pi-2)i|^2 = \pi^2 + (\pi-2)^2$$
and
$$|z_0 - \pi/2|^2 = |\pi/2 + \pi i|^2 = (\pi/2)^2 + \pi^2.$$
As $\pi/2 > \pi - 2$ the nearest singularity to z_0 is at $2i$ The radius of convergence is hence $\sqrt{\pi^2 + (\pi-2)^2}$.

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7. Obtain the first 3 non-zero terms in the Maclaurin expansion of the following stating in each case where the series converges.
$$\tanh z = \frac{\sinh z}{\cosh z}.$$

Solution

As $\sinh z$ is an odd function and $\cosh z$ is an even function the function $\tanh z$ is an odd function and the Maclaurin series only involves odd powers and is of the form
$$\tanh z = a_1 z + a_3 z^3 + a_5 z^5 + \cdots$$
As
$$(\tanh z)(\cosh z) = \sinh z$$
we have
$$(a_1 z + a_3 z^3 + a_5 z^5 + \cdots)\left(1 + \frac{z^2}{2} + \frac{z^4}{24} + \cdots\right) = z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots$$

We equate the coefficients of the powers of z on the left hand side with the right hand side.

z term: $a_1 = 1$.

z^3 term: $a_3 + a_1/2 = 1/6$. This gives $a_3 = 1/6 - 1/2 = -1/3$.

z^5 term: $a_5 + a_3/2 + a_1/24 = 1/120$. This gives

$$a_5 = \frac{1}{120} + \frac{1}{6} - \frac{1}{24} = \frac{1+20-5}{120} = \frac{16}{120} = \frac{2}{15}.$$

tanh z is not analytic when cosh $z = 0$ and

$$\cosh z = \frac{1}{2}(\mathrm{e}^z + \mathrm{e}^{-z}) = \frac{\mathrm{e}^{-z}}{2}(\mathrm{e}^{2z} + 1).$$

The points where tanh z is not analytic which are nearest to $z = 0$ are when $\mathrm{e}^{2z} = -1$, i.e. $z = \pm i\pi/2$. The Maclaurin series converges for $|z| < \pi/2$.

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8. By any means determine the radius of convergence of the following power series

$$1 + 2z + z^2 + (2z)^3 + z^4 + (2z)^5 + \cdots + z^{2n} + (2z)^{2n+1} + \cdots$$

Give an expression for the limit.

Solution

The ratio test and the root test both do not work in their standard form to determine the radius of convergence but we can consider the even powers and the odd powers separately as these are just geometric series. The even powers gives

$$1 + z^2 + z^4 + \cdots + \cdots = \frac{1}{1-z^2}, \quad |z| < 1.$$

The odd powers gives

$$2z(1 + (2z)^2 + (2z)^4 + \cdots) = \frac{2z}{1-(2z)^2}, \quad |2z| < 1.$$

Both series are valid when $|z| < 1/2$ and thus the radius of convergence is $1/2$.

The limit function is

$$\frac{1}{1-z^2} + \frac{2z}{1-(2z)^2}.$$

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9. Find the general term in the Maclaurin series of the following function.

$$\frac{\mathrm{e}^z}{1-z} = \sum_{n=0}^{\infty} c_n z^n.$$

Solution

The Maclaurin series for e^z and $1/(1-z)$ are standard and we have

$$\frac{\mathrm{e}^z}{1-z} = \left(1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots\right)\left(1 + z + z^2 + \cdots + z^n + \cdots\right) = \sum_{n=0}^{\infty} c_n z^n.$$

By the Cauchy product result we have

$$c_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

The series is valid for $|z| < 1$.

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10. *The following was question 3 of the May 2023 exam*

(a) Determine if the following power series define entire functions, and if this is not the case then find the circle of convergence. In each case you must justify your answer.

$$\sum_{n=1}^{\infty} n\left(\frac{z-5}{2}\right)^n, \qquad \sum_{n=0}^{\infty}\left(\frac{8^n+2}{3^n+7}\right)z^n, \qquad \sum_{n=1}^{\infty}\frac{(z+2)^n}{2^n n^n+1}.$$

(b) Let $f_1(z)$ and $f_2(z)$ be given by

$$f_1(z) = \frac{4}{z^2+4}, \quad \text{and} \quad f_2(z) = \frac{3}{2+\mathrm{e}^z}.$$

Both functions are analytic at $z = 0$ and have a Maclaurin expansion. Determine the radius of convergence of the Maclaurin series for the function $f_1(z)$ and also determine the radius of convergence of the Maclaurin series for the function $f_2(z)$. In each case you must justify your answer.

Suppose that the Maclaurin series for $f_2(z)$ is expressed in the form

$$f_2(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots .$$

Determine a_0, a_1, a_2 and a_3.

For your information, the first few terms of the Maclaurin series of e^z are given by

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \cdots .$$

(c) Let $b > 0$. Determine the Laurent series for

$$\frac{1}{z-b}$$

which is valid for large $|z|$ in an annulus with centre at 0. In your answer you need to give the general term and indicate the smallest value of $R > 0$ such that the series is valid for all $|z| > R$.

Let

$$g(z) = \frac{2}{(z-1)(z-2)(z-3)} = \frac{1}{z-1} - \frac{2}{z-2} + \frac{1}{z-3}.$$

Determine an expression for a_n such that

$$g(z) = \sum_{n=1}^{\infty} \frac{a_n}{z^n}, \quad |z| > 3.$$

Solution

(a)
$$\sum_{n=1}^{\infty} n\left(\frac{z-5}{2}\right)^n = \sum_{n=1}^{\infty} b_n \quad \text{when} \quad b_n = a_n(z-5)^n \quad \text{and} \quad a_n = \frac{n}{2^n}.$$

We use the ratio test.

