# Exercises involving series

1. Find the circle of convergence of the following power series justifying your answer in each case.

   (a)
   $$\sum_{n=0}^{\infty} \frac{(z-i)^n}{2^n}.$$

   (b)
   $$\sum_{n=0}^{\infty} (z+4i)^{2n}(n-1)^2.$$

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2. Suppose that a pseudo random number generator generates a sequence $a_0$, $a_1$, $\cdots$ with $|a_n| \le 1$. Find the circle of convergence of the power series
   $$\sum_{n=0}^{\infty} (2+a_n) z^n.$$

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3. For what values of $z$, if any, does the following series converge.

   (a)
   $$\sum_{n=1}^{\infty} \frac{z^n}{n^2} + \sum_{n=1}^{\infty} \frac{n^2}{z^n}.$$

   (b)
   $$\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n + \sum_{n=1}^{\infty} \frac{1}{z^n}.$$

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4. Prove that the Taylor series of $1/(\zeta - z)$ about $z_0 \ne \zeta$ is given by
   $$\frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}} \quad \text{for } |z-z_0| < |\zeta - z_0|.$$

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5. Find the first three non-zero terms of the Laurent series for each of the following functions in the specified domains.

   (a)
   $$\frac{\mathrm{e}^{1/z}}{z^2-1}, \quad |z| > 1.$$

   (b)
   $$\mathrm{cosec}(z) = \frac{1}{\sin(z)}, \quad 0 < |z| < \pi.$$

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6. Let

$$f(z) = \frac{\tan z}{(z^2+1)(z^2+4)}$$

What is the radius of convergence of the following series.

(a) The Maclaurin series.

(b) The Taylor series about the point $z_0 = 1$.

(c) The Taylor series about the point $z_0 = \pi(1+i)$.

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7. Obtain the first 3 non-zero terms in the Maclaurin expansion of the following stating in each case where the series converges.

$$\tanh z = \frac{\sinh z}{\cosh z}.$$

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8. By any means determine the radius of convergence of the following power series

$$1 + 2z + z^2 + (2z)^3 + z^4 + (2z)^5 + \cdots + z^{2n} + (2z)^{2n+1} + \cdots$$

Give an expression for the limit.

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9. Find the general term in the Maclaurin series of the following function.

$$\frac{\mathrm{e}^z}{1-z} = \sum_{n=0}^{\infty} c_n z^n.$$

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10. *The following was question 3 of the May 2023 exam*

(a) Determine if the following power series define entire functions, and if this is not the case then find the circle of convergence. In each case you must justify your answer.

$$\sum_{n=1}^{\infty} n\left(\frac{z-5}{2}\right)^n, \qquad \sum_{n=0}^{\infty}\left(\frac{8^n+2}{3^n+7}\right)z^n, \qquad \sum_{n=1}^{\infty}\frac{(z+2)^n}{2^n n^n+1}.$$

(b) Let $f_1(z)$ and $f_2(z)$ be given by

$$f_1(z) = \frac{4}{z^2+4}, \quad \text{and} \quad f_2(z) = \frac{3}{2+\mathrm{e}^z}.$$

Both functions are analytic at $z = 0$ and have a Maclaurin expansion. Determine the radius of convergence of the Maclaurin series for the function $f_1(z)$ and also determine the radius of convergence of the Maclaurin series for the function $f_2(z)$. In each case you must justify your answer.

Suppose that the Maclaurin series for $f_2(z)$ is expressed in the form

$$f_2(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots .$$

Determine $a_0$, $a_1$, $a_2$ and $a_3$.

For your information, the first few terms of the Maclaurin series of $\mathrm{e}^z$ are given by

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \cdots .$$

(c) Let $b > 0$. Determine the Laurent series for

$$\frac{1}{z-b}$$

which is valid for large $|z|$ in an annulus with centre at 0. In your answer you need to give the general term and indicate the smallest value of $R > 0$ such that the series is valid for all $|z| > R$.

Let

$$g(z) = \frac{2}{(z-1)(z-2)(z-3)} = \frac{1}{z-1} - \frac{2}{z-2} + \frac{1}{z-3}.$$

Determine an expression for $a_n$ such that

$$g(z) = \sum_{n=1}^{\infty}\frac{a_n}{z^n}, \quad |z| > 3.$$

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11. *The following was question 3 of the May 2022 exam*

(a) You need to consider the following series.

$$\sum_{n=0}^{\infty}(2n-1)z^n, \qquad \sum_{n=0}^{\infty}\left(\frac{1}{3^n+4^n}\right)(z-1)^n, \qquad \sum_{n=0}^{\infty}\frac{(z+4)^n}{(2n)!}.$$

For each of these three power series determine if it defines an entire function or not. If the series does not define an entire function then find the circle of convergence. In each case you must justify your answer.

(b) Let

$$f(z) = \frac{2-2z}{1+\cos(z)}.$$

Determine the radius of convergence of the Maclaurin series for the function $f(z)$. Also determine the radius of convergence of the Maclaurin series for the function $1/f(z)$. In each case you must justify your answer.

