Exercises involving elementary functions

1. *This question was in the class test in Dec 2022 and was worth 24 marks.*

   Determine the partial fraction representation of each of the following 3 functions and state the residue at any pole.

$$\begin{aligned} f_1(z) &= \frac{1}{(2z+1)(z-1)}, \\ f_2(z) &= \frac{-3z^2+4z+3}{z^2(z+1)}, \\ f_3(z) &= \frac{z^3}{(z+2)^5}. \end{aligned}$$

Solution

In the case of $f_1(z)$ the partial fraction representation has the form

$$f_1(z) = \frac{1}{(2z+1)(z-1)} = \frac{1/2}{(z+1/2)(z-1)} = \frac{A}{z+1/2} + \frac{B}{z-1}.$$

A is the residue at $z = -1/2$ and B is the residue at 1. Putting the right hand side on a common denominator and equating the numerators gives

$$\frac{1}{2} = A(z-1) + B(z+1/2).$$

Letting $z = 1$ gives $1/2 = B(3/2)$, $B = 1/3$.

Letting $z = -1/2$ gives $1/2 = A(-3/2)$, $A = -1/3$.

In the case of $f_2(z)$ the partial fraction representation has the form

$$f_2(z) = \frac{-3z^2+4z+3}{z^2(z+1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z+1}.$$

A is the residue at $z = 0$ and C is the residue at $z = -1$.

$$(z+1)f_2(z) = C + (z+1)(\text{a function analytic at } -1) \to C \quad \text{as } z \to -1.$$

$$C = \left.\frac{-3z^2+4z+3}{z^2}\right|_{z=-1} = \frac{-3-4+3}{1} = -4.$$

$$z^2 f_2(z) = B + Az + z^2\left(\frac{C}{z+1}\right) \to B \quad \text{as } z \to 0.$$

$$B = \left.\frac{-3z^2+4z+3}{z+1}\right|_{z=0} = 3.$$

With only A still to find we let $z = 1$. $f_2(1) = 4/2 = 2$. The value of the partial fraction representation is

$$A + B + \frac{C}{2} = A + 3 - 2 = A + 1.$$

Thus $2 = A + 1$ and $A = 1$.

In the case of $f_3(z)$ let $p(z) = z^3$. As the only singularity of $f_3(z)$ is a pole of order 5 at $z = -2$ we can get the partial fraction representation using a finite Taylor representation of $p(z)$ about $z = -2$.

$$p'(z) = 3z^2, \quad p''(z) = 6z, \quad p'''(z) = 6.$$

$$\begin{aligned} p(z) &= p(-2) + p'(-2)(z+2) + \frac{p''(-2)}{2}(z+2)^2 + \frac{p'''(-2)}{6}(z+2)^3 \\ &= -8 + 12(z+2) - 6(z+2)^2 + (z+2)^3. \end{aligned}$$

Thus

$$f_3(z) = -\frac{8}{(z+2)^5} + \frac{12}{(z+2)^4} - \frac{6}{(z+2)^3} + \frac{1}{(z+2)^2}.$$

The residue at $z = -2$ is 0 as there is no $1/(z + 2)$ term.

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2. *This question was in the class test in Dec 2021 and was worth 25 marks.*

In this question the functions $f_1(z)$, $f_2(z)$ and $f_3(z)$ that you consider depends on the 5th digit of your 7-digit student id..

If the 5th digit is one of the digits 0, 1, 2, 3, 4 then your functions $f_1(z)$, $f_2(z)$ and $f_3(z)$ are as follows.

$$f_1(z) = \frac{-13 - z}{z^2 + z - 6}, \quad f_2(z) = \frac{9}{z^2(3 + z)}, \quad f_3(z) = \frac{(z+2)^3}{(z-1)^3}.$$

If the 5th digit is one of the digits 5, 6, 7, 8, 9 then your functions $f_1(z)$, $f_2(z)$ and $f_3(z)$ are as follows.

$$f_1(z) = \frac{6z - 19}{z^2 - 3z - 4}, \quad f_2(z) = \frac{1}{z(z-1)^2}, \quad f_3(z) = \frac{(z+1)^3}{(z-3)^3}.$$

(i) For your version of $f_1(z)$ and $f_2(z)$ determine the partial fraction representation in each case and state the residue at each pole.

(ii) For your version of $f_3(z)$ determine the residue at the pole.

Solution

This is the version if the 5th digit is one of the digits of 0, 1, 2, 3, 4.

(i)

$$z^2 + z - 6 = (z-2)(z+3).$$

The partial fraction representation of $f_1(z)$ has the following form.

$$f_1(z) = \frac{-13-z}{(z-2)(z+3)} = \frac{A}{z-2} + \frac{B}{z+3}.$$

A is the residue at $z = 2$ and B is the residue at $z = -3$. Putting everything on a common denominator and comparing the numerators gives

$$-13 - z = A(z+3) + B(z-2).$$

Letting $z = 2$ gives $-15 = 5A$ and $A = -3$.
Letting $z = -3$ gives $-10 = -5B$ and $B = 2$.
The partial fraction representation of $f_2(z)$ has the following form.

$$f_2(z) = \frac{9}{z^2(3+z)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z+3}.$$

A is the residue at $z = 0$ and C is the residue at $z = -3$.

$$(z+3)f_2(z) = \frac{9}{z^2} = C + (z+3)(\text{other term}) \to C \quad \text{as } z \to -3.$$

Thus $C = 1$.

$$z^2 f_2(z) = \frac{9}{z+3} = B + Az + z^2(\text{other term}) \to B \quad \text{as } z \to 0.$$

Thus $B = 3$.
With B and C known we only have A to find. Let $z = 1$ in the expression to give

$$f_2(1) = \frac{9}{4} = A + B + \frac{C}{4} = A + 3 + \frac{1}{4}.$$

Thus $A = -1$.

(ii) Let $p(z) = (z+2)^3$.

$$f_3(z) = \frac{p(z)}{(z-1)^3} = \frac{p(1) + p'(1)(z-1) + \frac{p''(1)}{2}(z-1)^2 + \frac{p'''(1)}{6}(z-1)^3}{(z-1)^3}.$$

The residue at $z = 1$ is

$$\frac{p''(1)}{2},$$

$$p''(z) = 6(z+2).$$

$$\frac{p''(1)}{2} = \frac{18}{2} = 9.$$

This is the version if the 5th digit is one of the digits of 5, 6, 7, 8, 9.