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)\left(\frac{1}{2}\right) = \frac{1+1/n}{2} \to \frac{1}{2} \quad \text{as } n \to \infty.$$

Thus

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{a_n}(z-5) \to \frac{z-5}{2} \quad \text{as } n \to \infty.$$

By the ratio test the series converges when $|z-5| < 2$ and diverges when $|z-5| > 2$. The cirle of convergence is $\{z : \, |z-5| = 2\}$.

$$\sum_{n=0}^{\infty} \left(\frac{8^n+2}{3^n+7}\right) z^n = \sum_{n=1}^{\infty} b_n \quad \text{when} \quad b_n = a_n z^n \quad \text{and} \quad a_n = \frac{8^n+2}{3^n+7}.$$

We use the ratio test.

$$\frac{a_{n+1}}{a_n} = \left(\frac{8^{n+1}+2}{8^n+2}\right)\left(\frac{3^n+7}{3^{n+1}+7}\right) = \left(\frac{8+2/8^n}{1+2/8^n}\right)\left(\frac{1+7/3^n}{3+7/3^n}\right) \to \frac{8}{3} \quad \text{as } n \to \infty.$$

Thus

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{a_n} z \to \frac{8z}{3} \quad \text{as } n \to \infty.$$

By the ratio test the series converges when $|z| < 3/8$ and diverges when $|z| > 3/8$. The cirle of convergence is $\{z : \, |z| = 3/8\}$.

$$\sum_{n=1}^{\infty} \frac{(z+2)^n}{2^n n^n + 1} = \sum_{n=1}^{\infty} b_n \quad \text{when} \quad b_n = a_n(z+2)^n \quad \text{and} \quad a_n = \frac{1}{2^n n^n + 1}.$$

We use the root test.

$$a_n = \left(\frac{1}{2^n n^n}\right)\frac{1}{1+1/(2n)^n}, \quad a_n^{1/n} = \left(\frac{1}{2n}\right)\left(\frac{1}{1+1/(2n)^n}\right)^{1/n} \to 0 \quad \text{as } n \to \infty.$$

Thus

$$|b_n|^{1/n} = |a_n|^{1/n}|z+2| \to 0 \quad \text{as } n \to \infty.$$

The series converges for all z and defines an entire function.

(b) The only singularities of $f_1(z)$ are simple poles at $\pm 2i$. As $f_1(z)$ is analytic in $|z| < 2$ but not at two points on $|z| = 2$ the radius of convergence is 2.

The only singularities of $f_2(z)$ are simple poles at points satisfying

$$\mathrm{e}^z + 2 = 0, \quad z = \mathrm{Log}(-2) + k(2\pi i) = \ln(2) + i\pi + 2k\pi i, \quad k \text{ is an integer.}$$

The nearest simple poles to 0 are at

$$\ln(2) \pm \pi i.$$

The radius of convergence is $|\ln(2) \pm \pi i| = \sqrt{|\ln(2)|^2 + \pi^2}$.
Using the series for e^z

$$\frac{3}{2+e^z} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

re-arranges to

$$3 = \left(3 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots\right)\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right).$$

Using the Cauchy product technique and equating powers of z gives the following.
Equating the constant term gives

$$3 = 3a_0, \quad a_0 = 1.$$

Equating the z term gives

$$0 = 3a_1 + a_0 = 3a_1 + 1, \quad a_1 = -1/3.$$

Equating the z^2 term gives

$$0 = 3a_2 + a_1 + \frac{a_0}{2} = 3a_2 - \frac{1}{3} + \frac{1}{2} = 3a_2 + \frac{1}{6}, \quad a_2 = -\frac{1}{18}.$$

Equating the z^3 term gives

$$0 = 3a_3 + a_2 + \frac{a_1}{2} + \frac{a_0}{6} = 3a_3 - \frac{1}{18} - \frac{1}{6} + \frac{1}{6} = 3a_3 - \frac{1}{18}, \quad a_3 = \frac{1}{54}.$$

(c)

$$z - b = z\left(1 - \frac{b}{z}\right), \quad (z-b)^{-1} = \frac{1}{z}\left(1 - \frac{b}{z}\right)^{-1}$$

$$\frac{1}{z-b} = \frac{1}{z}\left(1 + \frac{b}{z} + \left(\frac{b}{z}\right)^2 + \cdots + \left(\frac{b}{z}\right)^n + \cdots\right).$$

The coefficient of $1/z^n$ is b^{n-1}. The series is valid for $|z| > b$.
As $g(z)$ is a combination of the 3 terms the coefficient of $1/z^n$ is

$$1 - 2(2^{n-1}) + 3^{n-1} = 1 - 2^n + 3^{n-1}.$$

11. *The following was question 3 of the May 2022 exam*

(a) You need to consider the following series.

$$\sum_{n=0}^{\infty}(2n-1)z^n, \qquad \sum_{n=0}^{\infty}\left(\frac{1}{3^n+4^n}\right)(z-1)^n, \qquad \sum_{n=0}^{\infty}\frac{(z+4)^n}{(2n)!}.$$

For each of these three power series determine if it defines an entire function or not. If the series does not define an entire function then find the circle of convergence. In each case you must justify your answer.

(b) Let

$$f(z) = \frac{2-2z}{1+\cos(z)}.$$

Determine the radius of convergence of the Maclaurin series for the function $f(z)$. Also determine the radius of convergence of the Maclaurin series for the function $1/f(z)$. In each case you must justify your answer.

Suppose that the Maclaurin series for $f(z)$ is expressed in the form

$$1 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots .$$

Determine a_1, a_2, a_3 and a_4. For your information, the first few terms of the Maclaurin series of $\cos(z)$ are given by

$$\cos(z) = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots .$$

(c) Let

$$g(z) = \frac{1}{z-1} + \frac{1}{z-4}.$$

Determine the Laurent series of $g(z)$ valid in $|z| > 4$. In your answer you should give the coefficient of $1/z^{n+1}$ for $n \ge 0$.

Solution

(a)

$$\sum_{n=0}^{\infty}(2n-1)z^n, \quad \text{let } a_n = 2n-1, \quad b_n = a_n z^n.$$

$$\frac{a_{n+1}}{a_n} = \frac{2n+1}{2n-1} = \frac{1+1/(2n)}{1-1/(2n)} \to 1 \quad \text{as } n \to \infty.$$

Thus

$$\left|\frac{b_{n+1}}{b_n}\right| \to |z| \quad \text{as } n \to \infty.$$

By the ratio test the circle of convergence is $\{z : |z| = 1\}$.

$$\sum_{n=0}^{\infty}\left(\frac{1}{3^n+4^n}\right)(z-1)^n, \quad \text{let } a_n = \frac{1}{3^n+4^n}, \quad b_n = a_n(z-1)^n.$$

$$\frac{a_{n+1}}{a_n} = \frac{3^n+4^n}{3^{n+1}+4^{n+1}} = \frac{(3/4)^n+1}{3(3/4)^n+4} \to \frac{1}{4} \quad \text{as } n \to \infty.$$

Thus

$$\left|\frac{b_{n+1}}{b_n}\right| \to \frac{|z-1|}{4} \quad \text{as } n \to \infty.$$

By the ratio test the circle of convergence is $\{z : |z-1| = 4\}$.

$$\sum_{n=0}^{\infty}\frac{(z+4)^n}{(2n)!}, \quad \text{let } a_n = \frac{1}{(2n)!} \quad b_n = a_n(z+4)^n.$$

$$\frac{a_{n+1}}{a_n} = \frac{(2n)!}{(2n+2)!} = \frac{1}{(2n+2)(2n+1)} \to 0 \quad \text{as } n \to \infty.$$

$$\left|\frac{b_{n+1}}{b_n}\right| \to 0 \quad \text{as } n \to \infty.$$

There is no restriction on z and the function is an entire function.