Suppose that the Maclaurin series for $f(z)$ is expressed in the form

$$1 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots .$$

Determine $a_1$, $a_2$, $a_3$ and $a_4$. For your information, the first few terms of the Maclaurin series of $\cos(z)$ are given by

$$\cos(z) = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots .$$

(c) Let

$$g(z) = \frac{1}{z-1} + \frac{1}{z-4}.$$

Determine the Laurent series of $g(z)$ valid in $|z| > 4$. In your answer you should give the coefficient of $1/z^{n+1}$ for $n \ge 0$.

12. *The following was question 3 of the May 2021 exam*

(a) In the following the series to consider depends on the 5th digit of your 7-digit student id.. If your 5th digit is one of the digits 0, 1, 2, 3, 4 then your three series are as follows.

$$\sum_{n=0}^{\infty} \frac{(2z)^n}{(n+2)!}, \qquad \sum_{n=0}^{\infty} \frac{2^n}{4^n+1} z^n, \qquad \sum_{n=0}^{\infty} \left(2^n + \sin^2(n)\right)(z-1)^n.$$

If your 5th digit is one of the digits 5, 6, 7, 8, 9 then your three series are as follows.

$$\sum_{n=0}^{\infty} \frac{(z-3)^{3n}}{(n+1)!}, \qquad \sum_{n=0}^{\infty} \frac{n^2}{3^n+1} z^n, \qquad \sum_{n=0}^{\infty} \left(3^n + \cos^2(n)\right)(z+1)^n.$$

For each of your three power series determine if it defines an entire function or not. If the series does not define an entire function then find the circle of convergence. In each case you must justify your answer.

(b) In the following the function to consider depends on the 5th digit of your 7-digit student id.. If your 5th digit is one of the digits 0, 2, 4, 6, 8 then the function is

$$f_1(z) = \frac{3z + z^2}{1 - 2\cos(z)}$$

whilst if your 5th digit is one of the digits 1, 3, 5, 7, 9 then the function is

$$f_2(z) = \frac{-2z + z^2}{1 + 2\sin(z)}.$$

For your version determine the radius of convergence of the Maclaurin series. You must justify your answer.

Suppose that the Maclaurin series for your function is expressed in the form

$$a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots .$$

Determine $a_1$, $a_2$, $a_3$ and $a_4$.

(c) This part of the question is for all student numbers.

Let

$$f(z) = \frac{z^3}{z^2 - 4}.$$

Determine each of the following two series:

i. The power series of $f(z)$ in $|z| < 2$.

ii. The Laurent series of $f(z)$ valid in the annulus $\{z : |z| > 2\}$.

13. *The following was question 3 of the May 2020 exam*

(a) Determine the circle of convergence of each of the following power series. In each case you must justify your answer.

i.
$$-4+100z+\sum_{n=2}^{\infty}\left(\frac{z+1}{3}\right)^n.$$

ii.
$$\sum_{n=0}^{\infty}\frac{nz^n}{2^n+1}.$$

(b) Determine the largest annulus of the form $0 \le r < |z| < R \le \infty$ for which the following Laurent series converges. You must justify your answer.

$$\sum_{n=1}^{\infty}\frac{n^2}{(3z)^n}+\sum_{n=0}^{\infty}\frac{z^n}{(2n)!}.$$

(c) The function $\sin(z)$ is an entire function with the Maclaurin series representation

$$\sin(z)=\sum_{m=1}^{\infty}(-1)^{m+1}\frac{z^{2m-1}}{(2m-1)!}=z-\frac{z^3}{6}+\frac{z^5}{120}-\frac{z^7}{5040}+\cdots$$

Let $f_1(z)$ and $f_2(z)$ denote the following functions.

$$f_1(z)=\frac{\sin(z)}{1-z^2}\quad\text{and}\quad f_2(z)=\frac{1-z^2}{\sin(z)}.$$

i. State, giving reasons, the radius of convergence $R_1$ of the Maclaurin series for $f_1(z)$.

ii. State, giving reasons, the largest value of $R_2>0$ such that $f_2(z)$ is analytic in the annulus $0<|z|<R_2$.

iii. Determine the first 3 coefficients $a_1$, $a_3$ and $a_5$ in the Maclaurin series

$$f_1(z)=a_1z+a_3z^3+a_5z^5+\cdots.$$

You must indicate the method used and give appropriate workings.

iv. Determine the first 3 coefficients $b_{-1}$, $b_1$ and $b_3$ in the Laurant series

$$f_2(z)=\frac{b_{-1}}{z}+b_1z+b_3z^3+\cdots.$$

You must indicate the method used and give appropriate workings.

(d) Let

$$g(z)=\frac{1}{z-1}+\frac{1}{2z-1}.$$

Determine the general term in the Laurent series for $g(z)$ valid in $|z|>1$.

14. *The following was question 3 of the May 2019 exam*

(a) Determine if the following power series define entire functions, and if this is not the case then find the circle of convergence. In each case you must justify your answer.

i.
$$\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}.$$

ii.
$$\sum_{n=0}^{\infty} \frac{n^2(z+1)^n}{3^n}.$$

iii.
$$\sum_{n=0}^{\infty} c_n z^n,$$

where $c_0,\ c_1,\ c_2,\ldots$ is any bounded sequence of complex numbers which are such that $|c_n| \ge 1$ for $n = 0, 1, \ldots$.