(i)
$$z^2 - 3z - 4 = (z+1)(z-4).$$

The partial fraction representation of $f_1(z)$ has the following form.

$$f_1(z) = \frac{6z-19}{z^2-3z-4} = \frac{A}{z+1} + \frac{B}{z-4}.$$

A is the residue at $z=-1$ and B is the residue at $z=4$. Putting everything on a common denominator and comparing the numerators gives

$$6z - 19 = A(z-4) + B(z+1).$$

Letting $z=-1$ gives $-25 = -5A$ and $A=5$.
Letting $z=4$ gives $5 = 5B$ and $B=1$.
The partial fraction representation of $f_2(z)$ has the following form.

$$f_2(z) = \frac{1}{z(z-1)^2} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2}.$$

A is the residue at $z=0$ and B is the residue at $z=1$.

$$zf_2(z) = \frac{1}{(z-1)^2} = A + z(\text{other term}) \to A \quad \text{as } z \to 0.$$

Thus $A=1$.

$$(z-1)^2 f_2(z) = \frac{1}{z} = C + B(z-1) + (z-1)^2(\text{other term}) \to C \quad \text{as } z \to 1.$$

Thus $C=1$.
With A and C known we only have B to find. Let $z=2$ in the expression to give

$$f_2(2) = \frac{1}{4} = \frac{A}{4} + B + C.$$

Thus $B = -A = -1$.

(ii) Let $p(z) = (z+1)^3$.

$$f_3(z) = \frac{p(z)}{(z-3)^3} = \frac{p(3) + p'(3)(z-3) + \frac{p''(3)}{2}(z-3)^2 + \frac{p'''(3)}{6}(z-3)^3}{(z-3)^3}.$$

The residue at $z=1$ is

$$\frac{p''(3)}{2},$$

$$p''(z) = 6(z+1).$$

$$\frac{p''(3)}{2} = \frac{24}{2} = 12.$$

3. *This question was in the class test in Jan 2021 and was worth 22 marks.*

   In the following there are three rational functions $f_1(z)$, $f_2(z)$ and $f_3(z)$ that you need to consider and your version of these depends on the 6th digit of your 7-digit student id. as follows.

   If the 6th digit is one of $0, 3, 6, 9$ then you have

$$f_1(z) = \frac{5z - i}{z^2 + 1}, \quad f_2(z) = \frac{1}{z^3 - 8}, \quad f_3(z) = \frac{z^4 - 2z^3}{(z+1)^3}.$$

   If the 6th digit is one of $1, 4, 7$ then you have

$$f_1(z) = \frac{7z - 2i}{z^2 + 4}, \quad f_2(z) = \frac{1}{z^3 + 8}, \quad f_3(z) = \frac{z^5}{(z-1)^3}.$$

   If the 6th digit is one of $2, 5, 8$ then you have

$$f_1(z) = \frac{-z - 9i}{z^2 + 9}, \quad f_2(z) = \frac{1}{z^4 - 1}, \quad f_3(z) = \frac{-z^4 + 3z^3}{(z-1)^3}.$$

   (a) For your version of $f_1(z)$ determine the partial fraction representation and state the residue at any pole.

   (b) For your version of $f_2(z)$ state in cartesian form the points where it has simple poles and determine the residue at one of these points.

   (c) For your version of $f_3(z)$ determine the residue at the pole of the function.

Solution

The version if the 6th digit is one of the digits $0, 3, 6, 9$.

$z^2 + 1 = (z + i)(z - i)$.

$$f_1(z) = \frac{5z - i}{z^2 + 1} = \frac{A}{z + i} + \frac{B}{z - i}.$$

$$A = \lim_{z \to -i} (z + i) f_1(z) = (5z - i)|_{z=-i} \lim_{z \to -i} \frac{z + i}{z^2 + 1} = \frac{-6i}{2(-i)} = 3.$$

This is the residue at $-i$.

$$B = \lim_{z \to i} (z - i) f_1(z) = (5z - i)|_{z=i} \lim_{z \to i} \frac{z - i}{z^2 + 1} = \frac{4i}{2i} = 2.$$

This is the residue at i.

$$f_2(z)=\frac{1}{z^3-8}$$

This has simple poles at the 3 points satisfying $z^3=8=2^3$. These 3 points are

$$2,\quad 2\mathrm{e}^{i2\pi/3}=2(-\frac{1}{2}+i\frac{\sqrt{3}}{2})=-1+i\sqrt{3},\quad 2\mathrm{e}^{-i2\pi/3}=-1-i\sqrt{3}.$$

The residue at 2 is

$$\lim_{z\to 2}\frac{z-2}{z^3-8}=\frac{1}{3z^2}{}_{z=2}=\frac{1}{12}.$$

$$f_3(z)=\frac{z^4-2z^3}{(z+1)^3}.$$

The pole is at -1 and is of order 3. Let $p(z)=z^4-2z^3$. The finite Taylor representation about -1 is

$$p(z)=p(-1)+p'(-1)(z+1)+\frac{p''(-1)}{2}(z+1)^2+\frac{p'''(-1)}{6}(z+1)^3+\frac{p''''(-1)}{24}(z+1)^4$$

from which it follows that the residue at -1 is

$$\frac{p''(-1)}{2}.$$

$p'(z)=4z^3-6z^2$, $p''(z)=12z^2-12z$ giving the residue as 12.

The version if the 6th digit is one of the digits $1,4,7$.

$z^2+4=(z+2i)(z-2i)$.

$$f_1(z)=\frac{7z-2i}{z^2+4}=\frac{A}{z+2i}+\frac{B}{z-2i}.$$

$$A=\lim_{z\to -2i}(z+2i)f_1(z)=(7z-2i)|_{z=-2i}\lim_{z\to -2i}\frac{z+2i}{z^2+4}=\frac{-16i}{2(-2i)}=4.$$

This is the residue at $-2i$.

$$B=\lim_{z\to 2i}(z-2i)f_1(z)=(7z-2i)|_{z=2i}\lim_{z\to 2i}\frac{z-2i}{z^2+4}=\frac{12i}{4i}=3.$$

This is the residue at $2i$.

$$f_2(z)=\frac{1}{z^3+8}$$

This has simple poles at the 3 points satisfying $z^3 = -8 = (-2)^3$. These 3 points are

$$-2, \quad -2\mathrm{e}^{i2\pi/3} = -2(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 1 - i\sqrt{3}, \quad -2\mathrm{e}^{-i2\pi/3} = 1 + i\sqrt{3}.$$

The residue at -2 is

$$\lim_{z\to -2} \frac{z+2}{z^3+8} = \frac{1}{3z^2}_{z=-2} = \frac{1}{12}.$$

$$f_3(z) = \frac{z^5}{(z-1)^3}$$

The pole is at 1 and is of order 3. Let $p(z) = z^5$. The finite Taylor representation about 1 is

$$p(z) = p(1) + p'(1)(z-1) + \frac{p''(1)}{2}(z-1)^2 + \frac{p'''(1)}{6}(z-1)^3 + \frac{p''''(1)}{24}(z-1)^4 + \frac{p'''''(1)}{120}(z-1)^5$$

from which it follows that the residue at -1 is

$$\frac{p''(1)}{2}.$$

$p'(z) = 5z^4$, $p''(z) = 20z^3$ giving the residue as 10.