(b)

$$f(z) = \frac{2-2z}{1+\cos(z)}, \quad \frac{1}{f(z)} = \frac{1+\cos(z)}{2-2z}.$$

The functions have poles at points where the denominator is zero. For $f(z)$ this is when $\cos(z) = -1$ and the nearest such points to 0 are $\pm\pi$. The radius of convergence of the Maclaurin series for $f(z)$ is π.

In the case of $1/f(z)$ there is a pole at $z = 1$ and thus the radius of convergence of the Maclaurin series for $1/f(z)$ is 1.

$$\begin{aligned}
2-2z &= (1+\cos(z))f(z) \\
&= \left(2-\frac{z^2}{2}+\frac{z^4}{24}+\cdots\right)\left(1+a_1z+a_2z^2+a_3z^3+a_4z^4+\cdots\right).
\end{aligned}$$

Equating the coefficient of z gives

$$-2 = 2a_1, \quad a_1 = -1.$$

Equating the coefficient of z^2 gives

$$0 = 2a_2 - \frac{1}{2}, \quad a_2 = \frac{1}{4}.$$

Equating the coefficient of z^3 gives

$$0 = 2a_3 - \frac{a_1}{2} = 2a_3 + \frac{1}{2}, \quad a_3 = -\frac{1}{4}.$$

Equating the coefficient of z^4 gives

$$0 = 2a_4 - \frac{a_2}{2} + \frac{1}{24} = 2a_4 - \frac{1}{8} + \frac{1}{24} = 2a_4 - \frac{1}{12}, \quad a_4 = \frac{1}{24}.$$

(c) Now

$$z - 1 = z\left(1 - \frac{1}{z}\right), \quad \frac{1}{z-1} = \left(\frac{1}{z}\right)\left(1 - \frac{1}{z}\right)^{-1}.$$

For $|z| > 1$,

$$\frac{1}{z-1} = \left(\frac{1}{z}\right)\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots + \frac{1}{z^n} + \cdots\right).$$

Similarly

$$z - 4 = z\left(1 - \frac{4}{z}\right), \quad \frac{1}{z-4} = \left(\frac{1}{z}\right)\left(1 - \frac{4}{z}\right)^{-1}.$$

For $|z| > 4$,

$$\frac{1}{z-4} = \left(\frac{1}{z}\right)\left(1 + \frac{4}{z} + \frac{4^2}{z^2} + \cdots + \frac{4^n}{z^n} + \cdots\right).$$

Combining the two series gives a Laurent series with a coefficient of $1/z^{n+1}$ of

$$1 + 4^n, \quad n \geq 0.$$

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12. *The following was question 3 of the May 2021 exam*

(a) In the following the series to consider depends on the 5th digit of your 7-digit student id.. If your 5th digit is one of the digits 0, 1, 2, 3, 4 then your three series are as follows.

$$\sum_{n=0}^{\infty} \frac{(2z)^n}{(n+2)!}, \qquad \sum_{n=0}^{\infty} \frac{2^n}{4^n+1} z^n, \qquad \sum_{n=0}^{\infty} \left(2^n + \sin^2(n)\right)(z-1)^n.$$

If your 5th digit is one of the digits 5, 6, 7, 8, 9 then your three series are as follows.

$$\sum_{n=0}^{\infty} \frac{(z-3)^{3n}}{(n+1)!}, \qquad \sum_{n=0}^{\infty} \frac{n^2}{3^n+1} z^n, \qquad \sum_{n=0}^{\infty} \left(3^n + \cos^2(n)\right)(z+1)^n.$$

For each of your three power series determine if it defines an entire function or not. If the series does not define an entire function then find the circle of convergence. In each case you must justify your answer.

(b) In the following the function to consider depends on the 5th digit of your 7-digit student id.. If your 5th digit is one of the digits 0, 2, 4, 6, 8 then the function is

$$f_1(z) = \frac{3z + z^2}{1 - 2\cos(z)}$$

whilst if your 5th digit is one of the digits 1, 3, 5, 7, 9 then the function is

$$f_2(z) = \frac{-2z + z^2}{1 + 2\sin(z)}.$$

For your version determine the radius of convergence of the Maclaurin series. You must justify your answer.

Suppose that the Maclaurin series for your function is expressed in the form

$$a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots .$$

Determine a_1, a_2, a_3 and a_4.

(c) This part of the question is for all student numbers.

Let

$$f(z) = \frac{z^3}{z^2 - 4}.$$

Determine each of the following two series:

i. The power series of $f(z)$ in $|z| < 2$.

ii. The Laurent series of $f(z)$ valid in the annulus $\{z : |z| > 2\}$.

Solution

(a) This is the version for a 5th digit of 0, 1, 2, 3, 4.

For the first series let

$$a_n = \frac{2^n}{(n+2)!}, \quad b_n = a_n z^n.$$

$$\frac{a_{n+1}}{a_n} = \frac{2}{n+3} \to 0 \quad \text{as } n \to \infty.$$

Thus

$$\frac{b_{n+1}}{b_n} = \left(\frac{a_{n+1}}{a_n}\right) z \to 0 \quad \text{as } n \to \infty.$$

By the ratio test the series converges for all z. The function is an entire function.