**[7 marks]**

(b) Determine the largest annulus of the form $0 \le r < |z| < R$ for which the following Laurent series converges. You must justify your answer.

$$\sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{n}{3^n} z^n.$$ **[3 marks]**

(c) Let $f(z)$ and $g(z)$ be defined by

$$f(z) = \frac{1-\cos(z)}{1+2\cos(z)} = a_2 z^2 + a_4 z^4 + \cdots,$$
$$g(z) = \frac{1}{f(z)} = \frac{b_{-2}}{z^2} + b_0 + b_2 z^2 + \cdots.$$

Giving justification for your answer in each case, do the following.

i. Determine the radius of convergence of the Maclaurin series for $f(z)$.

ii. Determine the largest annulus of the form $0 < |z| < R$ for which the Laurent series for $g(z)$ converges.

iii. Determine the coefficients $a_2$ and $a_4$ in the Maclaurin series for $f(z)$.

iv. Determine the coefficients $b_{-2}$ and $b_0$ in the Laurent series for $g(z)$.

**[7 marks]**

(d) Let

$$f(z) = \frac{1}{1+z} + \frac{1}{2+z}.$$

Determine the Laurent series for this function valid in $|z| > 2$, giving the coefficient $a_n$ in the term $a_n/z^n$ for $n \ge 1$.

**[3 marks]**

15. *The following was question 3 of the May 2018 exam*

(a) Determine if the following power series' define entire functions, and if this is not the case then find the circle of convergence. In each case you must justify your answer.

i.
$$\sum_{n=1}^{\infty}(3iz)^n.$$

ii.
$$\sum_{n=0}^{\infty}(n+2)(2^n+1)(z+5)^n.$$

iii.
$$\sum_{n=1}^{\infty} b_n\left(\frac{z}{n}\right)^n, \quad \text{where } b_n = \begin{cases} n^2, & \text{when } n \text{ is odd,} \\ n^3, & \text{when } n \text{ is even.} \end{cases}$$

(b) Let
$$f_1(z) = \frac{\mathrm{e}^z - \mathrm{e}^\pi}{z^2 - 5z + 6} \quad \text{and} \quad f_2(z) = \frac{1}{f_1(z)}.$$

For the function $f_1(z)$, give all the points where it is not analytic.

For the function $f_2(z)$, give all the points where it is not analytic.

Let $z_0 = 0$ and $z_1 = 5/2 + 2\pi i$. Give the radius of convergence of the Taylor series of $f_1(z)$ about each of the points $z_0$ and $z_1$. Similarly give the radius of convergence of the Taylor series of $f_2(z)$ about each of the points $z_0$ and $z_1$. In each of these cases you need to give reasons to justify your answers.

(c) Let
$$f(z) = \frac{1}{\sinh(z)} = \frac{2}{\mathrm{e}^z - \mathrm{e}^{-z}}.$$

This function is analytic in an annulus of the form $0 < |z| < R$. State the largest value of $R$ for which this is true and determine $a_{-1}$, $a_1$ and $a_3$ in the Laurent series representation of the form
$$f(z) = \frac{a_{-1}}{z} + a_1 z + a_3 z^3 + \cdots$$

(d) Let
$$g(z) = \frac{z}{z^2 - 9}.$$

Give the Laurent series representation of $g(z)$ valid in the annulus $|z| > 3$.

16. *The following was question 3 of the May 2017 exam*

(a) Determine if the following power series' define entire functions and if this is not the case then find the circle of convergence. In each case you must justify your answer.

i.
$$\sum_{n=0}^{\infty} n^2 z^n.$$

ii.
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)} \left(\frac{z+1}{2}\right)^n.$$

iii.
$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n = \begin{cases} 1/n^n, & \text{if } n \text{ is even,} \\ 2/n^n, & \text{if } n \text{ is odd.} \end{cases}$$

(b) Let $f(z)$ and $g(z)$ be defined by

$$f(z) = \cosh z - \cos z \quad \text{and} \quad g(z) = \frac{1}{f(z)}.$$

Give the first 2 non-zero terms of the Maclaurin series of $f(z)$.

The function $g(z)$ has a Laurent series representation close to 0 of the form

$$g(z) = \frac{c_{-2}}{z^2} + c_0 + c_2 z^2 + \cdots$$

Determine $c_{-2}$, $c_0$ and $c_2$.

For all complex numbers $a$ and $b$ the following identity holds.

$$\cos a - \cos b = -2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right).$$

By using this relation, and by expressing $\cosh(z)$ in the form $\cos(kz)$ for a suitable constant $k$, find the largest value of $R$ such that $g(z)$ is analytic in the annulus $0 < |z| < R$.

(c) Let

$$\phi(z) = \frac{1}{1-z} + \frac{2}{2+z}.$$

Determine the the Laurent series valid for $1 < |z| < 2$. In your answer you must give the coefficient of $1/z^n$ for $n \geq 1$ and the coefficient of $z^n$ for $n \geq 0$.