The version if the 6th digit is one of the digits $2, 5, 8$.

$z^2 + 9 = (z+3i)(z-3i)$.

$$f_1(z) = \frac{-z-9i}{z^2+9} = \frac{A}{z+3i} + \frac{B}{z-3i}.$$

$$A = \lim_{z\to -3i}(z+3i)f_1(z) = (-z-9i)|_{z=-3i} \lim_{z\to -3i} \frac{z+3i}{z^2+9} = \frac{-6i}{2(-3i)} = 1.$$

This is the residue at $-3i$.

$$B = \lim_{z\to 3i}(z-3i)f_1(z) = (-z-9i)|_{z=3i} \lim_{z\to 3i} \frac{z-3i}{z^2+9} = \frac{-12i}{6i} = -2.$$

This is the residue at $3i$.

$$f_2(z) = \frac{1}{z^4-1},$$

This has simple poles at the 4 points satisfying $z^4 = 1$ These 4 points are

$$\pm 1, \quad \pm i.$$

The residue at 1 is

$$\lim_{z\to 1} \frac{z-1}{z^4-1} = \left.\frac{1}{4z^3}\right|_{z=1} = \frac{1}{4}.$$

$$f_3(z) = \frac{-z^4+3z^3}{(z-1)^3}.$$

The pole is at 1 and is of order 3. Let $p(z) = -z^4 + 3z^3$. The finite Taylor representation about 1 is

$$p(z) = p(1) + p'(1)(z-1) + \frac{p''(1)}{2}(z-1)^2 + \frac{p'''(1)}{6}(z-1)^3 + \frac{p''''(1)}{24}(z-1)^4$$

from which it follows that the residue at 1 is

$$\frac{p''(1)}{2}.$$

$p'(z) = -4z^3 + 9z^2$, $p''(z) = -12z^2 + 18z$ giving the residue as 3.

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4. *This question was in the class test in 2019/2020 and was worth 25 marks.*

Let

$$f_1(z) = \frac{2z}{z^2-3z+2}, \quad f_2(z) = \frac{16}{z^2(z-4)} \quad \text{and} \quad f_3(z) = \frac{z^4}{(z-2)^3}.$$

(a) Determine the partial fraction representation of $f_1(z)$ and state the residue at each pole.

(b) Determine the partial fraction representation of $f_2(z)$ and state the residue at each pole.

(c) Determine the residue of $f_3(z)$ at $z = 2$.

Solution

(a)

$$z^2 - 3z + 2 = (z-1)(z-2)$$

and hence the partial fraction representation is of the form

$$f_1(z) = \frac{2z}{z^2-3z+2} = \frac{A}{z-1} + \frac{B}{z-2}.$$

$$\begin{aligned} A &= \lim_{z\to 1}(z-1)f_1(z) = (2z)|_{z=1} \lim_{z\to 1} \frac{z-1}{z^2-3z+2} = 2\lim_{z\to 1}\frac{1}{2z-3} = -2, \\ B &= \lim_{z\to 2}(z-2)f_1(z) = (2z)|_{z=2} \lim_{z\to 2} \frac{z-2}{z^2-3z+2} = 4\lim_{z\to 2}\frac{1}{2z-3} = 4. \end{aligned}$$

The residue at $z = 1$ is -2 and the residue at $z = 2$ is 4.

(b) $f_2(z)$ has a double pole at $z = 0$ and the partial fraction representation is of the form

$$f_2(z) = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-4}.$$

$$\begin{aligned} B &= \lim_{z\to 0} z^2 f_2(z) = \left.\frac{16}{z-4}\right|_{z=0} = -4, \\ C &= \lim_{z\to 4} (z-4) f_2(z) = \left.\frac{16}{z^2}\right|_{z=4} = 1. \end{aligned}$$

$C = 1$ is the residue at $z = 4$. As

$$z^2 f_2(z) = \frac{16}{z-4} = B + Az + z^2(\text{a function analytic at } z = 0),$$

by differentiating with respect to z we get

$$-\frac{16}{(z-4)^2} = A + z(\text{another function analytic at } z = 0).$$

Letting $z \to 0$ gives $A = -1$. $A = -1$ is the residue at $z = 0$.

(c) As the only pole is at $z = 2$ we can get the residue by considering the finite Taylor representation of $p(z) = z^4$ about $z = 2$.

$$p(z) = p(2) + p'(2)(z-2) + \frac{p''(2)}{2}(z-2)^2 + \frac{p'''(2)}{6}(z-2)^3 + \frac{p''''(2)}{24}(z-2)^4.$$

The residue is $p''(2)/2$. $p''(z) = 12z^2$ and hence the residue is 24.

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5. *This question was in the class test in 2018/2019 and was worth 20 marks.*

Let f_1 and f_2 denote the following rational functions.

$$f_1(z) = \frac{z+11}{(z-1)(z+2)}, \quad f_2(z) = \frac{4z(2z-1)}{(z-1)^2(z+1)}.$$

In each case determine the partial fraction representation and state the residue at any pole.

Solution

$$f_1(z) = \frac{z+11}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}.$$

$$A = \lim_{z\to 1}(z-1)f_1(z) = \left.\frac{z+11}{z+2}\right|_{z=1} = \frac{12}{3} = 4.$$

$$B = \lim_{z\to -2}(z+2)f_1(z) = \left.\frac{z+11}{z-1}\right|_{z=-2} = \frac{9}{-3} = -3.$$

$$f_1(z) = \frac{4}{z-1} - \frac{3}{z+2}.$$

4 is the residue of f_1 at $z=1$ and -3 is the residue of f_1 at $z=-2$.

$$f_2(z) = \frac{4z(2z-1)}{(z-1)^2(z+1)} = \frac{A_1}{z-1} + \frac{A_2}{(z-1)^2} + \frac{B}{z+1}.$$

$$B = \lim_{z\to -1}(z+1)f_2(z) = \left.\frac{4z(2z-1)}{(z-1)^2}\right|_{z=-1} = \frac{12}{4} = 3.$$

$$(z-1)^2 f_2(z) = \frac{4z(2z-1)}{z+1} = A_2 + A_1(z-1) + (z-1)^2(\text{function analytic at } z=1).$$

$$A_2 = \left.\frac{4z(2z-1)}{z+1}\right|_{z=1} = 2.$$

$$A_1 = \left.\left(\frac{4z(2z-1)}{z+1}\right)'\right|_{z=1}.$$

Now

$$\left(\frac{4z(2z-1)}{z+1}\right)' = \frac{(z+1)(16z-4)-4z(2z-1)}{(z+1)^2}.$$

$$A_1 = \frac{2(12)-4}{4} = 5.$$

$$f_2(z) = \frac{5}{z-1} + \frac{2}{(z-1)^2} + \frac{3}{z+1}.$$

The residue of f_2 at $z=1$ is 5 and the residue of f_2 at $z=-1$ is 3.