For the second series let

$$a_n = \frac{2^n}{4^n+1}, \quad b_n = a_n z^n.$$

$$\left(\frac{a_{n+1}}{a_n}\right) = \left(\frac{2^{n+1}}{2^n}\right)\left(\frac{4^n+1}{4^{n+1}+1}\right) = 2\left(\frac{1+1/4^n}{4+1/4^n}\right) \to \frac{2}{4} = \frac{1}{2} \quad \text{as } n \to \infty.$$

Thus

$$\frac{b_{n+1}}{b_n} = \left(\frac{a_{n+1}}{a_n}\right) z \to \frac{z}{2} \quad \text{as } n \to \infty.$$

By the ratio test the circle of convergence is $\{z : \ |z| = 2\}$.

$\boxed{\textbf{3 marks}}$

For the third series let

$$a_n = 2^n + \sin^2(n) = 2^n\left(1 + \frac{\sin^2(n)}{2^n}\right), \quad b_n = a_n (z-1)^n.$$

$$a_n^{1/n} = 2\left(1 + \frac{\sin^2(n)}{2^n}\right)^{1/n},$$

As

$$\frac{\sin^2(n)}{2^n} \le 1, \quad 2 \le a_n^{1/n} \le 2(1+1)^{1/n} \to 2 \quad \text{as } n \to \infty.$$

Thus $a_n^{1/n} \to 2$ as $n \to \infty$ and

$$|b_n|^{1/n} = a_n^{1/n}|z-1| \to 2|z-1| \quad \text{as } n \to \infty.$$

By the root test the circle of convergence is $\{z : \ |z-1| = 1/2\}$.

This is the version for a 5th digit of 5, 6, 7, 8, 9.
For the first series let

$$a_n = \frac{1}{(n+1)!}, \quad b_n = a_n (z-3)^{3n}.$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{n+2} \to 0 \quad \text{as } n \to \infty.$$

Thus

$$\frac{b_{n+1}}{b_n} = \left(\frac{a_{n+1}}{a_n}\right)(z-3)^3 \to 0 \quad \text{as } n \to \infty.$$

By the ratio test the series converges for all z. The function is an entire function.
For the second series let

$$a_n = \frac{n^2}{3^n+1}, \quad b_n = a_n z^n.$$

$$\left(\frac{a_{n+1}}{a_n}\right) = \left(\frac{(n+1)^2}{n^2}\right)\left(\frac{3^n+1}{3^{n+1}+1}\right) = \left(1+\frac{1}{n}\right)^2\left(\frac{1+1/3^n}{3+1/3^n}\right) \to \frac{1}{3} \quad \text{as } n \to \infty.$$

Thus

$$\frac{b_{n+1}}{b_n} = \left(\frac{a_{n+1}}{a_n}\right) z \to \frac{z}{3} \quad \text{as } n \to \infty.$$

By the ratio test the circle of convergence is $\{z : \ |z| = 3\}$.
For the third series let

$$a_n = 3^n + \cos^2(n) = 3^n\left(1 + \frac{\cos^2(n)}{3^n}\right), \quad b_n = a_n (z+1)^n.$$

$$a_n^{1/n} = 3\left(1+\frac{\cos^2(n)}{3^n}\right)^{1/n},$$

As

$$\frac{\cos^2(n)}{3^n} \le 1, \quad 3 \le a_n^{1/n} \le 3(1+1)^{1/n} \to 3 \quad \text{as } n \to \infty.$$

Thus $a_n^{1/n} \to 3$ as $n \to \infty$ and

$$|b_n|^{1/n} = a_n^{1/n}|z+1| \to 3|z+1| \quad \text{as } n \to \infty.$$

By the root test the circle of convergence is $\{z : \ |z+1| = 1/3\}$.

(b) This is the version for a 5th digit of 0, 2, 4, 6, 8.
$1 - 2\cos(z) = 0$ when $\cos(z) = 1/2$ and the nearest points to 0 are at $\pm\pi/3$. $f_1(z)$ is analytic in $|z| < \pi/3$ and has two simple poles on $|z| = \pi/3$. The radius of convergence of the Maclaurin series is $\pi/3$.

2 marks

We consider

$$(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)(1 - 2\cos(z)) = 3z + z^2$$

$$1 - 2\cos(z) = 1 - 2\left(1 - \frac{z^2}{2} + \frac{z^4}{24} + \cdots\right) = -1 + z^2 - \frac{z^4}{12} + \cdots$$

Hence

$$(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)\left(-1 + z^2 - \frac{z^4}{12} + \cdots\right) = 3z + z^2.$$

Equating the coefficients of z gives

$$-a_1 = 3, \quad a_1 = -3.$$

Equating the coefficients of z^2 gives

$$-a_2 = 1, \quad a_2 = -1.$$

Equating the coefficients of z^3 gives

$$-a_3 + a_1 = 0, \quad a_3 = -3.$$

Equating the coefficients of z^4 gives

$$-a_4 + a_2 = 0, \quad a_4 = -1.$$

This is the version for a 5th digit of 1, 3, 5, 7, 9.
$1 + 2\sin(z) = 0$ when $\sin(z) = -1/2$ and the nearest points to 0 is at $-\pi/6$. $f_2(z)$ is analytic in $|z| < \pi/6$ and has a simple pole on $|z| = \pi/6$. The radius of convergence of the Maclaurin series is $\pi/6$.
We consider

$$(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)(1 + 2\sin(z)) = -2z + z^2$$

$$1 + 2\sin(z) = 1 + 2\left(z - \frac{z^3}{6} + \frac{z^5}{120} + \cdots\right) = 1 + 2z - \frac{z^3}{3} + \frac{z^5}{60} + \cdots$$

Hence

$$(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)\left(1 + 2z - \frac{z^3}{3} + \frac{z^5}{60} + \cdots\right) = -2z + z^2.$$

Equating the coefficients of z gives

$$a_1 = -2.$$

Equating the coefficients of z^2 gives

$$a_2 + 2a_1 = a_2 - 4 = 1, \quad a_2 = 5.$$

Equating the coefficients of z^3 gives

$$a_3 + 2a_2 = a_3 + 10 = 0, \quad a_3 = -10.$$

Equating the coefficients of z^4 gives

$$a_4 + 2a_3 - \frac{a_1}{3} = a_4 - 20 + \frac{2}{3} = 0, \quad a_4 = 20 - \frac{2}{3} = \frac{58}{3}.$$

(c) This part is for all students.

$$z^2 - 4 = z^2\left(1 - \frac{4}{z^2}\right) = -4\left(1 - \frac{z^2}{4}\right).$$

For $|z| < 2$ we have the geometric series

$$z^3(z^2-4)^{-1} = -\frac{z^3}{4}\left(1 + \left(\frac{z^2}{4}\right) + \left(\frac{z^2}{4}\right)^2 + \cdots + \left(\frac{z^2}{4}\right)^k + \cdots\right).$$

For $|z| > 2$ we have the geometric series

$$\begin{aligned} z^3(z^2-4)^{-1} &= \frac{z^3}{z^2}\left(1 - \frac{4}{z^2}\right)^{-1}, \\ &= z\left(1 + \left(\frac{4}{z^2}\right) + \left(\frac{4}{z^2}\right)^2 + \cdots + \left(\frac{4}{z^2}\right)^k + \cdots\right). \end{aligned}$$

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13. *The following was question 3 of the May 2020 exam*

(a) Determine the circle of convergence of each of the following power series. In each case you must justify your answer.

i.
$$-4 + 100z + \sum_{n=2}^{\infty} \left(\frac{z+1}{3}\right)^n.$$

ii.
$$\sum_{n=0}^{\infty} \frac{nz^n}{2^n+1}.$$

(b) Determine the largest annulus of the form $0 \le r < |z| < R \le \infty$ for which the following Laurent series converges. You must justify your answer.

$$\sum_{n=1}^{\infty} \frac{n^2}{(3z)^n} + \sum_{n=0}^{\infty} \frac{z^n}{(2n)!}.$$

(c) The function $\sin(z)$ is an entire function with the Maclaurin series representation

$$\sin(z) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{z^{2m-1}}{(2m-1)!} = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \cdots$$

Let $f_1(z)$ and $f_2(z)$ denote the following functions.

$$f_1(z) = \frac{\sin(z)}{1-z^2} \quad \text{and} \quad f_2(z) = \frac{1-z^2}{\sin(z)}.$$

i. State, giving reasons, the radius of convergence R_1 of the Maclaurin series for $f_1(z)$.

ii. State, giving reasons, the largest value of $R_2 > 0$ such that $f_2(z)$ is analytic in the annulus $0 < |z| < R_2$.

iii. Determine the first 3 coefficients a_1, a_3 and a_5 in the Maclaurin series

$$f_1(z) = a_1 z + a_3 z^3 + a_5 z^5 + \cdots .$$

You must indicate the method used and give appropriate workings.

iv. Determine the first 3 coefficients b_{-1}, b_1 and b_3 in the Laurant series

$$f_2(z) = \frac{b_{-1}}{z} + b_1 z + b_3 z^3 + \cdots .$$

You must indicate the method used and give appropriate workings.

(d) Let
$$g(z) = \frac{1}{z-1} + \frac{1}{2z-1}.$$

Determine the general term in the Laurent series for $g(z)$ valid in $|z| > 1$.

Solution

(a) i. The series part from $n = 2$ is a geometric series which converges if and only if $|z+1| < 3$. The circle of convergence is $\{z : |z+1| = 3\}$.

ii. Let

$$a_n = \frac{n}{2^n + 1} \quad \text{and} \quad b_n = a_n z^n.$$

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \left(\frac{n+1}{n}\right)\left(\frac{2^n+1}{2^{n+1}+1}\right)\\ &= \left(1+\frac{1}{n}\right)\left(\frac{1+1/2^n}{2+1/2^n}\right) \to \frac{1}{2} \quad \text{as } n \to \infty.\end{aligned}$$

Thus

$$\frac{b_{n+1}}{b_n} \to \frac{z}{2} \quad \text{as } n \to \infty.$$

By the ratio test the circle of convergence is $\{z : |z| = 2\}$.

(b) Now let $a_n = n^2/3^n$.

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^2 \left(\frac{1}{3}\right) \to \left(\frac{1}{3}\right) \quad \text{as } n \to \infty.$$

By the ratio test the series with the negative powers converges when $1/|3z| < 1$ and thus $r = 1/3$.

Now let $a_n = \dfrac{1}{(2n)!}$.

$$\frac{a_{n+1}}{a_n} = \frac{(2n)!}{(2n+2)!} = \frac{1}{(2n+1)(2n+2)} \to 0 \quad \text{as } n \to \infty.$$

By the ratio test the series with the positive powers converges for all z and thus $R = \infty$.

(c) i. The denominator is 0 when $z = \pm 1$ and these points are the only non-analytic points of $f(z)$. Thus $R_1 = 1$.

ii. The zeros of $\sin(z)$ are at $z = k\pi$, $k \in \mathbb{Z}$. The denominator is non-zero when $0 < |z| < \pi$ and thus the function is analytic here with simple poles at $z = \pm\pi$. Hence $R_2 = \pi$.

iii. By the geometric series

$$\frac{1}{1-z^2} = 1 + z^2 + z^4 + \cdots .$$

Thus by the Cauchy product

$$\begin{aligned}f_1(z) &= \left(1 + z^2 + z^4 + \cdots\right)\left(z - \frac{z^3}{6} + \frac{z^5}{120} + \cdots\right)\\ &= z + \left(1 - \frac{1}{6}\right) z^3 + \left(1 - \frac{1}{6} + \frac{1}{120}\right) z^5 + \cdots\\ &= z + \frac{5}{6} z^3 + \frac{101}{120} z^5 + \cdots\end{aligned}$$

Hence $a_1 = 1$, $a_3 = 5/6$ and $a_5 = 101/120$.

iv.

$$1 - z^2 = f_2(z)\sin(z) = \left(\frac{b_{-1}}{z} + b_1 z + b_3 z^3 + \cdots\right)\left(z - \frac{z^3}{6} + \frac{z^5}{120} + \cdots\right).$$

By equating constant terms $b_{-1} = 1$.
By equating coefficients of z^2 we have

$$-1 = b_1 - \frac{b_{-1}}{6} = b_1 - \frac{1}{6}, \quad \text{thus} \quad b_1 = -\frac{5}{6}.$$

By equating coefficients of z^4 we have

$$0 = b_3 - \frac{b_1}{6} + \frac{b_{-1}}{120} = b_3 + \frac{5}{36} + \frac{1}{120} \quad \text{thus} \quad b_3 = -\frac{5}{36} - \frac{1}{120} = -\frac{53}{360}.$$