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6. Let

$$R(z) = \frac{p(z)}{(z-z_1)^{r_1}(z-z_2)^{r_2}\cdots(z-z_n)^{r_n}}$$

denote a rational function in which $z_1,\ldots,z_n$ are distinct points, where each $r_k \ge 1$ is an integer and where $p(z)$ is a polynomial which is non-zero at these n points. What can you say about the order of the poles of $R'(z)$ and $R''(z)$ and what can you say about the residues of the function $R'(z)$?

Solution

To cater for a numerator of any degree we have a representation of the form

$$R(z) = (\text{polynomial}) + \sum_{k=1}^{n}\left(\frac{A_{k,1}}{z-z_k} + \cdots + \frac{A_{k,r_k}}{(z-z_k)^{r_k}}\right).$$

The coefficient A_{k,r_k} is given by

$$\begin{aligned} A_{k,r_k} &= \lim_{z\to z_k}(z-z_k)^{r_k}R(z) \\ &= \frac{p(z_k)}{(z_k-z_1)^{r_1}\cdots(z_k-z_{k-1})^{r_{k-1}}(z_k-z_{k+1})^{r_{k+1}}\cdots(z_k-z_n)^{r_n}} \ne 0 \end{aligned}$$

as $p(z_k)\ne 0$. The order of the pole of $R(z)$ at z_k is r_k.

Differentiating gives

$$R'(z) = (\text{deriv of polynomial}) - \left(\sum_{k=1}^{n} \left(\frac{A_{k,1}}{(z-z_k)^2} + \cdots + \frac{r_k A_{k,r_k}}{(z-z_k)^{r_k+1}} \right) \right).$$

The order of the pole of $R'(z)$ at z_k is one more than the order of the pole of $R(z)$ at z_k and in particular this implies that $R'(z)$ has no terms of the form $1/(z-z_k)$ in the representation and hence all the residues are 0.

Similarly $R''(z)$ has poles which are of a order which is two more that of $R(z)$ and it has no residues.

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7. Let

$$q(z) = (z-z_1)^{r_1}(z-z_2)^{r_2}\cdots(z-z_n)^{r_n}$$

where $z_1, \ldots, z_n$ are distinct points. What can you say about the multiplicity of the zeros of $q'(z)$ at the points $z_1, \ldots, z_n$? Using a derivation based on partial fractions show that

$$\frac{q'(z)}{q(z)} = \frac{r_1}{z-z_1} + \frac{r_2}{z-z_2} + \cdots + \frac{r_n}{z-z_n}.$$

(Observe that the result is consistent with the result in an exericise of the previous exercise sheet which involved a proof by induction.)

Solution

$q(z)$ has a zero at z_k of multiplicity r_k and the derivative has a zero at z_k of multiplicity $r_k - 1$ for $k = 1, 2, \ldots, n$. The ratio $q(z)/q'(z)$ thus has a simple zero at z_k and the ratio $q'(z)/q(z)$ has a simple pole at z_k. From this it follows that the partial fraction representation is hence of the form

$$\frac{q'(z)}{q(z)} = \sum_{k=1}^{n} \frac{A_k}{z-z_k}$$

and

$$A_k = \lim_{z\to z_k} \frac{(z-z_k)q'(z)}{q(z)}.$$

Now to get A_k observe that

$$\begin{aligned} q(z) &= (z-z_k)^{r_k} g(z), \quad \text{with } g(z_k) \ne 0, \\ q'(z) &= r_k(z-z_k)^{r_k-1} g(z) + (z-z_k)^{r_k} g'(z), \\ (z-z_k)q'(z) &= r_k(z-z_k)^{r_k} g(z) + (z-z_k)^{r_k+1} g'(z) \end{aligned}$$

and for the ratio

$$\frac{(z-z_k)q'(z)}{q(z)} = r_k + (z-z_k)\frac{g'(z)}{g(z)} \to r_k \quad \text{as } z \to z_k$$

and we have the required result.

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8. *This question was in the class test in Dec 2022 and was worth 16 marks.*

   (a) Give in cartesian form the principal values of the following.

      i. $\mathrm{Log}(1+i\sqrt{3})$.

      ii. $i^{1/2}$.

   In the above Log denotes the principal value logarithm.

   In your answer you must give an exact representation of the real and imaginary parts which may involve the square root of a positive number, π and the natural logarithm of a positive number.

   (b) Let $\theta \in \mathbb{R}$. Give in cartesian form the principal value of i^α when $\alpha = \mathrm{e}^{i\theta}$. Give all values of $\theta \in (-\pi, \pi]$ such that i^α is pure imaginary.

   Full reasoning must be given to get all the marks.

Solution

(a) i. Let $z = 1 + \sqrt{3}$.

$$|z|^2 = 1 + 3 = 4 = 2^2, \quad \text{and} \quad \mathrm{Arg}(z) = \frac{\pi}{3}.$$

$$\mathrm{Log}(z) = \ln(|z|) + i\mathrm{Arg}(z) = \ln(2) + i\frac{\pi}{3}.$$

ii.

$$z^\alpha = \exp(\alpha \mathrm{Log}(z)).$$

With $z = i$, $\mathrm{Log}(z) = i\pi/2$. In our case $\alpha = 1/2$ and thus

$$i^{1/2} = \exp(i\pi/4) = \frac{1}{\sqrt{2}}(1+i).$$

(b)

$$z^\alpha = \exp(\alpha \mathrm{Log}(z)).$$

With $z = i$, $\mathrm{Log}(z) = i\pi/2$ as in the previous part. In this case $\alpha = \mathrm{e}^{i\theta} = c + is$, $c = \cos(\theta)$, $s = \sin(\theta)$.

$$\alpha \mathrm{Log}(z) = \left(\frac{i\pi}{2}\right)(c + is) = \left(\frac{\pi}{2}\right)(-s + ic).$$

$$z^\alpha = \exp\left(\frac{-s\pi}{2}\right)\left(\cos\left(\frac{\pi c}{2}\right) + i\sin\left(\frac{\pi c}{2}\right)\right).$$

This is pure imaginary when

$$\cos\left(\frac{\pi\cos(\theta)}{2}\right) = 0.$$

This is the case when

$$\cos(\theta) = \pm 1.$$

The solutions to this in $(-\pi, \pi]$ are $\theta = 0$ and $\theta = \pi$. Hence for α on the unit circle i^α is only pure imaginary when $\alpha = \pm 1$.