(d)

$$z - 1 = z\left(1 - \frac{1}{z}\right), \quad \frac{1}{z-1} = \left(\frac{1}{z}\right)\left(1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots\right)$$

Similarly

$$2z - 1 = 2z\left(1 - \frac{1}{2z}\right), \quad \frac{1}{2z-1} = \left(\frac{1}{2z}\right)\left(1 + \left(\frac{1}{2z}\right) + \left(\frac{1}{2z}\right)^2 + \left(\frac{1}{2z}\right)^3 + \cdots\right)$$

Thus

$$g(z) = \sum_{n=1}^{\infty} \frac{c_n}{z^n} \quad \text{with} \quad c_n = 1 + \frac{1}{2^n}.$$

14. *The following was question 3 of the May 2019 exam*

(a) Determine if the following power series define entire functions, and if this is not the case then find the circle of convergence. In each case you must justify your answer.

i.
$$\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}.$$

ii.
$$\sum_{n=0}^{\infty} \frac{n^2(z+1)^n}{3^n}.$$

iii.
$$\sum_{n=0}^{\infty} c_n z^n,$$

where $c_0,\ c_1,\ c_2, \ldots$ is any bounded sequence of complex numbers which are such that $|c_n| \ge 1$ for $n = 0, 1, \ldots$.

[7 marks]

(b) Determine the largest annulus of the form $0 \le r < |z| < R$ for which the following Laurent series converges. You must justify your answer.

$$\sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{n}{3^n} z^n.$$

[3 marks]

(c) Let $f(z)$ and $g(z)$ be defined by

$$\begin{aligned} f(z) &= \frac{1-\cos(z)}{1+2\cos(z)} = a_2 z^2 + a_4 z^4 + \cdots, \\ g(z) &= \frac{1}{f(z)} = \frac{b_{-2}}{z^2} + b_0 + b_2 z^2 + \cdots. \end{aligned}$$

Giving justification for your answer in each case, do the following.

i. Determine the radius of convergence of the Maclaurin series for $f(z)$.

ii. Determine the largest annulus of the form $0 < |z| < R$ for which the Laurent series for $g(z)$ converges.

iii. Determine the coefficients a_2 and a_4 in the Maclaurin series for $f(z)$.

iv. Determine the coefficients b_{-2} and b_0 in the Laurent series for $g(z)$.

[7 marks]

(d) Let

$$f(z) = \frac{1}{1+z} + \frac{1}{2+z}.$$

Determine the Laurent series for this function valid in $|z| > 2$, giving the coefficient a_n in the term a_n/z^n for $n \ge 1$.

[3 marks]

Solution

(a) i. Let $a_n = 1/(n+1)!$ and $b_n = a_n z^n$.

$$\frac{a_{n+1}}{a_n} = \frac{1}{n+2} \quad \text{and} \quad \frac{b_{n+1}}{b_n} = \frac{z}{n+2} \to 0 \quad \text{as } n \to \infty.$$

By the ratio test the series converges for all z. The function is an entire function.

$\boxed{\textbf{2 marks}}$

ii. Let $a_n = n^2/3^n$ and $b_n = a_n z^n$.

$$\frac{a_{n+1}}{a_n} = \left(\frac{(n+1)^2}{n^2}\right)\frac{1}{3} = \left(1+\frac{1}{n}\right)^2\frac{1}{3} \to \frac{1}{3} \quad \text{as } n \to \infty.$$

Thus

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{a_n}(z+1) \to \frac{z+1}{3} \quad \text{as } n \to \infty.$$

By the ratio test the series converges when $|z+1| < 3$ and diverges when $|z+1| > 3$. The circle of convergence is $|z+1| = 3$.

$\boxed{\textbf{2 marks}}$

iii. Let M be a bound for the sequence $|c_0|, |c_1|, \ldots$. As $|c_n| \ge 1$ we have

$$|z^n| \le |c_n z^n| \le M|z^n|.$$

Hence

$$|z| \le |c_n z^n|^{1/n} \le M^{1/n}|z| \to |z| \quad \text{as } n \to \infty.$$

By the root test the series converges when $|z| < 1$ and diverges when $|z| > 1$ and hence the circle of convergence is $|z| = 1$.

$\boxed{\textbf{3 marks}}$

(b) The geometric series

$$\sum_{n=1}^{\infty} \frac{1}{z^n}$$

converges for $|z| > 1$. For the other part let $a_n = n/3^n$ and $b_n = a_n z^n$.

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)\frac{1}{3} \to \frac{1}{3} \quad \text{as } n \to \infty.$$

Thus

$$\frac{b_{n+1}}{b_n} \to \frac{z}{3} \quad \text{as } n \to \infty$$

and by the ratio test the series with positive powers of z converges for $|z| < 3$. The annulus is hence $r = 1$ and $R = 3$.

$\boxed{\textbf{3 marks}}$

(c) i. $1 + 2\cos(z) = 0$ when $\cos(z) = -1/2$ and the nearest point to $z = 0$ is $z = 2\pi/3$. The radius of convergence of the Maclaurin series for $f(z)$ is $2\pi/3$.

2 marks

ii. $1 - \cos(z) = 0$ when $z = 2k\pi$ where k is an integer. Thus $g(z)$ is analytic in $0 < |z| < 2\pi$ and $R = 2\pi$.

1 mark

iii. We have

$$1 - \cos(z) = (1 + 2\cos(z))(a_2 z^2 + a_4 z^4 + \cdots),$$

i.e.

$$\frac{z^2}{2} - \frac{z^4}{24} + \cdots = \left(3 - z^2 + \frac{z^4}{12} + \cdots\right)(a_2 z^2 + a_4 z^4 + \cdots).$$

Equating coefficients of z^2 gives

$$\frac{1}{2} = 3a_2, \quad a_2 = \frac{1}{6}.$$

Equating coefficients of z^4 gives

$$-\frac{1}{24} = -a_2 + 3a_4 = -\frac{1}{6} + 3a_4 \quad \text{gives } a_4 = \frac{1}{24}.$$

2 marks

iv. We have

$$1 + 2\cos(z) = (1 - \cos(z))\left(\frac{b_{-2}}{z^2} + b_0 + b_2 z^2 + \cdots\right),$$

i.e.

$$3 - z^2 + \frac{z^4}{12} + \cdots = \left(\frac{z^2}{2} - \frac{z^4}{24} + \cdots\right)\left(\frac{b_{-2}}{z^2} + b_0 + b_2 z^2 + \cdots\right).$$

Equating the constant terms gives

$$3 = \frac{b_{-2}}{2}, \quad b_{-2} = 6.$$

Equating the z^2 terms gives

$$-1 = -\frac{b_{-2}}{24} + \frac{b_0}{2} = -\frac{1}{4} + \frac{b_0}{2}, \quad b_0 = -\frac{3}{2}.$$

2 marks

(d)

$$1+z = z\left(1-\left(\frac{-1}{z}\right)\right) \quad \text{and} \quad 2+z = z\left(1-\left(\frac{-2}{z}\right)\right).$$

Thus

$$\begin{aligned} f(z) &= \frac{1}{1+z}+\frac{1}{2+z} \\ &= \frac{1}{z}\left(\left(1+\left(\frac{-1}{z}\right)+\left(\frac{-1}{z}\right)^2+\cdots\right)+\left(1+\left(\frac{-2}{z}\right)+\left(\frac{-2}{z}\right)^2+\cdots\right)\right). \end{aligned}$$

The coefficients a_n for $n \ge 1$ are

$$a_n = (-1)^{n-1} + (-2)^{n-1} = (-1)^{n-1}(1+2^{n-1}).$$

3 marks

15. *The following was question 3 of the May 2018 exam*

(a) Determine if the following power series' define entire functions, and if this is not the case then find the circle of convergence. In each case you must justify your answer.

i.
$$\sum_{n=1}^{\infty} (3iz)^n.$$

ii.
$$\sum_{n=0}^{\infty} (n+2)(2^n+1)(z+5)^n.$$

iii.
$$\sum_{n=1}^{\infty} b_n \left(\frac{z}{n}\right)^n, \quad \text{where } b_n = \begin{cases} n^2, & \text{when } n \text{ is odd,} \\ n^3, & \text{when } n \text{ is even.} \end{cases}$$

(b) Let
$$f_1(z) = \frac{e^z - e^\pi}{z^2 - 5z + 6} \quad \text{and} \quad f_2(z) = \frac{1}{f_1(z)}.$$

For the function $f_1(z)$, give all the points where it is not analytic.
For the function $f_2(z)$, give all the points where it is not analytic.
Let $z_0 = 0$ and $z_1 = 5/2 + 2\pi i$. Give the radius of convergence of the Taylor series of $f_1(z)$ about each of the points z_0 and z_1. Similarly give the radius of convergence of the Taylor series of $f_2(z)$ about each of the points z_0 and z_1. In each of these cases you need to give reasons to justify your answers.

(c) Let
$$f(z) = \frac{1}{\sinh(z)} = \frac{2}{e^z - e^{-z}}.$$

This function is analytic in an annulus of the form $0 < |z| < R$. State the largest value of R for which this is true and determine a_{-1}, a_1 and a_3 in the Laurent series representation of the form
$$f(z) = \frac{a_{-1}}{z} + a_1 z + a_3 z^3 + \cdots$$

(d) Let
$$g(z) = \frac{z}{z^2 - 9}.$$

Give the Laurent series representation of $g(z)$ valid in the annulus $|z| > 3$.

Solution

(a) i. This series is the geometric series and converges if and only if
$$|3iz| < 1.$$

The circle of convergence is $\{z : |z| = 1/3\}$.

ii. Let $b_n = (n+2)(2^n+1)$ and $a_n = b_n(z+5)^n$.

$$\frac{b_{n+1}}{b_n} = \frac{(n+3)(2^{n+1}+1)}{(n+2)(2^n+1)} = \frac{(1+3/n)(2+1/2^n)}{(1+2/n)(1+1/2^n)} \to 2 \quad \text{as } n \to \infty.$$

Thus

$$\left|\frac{a_{n+1}}{a_n}\right| \to 2|z+5| \quad \text{as } n \to \infty.$$

By the ratio test the circle of convergence is $\{z : \; |z+5| = 1/2\}$.

iii. Let

$$a_n = b_n \left(\frac{z}{n}\right)^n \quad \text{and note that } |a_n|^{1/n} = |b_n|^{1/n} \frac{|z|}{n}.$$

For all $n \geq 1$, $1 \leq |b_n| \leq n^3$ and

$$1 \leq |b_n|^{1/n} \leq n^{3/n} \to 1 \quad \text{as } n \to \infty.$$

Thus for all z

$$|a_n|^{1/n} \to 0 \quad \text{as } n \to \infty.$$

By the root test the series converges for all z and the power series defines an entire function.

(b) As $z^2 - 5z + 6 = (z-2)(z-3)$ the function $f_1(z)$ is not analytic at $z = 2$ and $z = 3$.

The exponential function is periodic with period $2\pi i$ and thus $\mathrm{e}^z = \mathrm{e}^\pi$ when

$$z = \pi + 2k\pi i, \quad k \in \mathbb{Z}.$$

These are the points where $f_2(z)$ is not analytic.

For $f_1(z)$ the nearest non-analytic point to $z_0 = 0$ is $z = 2$ and thus the radius of convergence of the Taylor series about $z_0 = 0$ is 2.

For $f_1(z)$ the nearest non-analytic points to $z_1 = 5/2 + 2\pi i$ are $z = 2$ and $z = 3$ and thus the radius of convergence of the Taylor series about z_1 is

$$\left|\pm\frac{1}{2} + 2\pi i\right| = \sqrt{\frac{1}{4} + 4\pi^2}.$$

For $f_2(z)$ the nearest non-analytic point to $z_0 = 0$ is $z = \pi$ and thus the radius of convergence of the Taylor series about $z_0 = 0$ is π.

1 mark

For $f_2(z)$ the nearest non-analytic point to $z_1 = 5/2 + 2\pi i$ is $z = \pi + 2\pi i$ and thus the radius of convergence of the Taylor series about z_1 is $\pi - 5/2$.