9. *This question was in the class test in Dec 2021 and was worth 9 marks.*

   This question is for all student numbers.

   Let $f(z)$ and $g(z)$ be defined as follows.

$$\begin{aligned} f(z) &= \tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{\mathrm{e}^z + \mathrm{e}^{-z}}, \\ g(z) &= \coth(z) = \frac{\cosh(z)}{\sinh(z)} = \frac{\mathrm{e}^z + \mathrm{e}^{-z}}{\mathrm{e}^z - \mathrm{e}^{-z}}. \end{aligned}$$

   Give the location of all the zeros and all the poles of both $f(z)$ and $g(z)$.

   Consider the straight line segment

$$\Gamma = \left\{ iy : \ \frac{\pi}{4} \le y \le \frac{\pi}{3} \right\}.$$

   Describe as concisely as possible the image of Γ under the function $f(z)$.

Solution

This version is for all student numbers.

The zeros are when $\mathrm{e}^z = \mathrm{e}^{-z}$, i.e. $\mathrm{e}^{2z} = 1$. As $\mathrm{e}^w = 1$ if and only if $w = 2k\pi i$, k being an integer it follows that the zeros are when $z = k\pi i$.

The poles are when $\mathrm{e}^z = -\mathrm{e}^{-z}$, i.e. $\mathrm{e}^{2z} = -1$. As $\mathrm{e}^w = -1$ if and only if $w = \pi + 2k\pi i$, k being an integer it follows that the poles are when $z = (k + 1/2)\pi i$.

The poles of $g(z)$ are at the zeros of $f(z)$ and the zeros of $g(z)$ are at the poles of $f(z)$

$$f(iy) = \tanh(iy) = \frac{\mathrm{e}^{iy} - \mathrm{e}^{-iy}}{\mathrm{e}^{iy} + \mathrm{e}^{-iy}} = i\frac{\sin(y)}{\cos(y)} = i\tan(y).$$

For all $y \in \mathbb{R}$ this is pure imaginary.

$$f(i\pi/4) = i, \quad f(i\pi/3) = \sqrt{3}i.$$

The image of Γ is the straight line segment joining i and $\sqrt{3}i$.

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10. *This question was in the class test in Jan 2021 and was worth 8 marks.*

    Let $z = x+iy$ with $x, y \in \mathbb{R}$. The definition of e^z, $\cosh(z)$ and $\sinh(z)$ are respectively

$$\mathrm{e}^z = \mathrm{e}^x(\cos(y) + i\sin(y)), \quad \cosh(z) = \frac{1}{2}\left(\mathrm{e}^z + \mathrm{e}^{-z}\right), \quad \sinh(z) = \frac{1}{2}\left(\mathrm{e}^z - \mathrm{e}^{-z}\right).$$

    In the following you need to show an identity and which one you need to show depends on the 6th digit of your 7-digit student id..

If the 6th digit is one of the digits $0, 2, 4, 6, 8$ then show that

$$\sinh(z + i\pi/3) + \sinh(z - i\pi/3) = \sinh(z).$$

If the 6th digit is one of the digits $1, 3, 5, 7, 9$ then show that

$$\cosh(z + i\pi/3) + \cosh(z - i\pi/3) = \cosh(z).$$

Solution

The version if the 6th digit is one of the digits $0, 2, 4, 6, 8$.
$\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$.

$$\begin{aligned} \exp(z + i\pi/3) &= \exp(z)\exp(i\pi/3) = \mathrm{e}^z\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \\ \exp(z - i\pi/3) &= \exp(z)\exp(-i\pi/3) = \mathrm{e}^z\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right). \end{aligned}$$

Thus

$$\exp(z + i\pi/3) + \exp(z - i\pi/3) = \mathrm{e}^z.$$

Replacing z by $-z$ gives

$$\exp(-z + i\pi/3) + \exp(-z - i\pi/3) = \mathrm{e}^{-z}.$$

The result follows by subtracting the second result from the first result and dividing by 2.

The version if the 6th digit is one of the digits $1, 3, 5, 7, 9$.
$\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$.

$$\begin{aligned} \exp(z + i\pi/3) &= \exp(z)\exp(i\pi/3) = \mathrm{e}^z\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \\ \exp(z - i\pi/3) &= \exp(z)\exp(-i\pi/3) = \mathrm{e}^z\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right). \end{aligned}$$

Thus

$$\exp(z + i\pi/3) + \exp(z - i\pi/3) = \mathrm{e}^z.$$

Replacing z by $-z$ gives

$$\exp(-z + i\pi/3) + \exp(-z - i\pi/3) = \mathrm{e}^{-z}.$$

The result follows by adding the last two results and dividing by 2.

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11. *The following was in the class test in 2019/2020 and was worth 13 marks.*

Let $z = x + iy$ with $x, y \in \mathbb{R}$. Given that

$$\mathrm{e}^z = \mathrm{e}^x(\cos(y) + i\sin(y)) \quad \text{and} \quad \cosh(z) = \frac{\mathrm{e}^z + \mathrm{e}^{-z}}{2}$$

show that

$$\cosh(x + iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y).$$

Let $a > 0$ and let

$$z_1 = a + i\frac{\pi}{2}, \quad z_2 = i\frac{\pi}{2}, \quad z_3 = 0, \quad \text{and} \quad z_4 = a.$$

State the image of each point z_1, z_2, z_3 and z_4 under the mapping $\cosh(z)$. Describe, as concisely as possible, the image of the polygonal path z_1 to z_2, z_2 to z_3 and z_3 to z_4 under the mapping $\cosh(z)$. You need to justify your answer.

Solution

$$\begin{aligned} 2\cosh(x + iy) &= \mathrm{e}^{x+iy} + \mathrm{e}^{-x-iy} \\ &= \mathrm{e}^x(\cos(y) + i\sin(y)) + \mathrm{e}^{-x}(\cos(-y) + i\sin(-y)) \\ &= \cos(y)(\mathrm{e}^x + \mathrm{e}^{-x}) + i\sin(y)(\mathrm{e}^x - \mathrm{e}^{-x}) \\ &= 2\cos(y)\cosh(x) + i2\sin(y)\sinh(x). \end{aligned}$$

Dividing by 2 gives the required expression.

Let $w_k = \cosh(z_k)$ for $k = 1, 2, 3, 4$. As $\cos(\pi/2) = 0$, $\cos(0) = 1$, $\sin(\pi/2) = 1$ and $\sin(0) = 0$ we have

$$w_1 = i\sinh(a), \quad w_2 = 0, \quad w_3 = 1, \quad w_4 = \cosh(a).$$

As $\cosh(x + i\pi/2) = i\sinh(x)$ it follows that the straight line segment z_1 to z_2 maps to the part of the imaginary axis from $\sinh(a) > 0$ to 0.

As $\cosh(iy) = \cos(y)$ it follows that the straight line segment z_2 to z_3 maps to the part of the real axis from 0 to 1.

The straight line segment z_3 to z_4 maps to the part of the real axis from 1 to $\cosh(a)$.

The image of the polygonal path is thus the union of the straight line segment joining $i\sinh(a)$ to 0 with the straight line segment joining 0 to $\cosh(a)$.

---

12. Show that if $y \in \mathbb{R}$ then

$$\left|\tan\left(\frac{\pi}{4} + iy\right)\right| = 1.$$

Describe in words the set

$$G = \left\{\tan\left(\frac{\pi}{4} + iy\right) \,:\, -\infty < y < \infty\right\}.$$

Solution

As

$$\tan z = \frac{\sin z}{\cos z}$$

we first consider the numerator and denominator when $z = \pi/4 + iy$.

$$\begin{aligned} 2i\sin\left(\frac{\pi}{4} + iy\right) &= \exp\left(i\left(\frac{\pi}{4} + iy\right)\right) - \exp\left(-i\left(\frac{\pi}{4} + iy\right)\right) \\ &= \mathrm{e}^{i\pi/4}\mathrm{e}^{-y} - \mathrm{e}^{-i\pi/4}\mathrm{e}^{y} = \mathrm{e}^{i\pi/4}\mathrm{e}^{-y}\left(1 - \mathrm{e}^{-i\pi/2}\mathrm{e}^{2y}\right), \\ 2\cos\left(\frac{\pi}{4} + iy\right) &= \mathrm{e}^{i\pi/4}\mathrm{e}^{-y} + \mathrm{e}^{-i\pi/4}\mathrm{e}^{y} = \mathrm{e}^{i\pi/4}\mathrm{e}^{-y}\left(1 + \mathrm{e}^{-i\pi/2}\mathrm{e}^{2y}\right). \end{aligned}$$

By taking the absolute value of the ratio of these terms and noting that $\mathrm{e}^{i\pi/2} = i$ and $\mathrm{e}^{-i\pi/2} = -i$ we get

$$\left|\tan\left(\frac{\pi}{4} + iy\right)\right| = \frac{|1 + i\mathrm{e}^{2y}|}{|1 - i\mathrm{e}^{2y}|}.$$

$1 - i\mathrm{e}^{2y}$ is the complex conjugate of $1 + i\mathrm{e}^{2y}$ with both values having the same magnitude and hence

$$\left|\tan\left(\frac{\pi}{4} + iy\right)\right| = 1.$$

For the set G we have just shown that we get points on the unit circle and we need to establish now what part of the unit circle we get. From the workings just done we have

$$\begin{aligned} i\tan\left(\frac{\pi}{4} + iy\right) &= \frac{\left(1 + i\mathrm{e}^{2y}\right)}{\left(1 - i\mathrm{e}^{2y}\right)} \\ &= \frac{\left(1 + i\mathrm{e}^{2y}\right)^2}{1 + i\mathrm{e}^{4y}} \\ &= \frac{1 + 2i\mathrm{e}^{2y} - \mathrm{e}^{4y}}{1 + \mathrm{e}^{4y}}. \end{aligned}$$

Hence

$$\tan\left(\frac{\pi}{4} + iy\right) = \frac{2\mathrm{e}^{2y} + i(\mathrm{e}^{4y} - 1)}{1 + \mathrm{e}^{4y}}.$$

As $\mathrm{e}^{2y} > 0$ the real part of the complex number is positive.

The imaginary part is

$$\frac{\mathrm{e}^{4y} - 1}{\mathrm{e}^{4y} + 1}$$

and this varies continuously with y with

$$\lim_{y\to\infty} \frac{\mathrm{e}^{4y} - 1}{\mathrm{e}^{4y} + 1} = 1$$

and

$$\lim_{y\to-\infty} \frac{\mathrm{e}^{4y}-1}{\mathrm{e}^{4y}+1} = -1$$

Thus as all values between -1 and $+1$ are attained it follows that the set G is the half circle from $-i$ to i in the right half plane and $-i$ and $+i$ are not in the set.

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13. *The following was in the class test in 2018/2019 and was worth 16 marks.*

The complex cos, sin, cosh and sinh functions are defined by

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cosh(z) = \frac{e^{z} + e^{-z}}{2}, \quad \sinh(z) = \frac{e^{z} - e^{-z}}{2}.$$

(a) By using these definitions and properties of the exponential function show that for all complex numbers z_1 and z_2

$$2\sin\left(\frac{z_1+z_2}{2}\right)\sin\left(\frac{z_1-z_2}{2}\right) = \cos(z_2) - \cos(z_1).$$

(b) By making use of the identities above (including part (a)), or otherwise, explain why

$$\cos(z) - \cosh(z) = 0$$

has solutions $z = (1-i)k\pi$ and $z = (1+i)k\pi$ for all integers k.

Solution

(a) By considering twice the expression and using the definitions gives

$$4\sin\left(\frac{z_1+z_2}{2}\right)\sin\left(\frac{z_1-z_2}{2}\right) = -\left(e^{i(z_1+z_2)/2} - e^{-i(z_1+z_2)/2}\right)\left(e^{i(z_1-z_2)/2} - e^{-i(z_1-z_2)/2}\right).$$

Now by properties of the exponential

$$\begin{aligned} e^{i(z_1+z_2)/2}e^{i(z_1-z_2)/2} &= e^{iz_1}, \\ e^{i(z_1+z_2)/2}e^{-i(z_1-z_2)/2} &= e^{iz_2}, \\ e^{-i(z_1+z_2)/2}e^{i(z_1-z_2)/2} &= e^{-iz_2}, \\ e^{-i(z_1+z_2)/2}e^{-i(z_1-z_2)/2} &= e^{-iz_1}. \end{aligned}$$

Hence

$$\begin{aligned} 4\sin\left(\frac{z_1+z_2}{2}\right)\sin\left(\frac{z_1-z_2}{2}\right) &= -(e^{iz_1} - e^{iz_2} - e^{-iz_2} + e^{-iz_1}) \\ &= 2\cos(z_2) - 2\cos(z_1) \end{aligned}$$

and the result follows.