(c)
$$\sinh(z) = 0 \quad \text{when } \mathrm{e}^z = \mathrm{e}^{-z}, \quad \text{i.e. when } \mathrm{e}^{2z} = 1.$$

The solutions are $z = k\pi i$, $k \in \mathbb{Z}$ and thus $R = \pi$.

$$1 = \sinh(z) f(z) = \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)\left(\frac{a_{-1}}{z} + a_1 z + a_3 z^3 + \cdots\right).$$

Equating constant terms gives

$$1 = a_{-1}.$$

Equating z^2 terms gives

$$0 = a_1 + \frac{a_{-1}}{6}, \quad a_1 = -\frac{1}{6}.$$

Equating z^4 terms gives

$$0 = a_3 + \frac{a_1}{6} + \frac{a_{-1}}{120} = a_3 - \frac{1}{36} + \frac{1}{120} = a_3 - \frac{10-3}{360}. \quad \text{Thus } a_3 = 7/360.$$

(d)
$$z^2 - 9 = z^2\left(1 - \frac{9}{z^2}\right).$$

When $|z| > 3$ the geometric series gives

$$\frac{1}{z^2 - 9} = \frac{1}{z^2}\left(1 + \frac{9}{z^2} + \frac{9^2}{z^4} + \cdots + \frac{9^n}{z^{2n}} + \cdots\right).$$

Hence
$$g(z) = \frac{1}{z^1} + \frac{9}{z^3} + \frac{9^2}{z^5} + \cdots + \frac{9^n}{z^{2n+1}} + \cdots.$$

16. *The following was question 3 of the May 2017 exam*

(a) Determine if the following power series' define entire functions and if this is not the case then find the circle of convergence. In each case you must justify your answer.

i.

$$\sum_{n=0}^{\infty} n^2 z^n.$$

ii.

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)} \left(\frac{z+1}{2}\right)^n.$$

iii.

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n = \begin{cases} 1/n^n, & \text{if } n \text{ is even,} \\ 2/n^n, & \text{if } n \text{ is odd.} \end{cases}$$

(b) Let $f(z)$ and $g(z)$ be defined by

$$f(z) = \cosh z - \cos z \quad \text{and} \quad g(z) = \frac{1}{f(z)}.$$

Give the first 2 non-zero terms of the Maclaurin series of $f(z)$.
The function $g(z)$ has a Laurent series representation close to 0 of the form

$$g(z) = \frac{c_{-2}}{z^2} + c_0 + c_2 z^2 + \cdots$$

Determine c_{-2}, c_0 and c_2.
For all complex numbers a and b the following identity holds.

$$\cos a - \cos b = -2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right).$$

By using this relation, and by expressing $\cosh(z)$ in the form $\cos(kz)$ for a suitable constant k, find the largest value of R such that $g(z)$ is analytic in the annulus $0 < |z| < R$.

(c) Let

$$\phi(z) = \frac{1}{1-z} + \frac{2}{2+z}.$$

Determine the the Laurent series valid for $1 < |z| < 2$. In your answer you must give the coefficient of $1/z^n$ for $n \ge 1$ and the coefficient of z^n for $n \ge 0$.

Solution

(a) i. Let $a_n = n^2 z^n$.

$$\frac{a_{n+1}}{a_n} = z\frac{(n+1)^2}{n^2} = z\left(1 + \frac{1}{n}\right)^2 \to z \quad \text{as } n \to \infty.$$

By the ratio test the series converges in $|z| < 1$ and diverges in $|z| > 1$ and thus $|z| = 1$ is the circle of convergence.

ii. Let

$$a_n = \frac{1}{(2n+1)}\left(\frac{z+1}{2}\right)^n.$$

$$\frac{a_{n+1}}{a_n} = \left(\frac{2n+1}{2n+3}\right)\frac{z+1}{2} = \left(\frac{2+1/n}{2+3/n}\right)\frac{z+1}{2} \to \frac{z+1}{2} \quad \text{as } n\to\infty.$$

By the ratio test $|z+1|=2$ is the circle of convergence.

iii. Let $b_n = a_n z^n$. As $|a_n| \le 2/n^n$ we have

$$0 < |b_n| \le 2\frac{|z|^n}{n^n}$$

and

$$0 < |b_n|^{1/n} \le 2^{1/n}\frac{|z|}{n}.$$

As the right hand side tends to 0 as $n\to\infty$ it follows that $|b_n|^{1/n}\to 0$ for all $|z|$. By the root test the function is an entire function.

(b) We have

$$\begin{aligned}
\cosh z &= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots, \\
\cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \\
\cosh z - \cos z &= z^2 + \frac{z^6}{360} + \cdots.
\end{aligned}$$

As $1 = f(z)g(z)$ we have

$$1 = \left(z^2 + \frac{z^6}{360} + \cdots\right)\left(\frac{c_{-2}}{z^2} + c_0 + c_2 z^2 + \cdots\right)$$

By equating the constant term gives $c_{-2}=1$.
By equating the z^2 coefficients

$$c_0 = 0.$$

By equating the z^4 coefficients

$$c_2 + \frac{c_{-2}}{360} = 0 \quad \text{and this } c_2 = -\frac{1}{360}.$$

As

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \cos(iz) = \frac{e^{-z}+e^{z}}{2} = \cosh z$$

and

$$f(z) = \cos(iz) - \cos(z) = -2\sin\left(\frac{(i+1)z}{2}\right)\sin\left(\frac{(i-1)z}{2}\right).$$

The zeros of $\sin w$ are $w = k\pi$ where k is an integer. Thus $f(z) = 0$ when

$$\left(\frac{(i+1)z}{2}\right) = \pm\pi \quad \text{and when} \quad \left(\frac{(i-1)z}{2}\right) = \pm\pi.$$

These points have magnitude

$$\frac{2\pi}{|i+1|} = \sqrt{2}\pi.$$

The function $f(z)$ is non-zero in $0 < |z| < \sqrt{2}\pi$ and hence $g(z)$ is analytic in this annulus. The required value is $R = \sqrt{2}\pi$.

(c) For $|z| > 1$,

$$\begin{aligned} 1 - z &= -z\left(1 - \frac{1}{z}\right), \\ \frac{1}{1-z} &= \left(\frac{-1}{z}\right)\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right). \end{aligned}$$

The coefficient of $1/z^n$ is -1.

For $|z| < 2$,

$$\begin{aligned} 2 + z &= 2\left(1 + \frac{z}{2}\right), \\ \frac{2}{2+z} &= \left(1 + \left(\frac{-z}{2}\right) + \left(\frac{-z}{2}\right)^2 + \cdots\right). \end{aligned}$$

The coefficient of z^n is $(-1/2)^n$.

The Laurent series in $1 < |z| < 2$ is the sum of the above two series.