(b) Now

$$\cosh(z) = \cos(iz)$$

and thus

$$\cos(z) - \cosh(z) = \cos(z) - \cos(iz) = -2\sin\frac{(1+i)z}{2}\sin\frac{(1-i)z}{2}.$$

$\sin w = 0$ if and only if $w = k\pi$ where k is an integer.

$$\frac{(1+i)z}{2} = k\pi \quad \text{gives} \quad z = \frac{2k\pi}{1+i} = (1-i)k\pi.$$

$$\frac{(1-i)z}{2} = k\pi \quad \text{gives} \quad z = \frac{2k\pi}{1-i} = (1+i)k\pi.$$

14. *The following was in the class test in 2017/2018 and was worth 8 marks.*

It can be shown that $\tan z$ can be written as

$$\tan z = (-i)\left(\frac{1-\mathrm{e}^{-2iz}}{1+\mathrm{e}^{-2iz}}\right).$$

By using this expression, or otherwise, describe in words the following sets.

$$\begin{aligned} S_1 &= \{\tan(iy) : y \in \mathbb{R}\}, \\ S_2 &= \{\tan(\pi/2 + iy) : y > 0\}. \end{aligned}$$

In your answer you need to indicate if the set is part of the real axis, or part of the imaginary axis or any other line segment.

Solution

When $z = iy$, $-iz = y$ and we have

$$\tan(iy) = (-i)\left(\frac{1-\mathrm{e}^{2y}}{1+\mathrm{e}^{2y}}\right) = (-i)\left(\frac{-1+\mathrm{e}^{-2y}}{1+\mathrm{e}^{-2y}}\right).$$

This is purely imaginary. When $y \to +\infty$ the number tends to i and when $y \to -\infty$ it tends to $-i$. Thus S_1 is the part of the imaginary axis from $-i$ to i with the end points not included.

When $z = \pi/2 + iy$, $-2iz = 2y - i\pi$ and

$$\exp(-2iz) = \mathrm{e}^{2y}\mathrm{e}^{-i\pi} = -\mathrm{e}^{2y}.$$

Thus

$$\tan(\pi/2 + iy) = (-i)\left(\frac{1+\mathrm{e}^{2y}}{1-\mathrm{e}^{2y}}\right) = i\left(\frac{\mathrm{e}^{2y}+1}{\mathrm{e}^{2y}-1}\right).$$

This is purely imaginary. When $y \to +\infty$ the number tends to i and when $y > 0$ and $y \to 0$ it tends to $+\infty$. Also note that

$$\left(\frac{\mathrm{e}^{2y}+1}{\mathrm{e}^{2y}-1}\right) > 1.$$

Thus S_2 is the part of the imaginary axis above i.

15. Give the definition of the principal value of z^α and show that

$$\frac{\mathrm{d}}{\mathrm{d}z} z^\alpha = \alpha\, z^{\alpha-1}.$$

Solution

The principal value of z^α is defined as

$$z^\alpha = \exp(\alpha \operatorname{Log} z).$$

Differentiating using the chain rule gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}z^{\alpha} &= \exp(\alpha \mathrm{Log}\, z)\frac{\mathrm{d}}{\mathrm{d}z}(\alpha \mathrm{Log}\, z) \\ &= \exp(\alpha \mathrm{Log}\, z)\left(\frac{\alpha}{z}\right) \\ &= \alpha\frac{\exp(\alpha \mathrm{Log}\, z)}{\exp(\mathrm{Log}\, z)} \\ &= \alpha \exp((\alpha-1)\mathrm{Log}\, z) \\ &= \alpha z^{\alpha-1}\end{aligned}$$

where the last part is a consequence of the properties of the exponential.

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16. *This question was in the class test in Dec 2021 and was worth 9 marks.*

This question is for all student numbers.

Let $z = r\mathrm{e}^{i\theta}$ where $r > 0$ and $\theta \in (-\pi, \pi]$.

(i) In terms of r and θ give the real and imaginary parts and the magnitude of $(\mathrm{Log}(z))^2$. Here $\mathrm{Log}(z)$ means the principal valued logarithm.

(ii) Show that the principal value of z^{1+i} is real when r and θ satisfy $r\mathrm{e}^{\theta} = \mathrm{e}^{\pi}$.

Solution

This version is for all student numbers.

(i)

$$\mathrm{Log}(z) = \ln(r) + i\theta, \quad (\mathrm{Log}(z))^2 = (\ln(r)+i\theta)^2 = (\ln(r))^2 - \theta^2 + 2i\theta\ln(r).$$

$(\ln(r))^2 - \theta^2$ is the real part and $2\theta\ln(r)$ is the imaginary part. The magnitude is

$$|(\mathrm{Log}(z))^2| = |\mathrm{Log}(z)|^2 = (\ln(r))^2 + \theta^2.$$

(ii)

$$(1+i)\mathrm{Log}(z) = (1+i)(\ln(r)+i\theta) = (\ln(r)-\theta) + i(\ln(r)+\theta).$$

$$z^{1+i} = \exp((1+i)\mathrm{Log}(z)) = \mathrm{e}^{\ln(r)-\theta}\left(\cos(\ln(r)+\theta) + i\sin(\ln(r)+\theta)\right).$$

This number is real when

$$\sin(\ln(r)+\theta) = 0, \quad \text{i.e. when } \ln(r)+\theta = k\pi, \quad k \in \mathbb{Z}.$$

By taking the exponential gives

$$r\mathrm{e}^{\theta} = \mathrm{e}^{k\pi}.$$

The case in the question is $k = 1$ and thus when $r\mathrm{e}^{\theta} = \mathrm{e}^{\pi}$ the value is real.

17. *This question was in the class test in Jan 2021 and was worth 12 marks.*

In the following you should attempt either part (a) or part (b) depending on the 5th digit of your 7-digit student id..

(a) If the 5th digit is one of the numbers $0, 1, 2, 3, 4$ then you do this case.

(i) Let $z_1 = \sqrt{3} + i$ and $z_2 = z_1^7$. With Log denoting the principal value logarithm determine $\mathrm{Log}(z_1)$ and $\mathrm{Log}(z_2)$ stating your answer in cartesian form.

(ii) Let

$$\Gamma = \left\{ z = r\mathrm{e}^{2\pi i/3} :\ 0 < r < \infty \right\}.$$

Give in polar form the image set $\{w = f(z) : z \in \Gamma\}$ when $f(z)$ denotes the principal value complex power

$$f(z) = z^{2+i},$$

i.e. the complex exponent is $2 + i$.
Give any value of $r \in (0, \infty)$ such that the value is pure imaginary.

(b) If the 5th digit is one of the numbers $5, 6, 7, 8, 9$ then you do this case.

(i) Let $z_1 = 1 + \sqrt{3}i$ and $z_2 = z_1^5$. With Log denoting the principal value logarithm determine $\mathrm{Log}(z_1)$ and $\mathrm{Log}(z_2)$ stating your answer in cartesian form.

(ii) Let

$$\Gamma = \left\{ z = r\mathrm{e}^{5\pi i/6} :\ 0 < r < \infty \right\}.$$

Give in polar form the image set $\{w = f(z) : z \in \Gamma\}$ when $f(z)$ denotes the principal value complex power

$$f(z) = z^{3+i},$$

i.e. the complex exponent is $3 + i$.
Give any value of $r \in (0, \infty)$ such that the value is real.

Solution

(a) The version if the 5th digit is one of the digits $0, 1, 2, 3, 4$.

i.

$$z_1 = \sqrt{3} + i, \quad |z_1|^2 = 4, \quad \mathrm{Arg}(z_1) = \frac{\pi}{6}.$$

Thus

$$\mathrm{Log}(z_1) = \ln(2) + i\frac{\pi}{6}.$$

$$z_2 = z_1^7 = 2^7 \exp(i7\pi/6) = 2^7 \exp(-i5\pi/6).$$

Hence

$$\mathrm{Log}(z_2) = 7\ln(2) - i\frac{5\pi}{6}.$$

ii. Let $\alpha = 2 + i$.

$$\mathrm{Log}(z) = \ln(r) + i\frac{2\pi}{3}.$$

$$\alpha \mathrm{Log}(z) = (2+i)\left(\ln(r) + i\frac{2\pi}{3}\right) = 2\ln(r) - \frac{2\pi}{3} + i\left(\ln(r) + \frac{4\pi}{3}\right).$$

Next using the exp function gives

$$f(z) = \exp\left(2\ln(r) - \frac{2\pi}{3}\right)(\cos(\theta) + i\sin(\theta)), \quad \theta = \ln(r) + \frac{4\pi}{3}.$$

This is pure imaginary when

$$\theta = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}.$$

When $k = 0$ we have

$$\ln(r) + \frac{4\pi}{3} = \frac{\pi}{2}, \quad \ln(r) = -\frac{5\pi}{6}, \quad r = \mathrm{e}^{-5\pi/6}.$$

(b) The version if the 5th digit is one of the digits $5, 6, 7, 8, 9$.

i.

$$z_1 = 1 + i\sqrt{3}, \quad |z_1|^2 = 4, \quad \mathrm{Arg}(z_1) = \frac{\pi}{3}.$$

Thus

$$\mathrm{Log}(z_1) = \ln(2) + i\frac{\pi}{3}.$$

$$z_2 = z_1^5 = 2^5\exp(i5\pi/3) = 2^5\exp(-i\pi/3).$$

Hence

$$\mathrm{Log}(z_2) = 5\ln(2) - i\frac{\pi}{3}.$$

ii. Let $\alpha = 3 + i$.

$$\mathrm{Log}(z) = \ln(r) + i\frac{5\pi}{6}.$$

$$\alpha \mathrm{Log}(z) = (3+i)\left(\ln(r) + i\frac{5\pi}{6}\right) = 3\ln(r) - \frac{5\pi}{6} + i\left(\ln(r) + \frac{5\pi}{2}\right).$$

Next using the exp function gives

$$f(z) = \exp\left(3\ln(r) - \frac{5\pi}{6}\right)(\cos(\theta) + i\sin(\theta)), \quad \theta = \ln(r) + \frac{5\pi}{2}.$$

This is real when

$$\theta = k\pi, \quad k \in \mathbb{Z}.$$

When $k = 0$ we have

$$\ln(r) + \frac{5\pi}{2} = 0, \quad \ln(r) = -\frac{5\pi}{2}, \quad r = \mathrm{e}^{-5\pi/2}.$$

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18. (a) Give in cartesian form the value of $\mathrm{Log}(2-2i)$.

    (b) Let $z = r\mathrm{e}^{i\theta}$ with $r > 0$ and with $-\pi < \theta \le \pi$. As concisely as possible, give an expression for the imaginary part of the principal value of z^{2i}. Explain why the imaginary part of the principal value of z^{2i} is 0 when $r = \mathrm{e}^{3\pi/2}$.

Solution

(a) If $z = 2(1-i)$ then $|z|^2 = 8$ and $|z| = \sqrt{8}$.
The principal argument of z is $-\pi/4$. Thus

$$\mathrm{Log}(z) = \frac{\ln(8)}{2} - i\frac{\pi}{4} = \frac{3\ln(2)}{2} - i\frac{\pi}{4}.$$

(b)

$$\mathrm{Log}(z) = \ln(r) + i\theta.$$

With $\alpha = 2i$,

$$\alpha \mathrm{Log}(z) = -2\theta + i(2\ln(r))$$

and

$$z^\alpha = \mathrm{e}^{-2\theta}(\cos(2\ln(r)) + i\sin(2\ln(r))).$$

The imaginary part is

$$\mathrm{e}^{-2\theta}\sin(2\ln(r)).$$

If $r = \mathrm{e}^{3\pi/2}$ then $2\ln(r) = 3\pi$ and $\sin(3\pi) = 0$.

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19. Let $z = x + iy = r\mathrm{e}^{i\theta}$ with x, y, r, $\theta \in \mathbb{R}$, $r > 0$ and $\theta \in (-\pi, \pi]$.

    (a) In terms of x and y give the real part of the principal value of i^z.

    (b) In terms of r and θ give the imaginary part of the principal value of z^i.

Solution

(a) From the definition

$$i^z = \exp(z \mathrm{Log}\, i) \quad \text{and } \mathrm{Log}\, i = i\frac{\pi}{2}.$$

Thus

$$i^z = \exp\left(\frac{i\pi(x+iy)}{2}\right) = \exp\left(\frac{-\pi y}{2}\right)\exp\left(\frac{i\pi x}{2}\right)$$

and the real part is

$$\exp(-\pi y/2)\cos\left(\frac{\pi x}{2}\right).$$

(b) From the definition, and with $\theta = \mathrm{Arg}\, z$,

$$z^i = \exp(i\mathrm{Log}\, z) = \exp(i(\ln r + i\theta)) = \mathrm{e}^{-\theta}\mathrm{e}^{i\ln r}.$$

The imaginary part is

$$\mathrm{e}^{-\theta}\sin(\ln r).$$